Pricing American-style Derivatives under the Heston Model Dynamics:
A Fast Fourier Transformation in the Geske–Johnson Scheme

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Introduction and Motivation

- European options: solution available for broad class of AJDs (Duffie, Pan, Singleton, 2000)
- Pricing American options has practical importance
- Popular approaches: finite difference schemes, simulation methods
- Efficient methods exist for Black–Scholes dynamics
Plan of Talk

- Notation and Model
- Geske–Johnson Scheme: “Bermudan” recursion
- Specifics of:
  - joint characteristic function
  - characteristic function inversion
- Empirical application: pricing of S&P 100 options
- Pros and cons of FFT
Model

Heston’s dynamics (under RNPM):

\[ dS_t = (r - \delta) S_t dt + \sqrt{v_t} S_t dW_{1t} \quad \Leftrightarrow \quad ds_t = \left( r - \delta - \frac{v_t}{2} \right) dt + \sqrt{v_t} dW_{1t} \]
\[ dv_t = (\alpha - \beta v_t) dt + \gamma \sqrt{v_t} dW_{2t} \]

\{W_{1t}, W_{2t}\}_{t \geq 0} : \text{Brownian motions on } \left( \Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0} , \hat{P} \right) \]

Imperfect correlation: \( d \langle W_1, W_2 \rangle_t = \rho dt \text{ and } |\rho| < 1 \)

Money-market fund: \( M_t = M_0 e^{rt} \), where \( M_0 > 0 \)
Geske–Johnson Scheme

Sequence of “Bermudan”-style derivatives: \( \{D_n (s_t, v_t, T - t)\}_{n=1}^{\infty} \)

\( D_n \) can be exercised at times \( t_j = t + \frac{j(T-t)}{n}, \ j = 1, ..., n \)

American option: \( D_\infty \)    European option: \( D_1 \)

“Bermudan” recursion:
\[
D_n (s_t, v_t, T - t) =
\]
\[
e^{-r(t_1-t)} \hat{E} \left[ \max \left\{ EX (s_{t_1}, v_{t_1}, T - t_1), D_{n-1} (s_{t_1}, v_{t_1}, T - t_1) \right\} \right]
\]

Exercise value for put: \( EX (s_{t'}, v_{t'}, T - t') = (X - e^{s_{t'}})^+ \)

Linear Richardson extrapolation: \( D_\infty \approx 2D_2 - D_1 \)
Joint Characteristic Function

Conditional ch. f.: $\Psi(t) = \hat{E} \left[ e^{i(\zeta_1 s_T + \zeta_2 v_T)} | \mathcal{F}_t \right] = \Psi(\zeta, s, v, \tau)$

Martingale property: $\hat{E} \left[ d\Psi(t) | \mathcal{F}_t \right] = 0$

P.d.e.:

$\Psi_r = \Psi_s \left( r - \delta - \frac{v_t}{2} \right) + \Psi_v (\alpha - \beta v_t) + \Psi_{ss} \frac{v_t}{2} + \Psi_{vv} \gamma^2 \frac{v_t}{2} + \Psi_{sv} \rho \gamma v_t$

Trial solution:

$\Psi(t) = \Psi(\zeta_1, \zeta_2; s, v, \tau) = \exp \left[ p(\tau; \zeta_1, \zeta_2) + q(\tau; \zeta_1, \zeta_2) v_t + i\zeta_1 s_t \right]$

Solve analytically for: $p(\tau; \zeta_1, \zeta_2)$ and $q(\tau; \zeta_1, \zeta_2)$

Solution is rather involved
Characteristic Function: Real Part
Joint Density Function: One Step

Inversion: \[ f(s_T, v_T; s_t, v_t, \tau) = \]
\[ = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\zeta_1 s_T + \zeta_2 v_T)} \Psi(\zeta_1, \zeta_2; s_t, v_t, \tau) \, d\zeta_1 \, d\zeta_2 \]

Discrete Fourier transform: \[ (2\pi)^2 f(s_T, k_1, v_T, k_2) \approx \]
\[ \approx \Delta_1 \Delta_2 e^{-i(s_T, k_1 a_1 + v_T, k_2 a_2)} \sum_{j_2=0}^{N_2-1} \sum_{j_1=0}^{N_1-1} e^{-i \cdot 2\pi \left( k_1 \frac{j_1}{N_1} + k_2 \frac{j_2}{N_2} \right)} \Psi_{j_1, j_2} \]

Apply discrete FFT algorithm to \[ \sum_{j_2=0}^{N_2-1} \sum_{j_1=0}^{N_1-1} \]
\[ \Rightarrow \]
\[ \Rightarrow \text{restore } f \text{ on } s_T, v_T \text{ grid from } \Psi \text{ on } \zeta_1, \zeta_2 \text{ grid in one step} \]
Joint Density Function
Empirical Application

Data:
- CBOE S&P 100 options: American (OEX), European (XEO)
- Closing prices on June 30th – July 2nd, July 6th – July 9th, 2004
- July 9th for out-of-sample predictions

Calibrated parameters (on XEO):

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
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<td>$v_{t,01}$</td>
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<td>$\alpha$</td>
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## Empirical Application: Results

<table>
<thead>
<tr>
<th></th>
<th>June 30</th>
<th>July 1</th>
<th>July 2</th>
<th>July 6</th>
<th>July 7</th>
<th>July 8</th>
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<tr>
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### Errors:

<p>| | | | | | | | | |</p>
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### RMSEs:

<table>
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<tr>
<th></th>
<th>June 30</th>
<th>July 1</th>
<th>July 2</th>
<th>July 6</th>
<th>July 7</th>
<th>July 8</th>
<th>July 9</th>
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<td>July</td>
<td>0.563594</td>
<td>0.729401</td>
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<td>August</td>
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</tbody>
</table>
Conclusion

Advantages of FFT:
- p.d.f. is recovered in one step
- allows to apply equivalent-martingale approach
- fast

Limitations of FFT:
- large RAM needed for speed and accuracy
- little flexibility in choosing grids