Hidden Secrets of Idiosyncratic Risk Premia: An Investigation on Cross-Sections of U.S. Stock Returns

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Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset such as T-bill

Asset prices are quoted continuously, but we will ultimately focus on only two dates: \( t = 0 \) and \( t = T \)

**Simplification:**

between 0 and \( T \), risk-free rate \( r \) is constant
Dynamics of market index:

\[
\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t
\]

where drift \( \mu_m \) is

\[
\mu_m = r + \delta \sigma_m
\]

- \( \sigma_m \): market volatility, \( \sigma_m > 0 \)
- \( \delta \): Sharpe ratio of market index, non-identifiable
- \( \{W_t\} \): systematic risk, modeled as standard Brownian motion
Dynamics of stock $i$ for $i = 1, 2, ...$:

\[
\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i
\]

where drift $\mu_i$ is

\[
\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i
\]

- $\beta_i \sim UNI [\kappa_\beta, \kappa_\beta + \lambda_\beta]$: systematic risk loading of stock $i$
- $\sigma_i \sim UNI [0, \lambda_\sigma]$: idiosyncratic volatility of stock $i$
- $\gamma$: idiosyncratic risk premium
- $\{Z_t^i\}$: idiosyncratic risk, modeled as standard Brownian motion
Contribution to Finance Literature

Estimating $\gamma$ helps inform debate over idiosyncratic premium:

- value of $\gamma$ affects construction of investment strategies

Estimating $\sigma_m$ from cross-sectional data is complementary to time-series approach:

- many pricing applications require volatility estimates

Remark:
Our estimation method differs from traditional regression technique of Fama & MacBeth (1973)
Cross-Sectional Dependence

Using Itô’s lemma:

\[
\frac{S_T^i}{S_0^i} = \exp \left[ \left( \mu_i - 0.5 \beta_i^2 \sigma_m^2 - 0.5 \sigma_i^2 \right) T + \beta_i \sigma_m W_T + \sigma_i Z_T^i \right]
\]

\[
\frac{M_T}{M_0} = \exp \left[ \left( \mu_m - 0.5 \sigma_m^2 \right) T + \sigma_m W_T \right]
\]

where \( W_T, Z_T^i \) for \( i = 1, 2, ... \) \( \sim i.i.d. \ N(0, T) \)

Common shock \( W_T \) induces dependence among \( \frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ... \)

But easy to see that \( \frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ... \) are conditionally i.i.d. given \( \frac{M_T}{M_0} \)

more on econometrics
GMM Implementation: Moment Restrictions

Let $\theta = (\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma)'$. Consider:

$$g_i (\xi; \theta) = \left( S_{iT}^i / S_{i0}^i \right)^{\xi} - E_\theta \left[ \left( S_{iT}^i / S_{i0}^i \right)^{\xi} \mid M_T / M_0 \right]$$

Given constants $\xi_1, ..., \xi_k$, let $k \times 1$ vector of moment restrictions be:

$$g_i (\theta) = (g_i (\xi_1; \theta), ..., g_i (\xi_k; \theta))'$$

Result:
For any finite $\xi$, $E_\theta \left[ \left( S_{iT}^i / S_{i0}^i \right)^{\xi} \mid M_T / M_0 \right]$ exists and can be expressed analytically.
GMM Estimators

**One-step** estimation using $k \times k$ positive definite $\Sigma$:

$$Q_{1,n}(\theta) = \left(n^{-1} \sum_i g_i(\theta)\right)' \Sigma^{-1} \left(n^{-1} \sum_i g_i(\theta)\right)$$

$$\hat{\theta}_{1,n} = \arg\min_{\theta \in \Theta} Q_{1,n}(\theta)$$

**Two-step** estimation using $\hat{\Sigma}_{1,n} = n^{-1} \sum_i g_i(\hat{\theta}_{1,n}) g_i(\hat{\theta}_{1,n})'$:

$$Q_{2,n}(\theta) = \left(n^{-1} \sum_i g_i(\theta)\right)' \hat{\Sigma}_{1,n}^{-1} \left(n^{-1} \sum_i g_i(\theta)\right)$$

$$\hat{\theta}_{2,n} = \arg\min_{\theta \in \Theta} Q_{2,n}(\theta)$$
Asymptotics

**Theorem:** Under very general regularity conditions:

\[
\hat{\theta}_{1,n} \xrightarrow{p} \theta_0 \\
\hat{\theta}_{2,n} \xrightarrow{p} \theta_0
\]

**Theorem:** Under additional regularity conditions:

\[
\sqrt{n} \left( \hat{\theta}_{1,n} - \theta_0 \right) \xrightarrow{d} MN \left( 0, V_{1,F_0} \right) \\
\sqrt{n} \left( \hat{\theta}_{2,n} - \theta_0 \right) \xrightarrow{d} MN \left( 0, V_{2,F_0} \right)
\]

$V_{1,F_0}, V_{2,F_0}$ are $p \times p$ positive definite **stochastic** matrices
Consider testing $r$ parametric restrictions:

$$H_0 : a(\theta_0) = 0$$

Let $A(\cdot)$ be Jacobian of $a(\cdot)$. Under $H_0$, **Wald test** statistic

$$W_n \equiv n a(\hat{\theta}_{2,n})' \left[ A(\hat{\theta}_{2,n}) V_{2,n} A(\hat{\theta}_{2,n})' \right]^{-1} a(\hat{\theta}_{2,n}) \rightarrow^d \chi^2(r)$$

If the model is correctly specified, **OIR test** statistic

$$J_n \equiv n \cdot Q_{2,n}(\hat{\theta}_{2,n}) \rightarrow^d \chi^2(k - p)$$
Data

Sources:

- stock data: Center for Research in Security Prices (CRSP)
- T-bill data: Federal Reserve Bank Reports

CRSP provides extensive information on securities traded on NYSE, AMEX, and NASDAQ, but not all securities are used.

We only include regularly traded stocks of **operating companies**:

- drop closed-end funds, ETFs, financial REITs
- include ADRs
- if company issues 2+ classes of shares, include only one class with largest number of outstanding shares
Sample of Estimation Results

- Date 0: January 10, 2008; date T: January 11, 2008
- Sample size \( n = 5,407 \)
- Number of moment restrictions \( k = 8 \)
- S&P 500 index return \( M_T/M_0 = 0.986 \)
- T-bill rate \( r = 3.26\% \)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>(P-value)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_m )</td>
<td>0.07</td>
<td>(0.00)</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>6.36</td>
<td>(0.00)</td>
</tr>
<tr>
<td>( \kappa_\beta )</td>
<td>2.42</td>
<td>(0.85)</td>
</tr>
<tr>
<td>( \lambda_\beta )</td>
<td>0.05</td>
<td>(0.99)</td>
</tr>
<tr>
<td>( \lambda_\sigma )</td>
<td>1.70</td>
<td>(0.00)</td>
</tr>
<tr>
<td>OIR test ( J_n )</td>
<td>5.93</td>
<td>(0.11)</td>
</tr>
</tbody>
</table>
Thank you!
Questions?
Recall:

\[ \mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i \]

If \( \gamma = 0 \), our price dynamics are in line with:

- ICAPM with constant invest. opportunity set: Merton (1973)
- APT with one market factor: Ross (1976)

**But** growing literature suggests that idiosyncratic risk is priced:

- Epstein & Schneider (2008): ambiguity premium

Green & Rydqvist (1997), Ang et al. (2006), Fu (2009): idiosyncratic premium \( \neq 0 \), but no consensus about sign
Large literature on **localized** common shocks:

- general approach: Conley (1999)
- spatial, group, social effects: e.g., Kelejian & Prucha (1999), Lee (2007), Bramoullé et al. (2009)

Sparse literature on **non-localized** common shocks:


We build on Andrews (2003) to develop GMM estimation theory under a non-localized common shock, which is induced by $W_T$.
Random variable $Y$ has **mixed normal distribution**

\[ Y \sim MN \left( 0, \eta^2 \right) \]

if characteristic function of $Y$ is

\[ \phi_Y (t) \equiv E \left[ \exp \left( itY \right) \right] = E \left[ \exp \left( -\frac{1}{2} \eta^2 t^2 \right) \right] \]

where $\eta$ is random variable

$Y$ can be represented as

\[ Y = \eta Z \]

where $Z \sim N \left( 0, 1 \right)$ and $Z$ is independent of $\eta$
\[ W_n \equiv n a (\hat{\theta}_{2,n})' \left[ A (\hat{\theta}_{2,n}) V_{2,n} A (\hat{\theta}_{2,n})' \right]^{-1} a (\hat{\theta}_{2,n}) \]

\[ V_{2,n} = \left[ G'_{2,n} \hat{\Sigma}_{2,n}^{-1} G_{2,n} \right]^{-1} \]

\[ G_{2,n} = n^{-1} \sum_i \partial g_i (\hat{\theta}_{2,n}) / \partial \theta' \]

\[ \hat{\Sigma}_{2,n} = n^{-1} \sum_i g_i (\hat{\theta}_{2,n}) g_i (\hat{\theta}_{2,n})' \]
Further Directions

Currently in progress:

- estimation of model parameters

Extensions of financial model:

- multi-factor stock price model
- stochastic volatility setting

Direction for future econometric research:

- MLE under common non-localized shocks