Hidden Secrets of Idiosyncratic Risk Premia: An Investigation on Cross-Sections of U.S. Stock Returns

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Financial Market Structure

Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset such as T-bill

Asset prices are quoted continuously, but we will ultimately focus on only two dates: $t = 0$ and $t = T$

Simplification:

between 0 and T, risk-free rate r is constant

Market Index Price Dynamics

Dynamics of market index:

$$\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t$$

where drift μ_m is

$$\mu_m = r + \delta\sigma_m$$

- σ_m: market volatility, $\sigma_m > 0$
- δ: Sharpe ratio of market index, non-identifiable
- $\{W_t\}$: systematic risk, modeled as standard Brownian motion

Stock Price Dynamics

Dynamics of stock i for $i = 1, 2, ...$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift μ_i is

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- $\beta_i \sim UNI\left[\kappa_\beta, \kappa_\beta + \lambda_\beta\right]$: systematic risk loading of stock i
- $\sigma_i \sim UNI\left[0, \lambda_\sigma\right]$: idiosyncratic volatility of stock i
- γ: idiosyncratic risk premium
- $\left\{Z_t^i\right\}$: idiosyncratic risk, modeled as standard Brownian motion

Contribution to Finance Literature

Estimating γ helps inform debate over idiosyncratic premium:

- value of γ affects construction of investment strategies

Estimating σ_m from cross-sectional data is complementary to time-series approach:

- many pricing applications require volatility estimates

Remark:

Our estimation method differs from traditional regression technique of Fama & MacBeth (1973)

▸ more on finance literature

Cross-Sectional Dependence

Using Itô's lemma:

$$\frac{S_T^i}{S_0^i} = \exp\left[\left(\mu_i - 0.5\beta_i^2\sigma_m^2 - 0.5\sigma_i^2\right)T + \beta_i\sigma_m W_T + \sigma_i Z_T^i\right]$$

$$\frac{M_T}{M_0} = \exp\left[\left(\mu_m - 0.5\sigma_m^2\right)T + \sigma_m W_T\right]$$

where W_T, Z_T^i for $i = 1, 2, ... \sim i.i.d.\ N(0, T)$

Common shock W_T induces **dependence** among $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ...$

But easy to see that $\frac{S_T^1}{S_0^1}, \frac{S_T^2}{S_0^2}, ...$ are **conditionally i.i.d.** given $\frac{M_T}{M_0}$

▸ more on econometrics

GMM Implementation: Moment Restrictions

Let $\boldsymbol{\theta} = \left(\sigma_m, \gamma, \kappa_\beta, \lambda_\beta, \lambda_\sigma\right)'$. Consider:

$$g_i\left(\xi; \boldsymbol{\theta}\right) = \left(S_T^i / S_0^i\right)^{\xi} - E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^{\xi} | M_T / M_0\right]$$

Given constants $\xi_1, ..., \xi_k$, let $k \times 1$ vector of moment restrictions be:

$$\underset{(k\times 1)}{\boldsymbol{g}_i}\left(\boldsymbol{\theta}\right) = \left(g_i\left(\xi_1; \boldsymbol{\theta}\right), ..., g_i\left(\xi_k; \boldsymbol{\theta}\right)\right)'$$

Result:

For any finite ξ, $E_{\boldsymbol{\theta}}\left[\left(S_T^i / S_0^i\right)^{\xi} | M_T / M_0\right]$ exists and can be expressed analytically

GMM Estimators

One-step estimation using $k \times k$ positive definite $\boldsymbol{\Sigma}$:

$$Q_{1,n}(\boldsymbol{\theta}) = \left(n^{-1} \sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)' \boldsymbol{\Sigma}^{-1} \left(n^{-1} \sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)$$

$$\widehat{\boldsymbol{\theta}}_{1,n} = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} Q_{1,n}(\boldsymbol{\theta})$$

Two-step estimation using $\widehat{\boldsymbol{\Sigma}}_{1,n} = n^{-1} \sum_i \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right) \boldsymbol{g}_i\left(\widehat{\boldsymbol{\theta}}_{1,n}\right)'$:

$$Q_{2,n}(\boldsymbol{\theta}) = \left(n^{-1} \sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)' \widehat{\boldsymbol{\Sigma}}_{1,n}^{-1} \left(n^{-1} \sum_i \boldsymbol{g}_i(\boldsymbol{\theta})\right)$$

$$\widehat{\boldsymbol{\theta}}_{2,n} = \arg\min_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} Q_{2,n}(\boldsymbol{\theta})$$

Asymptotics

Theorem: Under very general regularity conditions:

$$\widehat{\boldsymbol{\theta}}_{1,n} \to^{p} \boldsymbol{\theta}_0$$

$$\widehat{\boldsymbol{\theta}}_{2,n} \to^{p} \boldsymbol{\theta}_0$$

Theorem: Under additional regularity conditions:

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{1,n} - \boldsymbol{\theta}_0\right) \to^{d} MN\left(\mathbf{0}, \mathbf{V}_{1,\mathcal{F}_0}\right)$$

$$\sqrt{n}\left(\widehat{\boldsymbol{\theta}}_{2,n} - \boldsymbol{\theta}_0\right) \to^{d} MN\left(\mathbf{0}, \mathbf{V}_{2,\mathcal{F}_0}\right)$$

$\mathbf{V}_{1,\mathcal{F}_0}$, $\mathbf{V}_{2,\mathcal{F}_0}$ are $p \times p$ positive definite **stochastic** matrices

▸ mixed normality

Inference and Specification Test

Consider testing r parametric restrictions:

$$H_0 : \underset{(r\times 1)}{\mathbf{a}\left(\boldsymbol{\theta}_0\right)} = \mathbf{0}$$

Let $\mathbf{A}(\cdot)$ be Jacobian of $\mathbf{a}(\cdot)$. Under H_0, **Wald test** statistic

$$W_n \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)' \left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)\mathbf{V}_{2,n}\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\right]^{-1} \mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right) \to^d \chi^2(r)$$

▸ formulas

If the model is correctly specified, **OIR test** statistic

$$J_n \equiv n \cdot Q_{2,n}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right) \to^d \chi^2(k-p)$$

Data

Sources:

- stock data: Center for Research in Security Prices (**CRSP**)
- T-bill data: Federal Reserve Bank Reports

CRSP provides extensive information on securities traded on NYSE, AMEX, and NASDAQ, but not all securities are used

We only include regularly traded stocks of **operating companies**:

- drop closed-end funds, ETFs, financial REITs
- include ADRs
- if company issues 2+ classes of shares, include only one class with largest number of outstanding shares

Sample of Estimation Results

- Date 0: January 10, 2008; date T: January 11, 2008
- Sample size $n = 5,407$
- Number of moment restrictions $k = 8$
- S&P 500 index return $M_T / M_0 = 0.986$
- T-bill rate $r = 3.26\%$

Parameter	Estimate	(P-value)
σ_m	0.07	(0.00)
γ	6.36	(0.00)
κ_β	2.42	(0.85)
λ_β	0.05	(0.99)
λ_σ	1.70	(0.00)
OIR test	$J_n = 5.93$	(0.11)

further directions

Thank you!

Questions?

Relationship to Finance Literature

Recall:

$$\mu_i = r + \delta\beta_i\sigma_m + \gamma\sigma_i$$

If $\gamma = 0$, our price dynamics are in line with:

- ICAPM with constant invest. opportunity set: Merton (1973)
- APT with one market factor: Ross (1976)

But growing literature suggests that idiosyncratic risk is priced:

- Merton (1987), Malkiel & Xu (2006): incomplete diversification
- Epstein & Schneider (2008): ambiguity premium

Green & Rydqvist (1997), Ang et al. (2006), Fu (2009):
idiosyncratic premium $\neq 0$, but no consensus about sign

return to contribution

Relationship to Econometrics Literature

Large literature on **localized** common shocks:

- general approach: Conley (1999)
- spatial, group, social effects: e.g., Kelejian & Prucha (1999), Lee (2007), Bramoullé et al. (2009)

Sparse literature on **non-localized** common shocks:

- Andrews (2003, 2005)

We build on Andrews (2003) to develop GMM estimation theory under a non-localized common shock, which is induced by W_T

◂ return to dependence

Mixed Normal Distribution

Random variable Y has **mixed normal distribution**

$$Y \sim MN\left(0, \eta^2\right)$$

if characteristic function of Y is

$$\phi_Y(t) \equiv E\left[\exp\left(itY\right)\right] = E\left[\exp\left(-\frac{1}{2}\eta^2 t^2\right)\right]$$

where η is random variable

Y can be represented as

$$Y = \eta Z$$

where $Z \sim N(0, 1)$ and Z is **independent** of η

◂ return to asymptotics

Inference: Formulas

$$W_n \equiv n\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\left[\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)\mathbf{V}_{2,n}\mathbf{A}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'\right]^{-1}\mathbf{a}\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)$$

$$\mathbf{V}_{2,n} = \left[\mathbf{G}_{2,n}'\widehat{\boldsymbol{\Sigma}}_{2,n}^{-1}\mathbf{G}_{2,n}\right]^{-1}$$

$$\mathbf{G}_{2,n} = n^{-1}\sum_i \partial g_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)/\partial\boldsymbol{\theta}'$$

$$\widehat{\boldsymbol{\Sigma}}_{2,n} = n^{-1}\sum_i g_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)g_i\left(\widehat{\boldsymbol{\theta}}_{2,n}\right)'$$

◂ return to inference

Further Directions

Currently in progress:

- estimation of model parameters

Extensions of financial model:

- multi-factor stock price model
- stochastic volatility setting

Direction for future econometric research:

- MLE under common non-localized shocks

◂ return to results