Impact of Idiosyncratic Volatility on Stock Returns: A Cross-Sectional Study

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Goals:

- Develop a method to consistently estimate parameters of a financial model using a single cross-section of return data
- Apply the method to compute idiosyncratic volatility parameters, including idiosyncratic volatility premium

Novelty:

- Estimation method differs from the two-pass regression approach of Fama & MacBeth (1973)
- GMM estimation is implemented under strong cross-sectional data dependence
Why focus on stock-specific idiosyncratic volatility (IV)?

- Classical finance models (e.g., Sharpe, 1964; Lintner, 1965):
  - IV commands **no premium** in capital market equilibrium

- Growing literature indicates that **IV may be priced**:
  - Epstein & Schneider (2008)

- **No consensus** on IV premium in empirical literature:
  - Fu (2009), Huang et al. (2010): **positive** premium
  - Ang et al. (2006), Jiang et al. (2009): **negative** premium
Financial assets:

- many risky assets called **stocks**
- one diversified portfolio of stocks called **market index**
- one riskless asset such as T-Bill

Asset prices are quoted continuously, but we will eventually focus on only two dates: \( t = 0 \) and \( t = T \)

*Simplifying assumption:*
Between 0 and \( T \), risk-free rate \( r \) is constant
Market Index Price Dynamics

Dynamics of market index:

\[
\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t
\]

where drift \( \mu_m \) is:

\[
\mu_m = r + \delta \sigma_m
\]

- \( \sigma_m \): market volatility, \( \sigma_m > 0 \)
- \( \delta \): Sharpe ratio of market index, non-identifiable
- \( \{W_t\} \): systematic risk source, modeled as Brownian motion
Dynamics of stock $i$ for $i = 1, 2, \ldots$:

$$\frac{dS_t^i}{S_t^i} = \mu_i dt + \beta_i \sigma_m dW_t + \sigma_i dZ_t^i$$

where drift $\mu_i$ is:

$$\mu_i = r + \delta \beta_i \sigma_m + \gamma \sigma_i$$

- $\beta_i \sim UNI [\kappa_\beta, \kappa_\beta + \lambda_\beta]$: beta of stock $i$
- $\sigma_i \sim UNI [0, \lambda_\sigma]$: idiosyncratic volatility of stock $i$
- $\gamma$: idiosyncratic volatility premium
- $\{Z_t^i\}$: idiosyncratic risk source, modeled as Brownian motion
Estimation Challenge

Using Ito’s lemma:

\[
\frac{S_i^T}{S_i^0} = \exp \left[ \left( \mu_i - 0.5\beta_i^2\sigma_m^2 - 0.5\sigma_i^2 \right) T + \beta_i\sigma_m W_T + \sigma_i Z^i_T \right]
\]

\[
\frac{M_T}{M_0} = \exp \left[ \left( \mu_m - 0.5\sigma_m^2 \right) T + \sigma_m W_T \right]
\]

\( W_T, Z_T^i \) for \( i = 1, 2, ... \) \( \sim\) i.i.d. \( N(0, T) \)

Common shock \( W_T \) induces dependence among \( \frac{S_1^T}{S_0^1}, \frac{S_2^T}{S_0^2}, ... \) \( \Rightarrow \)

\( \Rightarrow \) standard LLNs and CLTs are not applicable

But \( \frac{S_1^T}{S_0^1}, \frac{S_2^T}{S_0^2}, ... \) are conditionally i.i.d. given \( \frac{M_T}{M_0} \)
GMM Implementation

Let $\theta = (\sigma_m, \gamma, \kappa, \lambda, \lambda_\sigma)'$. We construct a function $g_i(\xi; \theta)$:

$$g_i(\xi; \theta) = \left( \frac{S_T^i}{S_0^i} \right)^\xi - E_\theta \left[ \left( \frac{S_T^i}{S_0^i} \right)^\xi | M_T/M_0 \right]$$

Given constants $\xi_1, ..., \xi_k$, $k \times 1$ vector of moment restrictions is:

$$g_i(\theta) = (g_i(\xi_1; \theta), ..., g_i(\xi_k; \theta))'$$

For any finite $\xi$, $E_\theta \left[ \left( \frac{S_T^i}{S_0^i} \right)^\xi | M_T/M_0 \right]$ exists and can be expressed analytically.
GMM Estimation

GMM objective function:

\[ Q_n(\theta) = \left( \frac{1}{n} \sum_i g_i(\theta) \right)' \Sigma^{-1} \left( \frac{1}{n} \sum_i g_i(\theta) \right) \]

\( \Sigma \) is a \( k \times k \) positive definite matrix

GMM estimator:

\[ \hat{\theta}_n = \arg \min_{\theta \in \Theta} Q_n(\theta) \]

As \( n \to \infty \), \( Q_n(\theta) \) converges to a **stochastic** function dependent on common shock
Properties of Estimator

Under general regularity conditions:

\[ \hat{\theta}_n \rightarrow^p \theta_0 \]

\(\hat{\theta}_n\) is **consistent** as \(n \rightarrow \infty\)

*Note:* two-pass regression requires \(T \rightarrow \infty\) (Shanken, 1992)

Under additional regularity conditions:

\[ \sqrt{n} \left( \hat{\theta}_n - \theta_0 \right) \rightarrow^d MN(0, V_{M_T/M_0}) \]

\(\hat{\theta}_n\) is asymptotically **mixed normal**. \(V_{M_T/M_0}\) is **stochastic**
Stock data are from **CRSP** database:
- include stocks of operating companies
- exclude bankruptcy cases, closed-end funds, ETFs, REITs

Market index is approximated by S&P 500 index

Risk-free rate is derived from T-Bill data

I show results for **weekly** return data from two months:
- **January 2008**: a month of relatively low market volatility
- **October 2008**: a month of relatively high market volatility
## Selected Empirical Results: Full Model Estimates

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_m )</td>
<td>0.0537 0.00</td>
<td>0.0672 0.00</td>
</tr>
<tr>
<td>( \gamma )</td>
<td>-2.1117 0.34</td>
<td>-1.2936 0.56</td>
</tr>
<tr>
<td>( \kappa \beta )</td>
<td>0.3417 0.74</td>
<td>-0.3058 0.74</td>
</tr>
<tr>
<td>( \lambda \beta )</td>
<td>3.0475 0.00</td>
<td>2.8367 0.00</td>
</tr>
<tr>
<td>( \lambda \sigma )</td>
<td>1.0580 0.00</td>
<td>1.7478 0.00</td>
</tr>
</tbody>
</table>

**Notes:**

- \( \sigma_m \): market volatility
- \( \gamma \): idiosyncratic volatility premium
- Beta of stock \( i, \beta_i \sim i.i.d.UNI[\kappa \beta, \kappa \beta + \lambda \beta] \)
- Idiosyncratic volatility of stock \( i, \sigma_i \sim i.i.d.UNI[0, \lambda \sigma] \)
- Moment order vector \( \xi = (-2, -1.5, -1, -0.5, 0.5, 1, 1.5, 2)' \)
## Selected Empirical Results: January 2008

<table>
<thead>
<tr>
<th>Interval</th>
<th>Idiosyncratic volatility premium, $\gamma$</th>
<th>Average idiosyncratic volatility, $\lambda_\sigma/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>P-value</td>
</tr>
<tr>
<td>January 02-09</td>
<td>$-4.7251$</td>
<td>0.00</td>
</tr>
<tr>
<td>03-10</td>
<td>$-5.0907$</td>
<td>0.00</td>
</tr>
<tr>
<td>04-11</td>
<td>$-8.0336$</td>
<td>0.00</td>
</tr>
<tr>
<td>07-14</td>
<td>$-0.8656$</td>
<td>0.00</td>
</tr>
<tr>
<td>08-15</td>
<td>$-4.4627$</td>
<td>0.00</td>
</tr>
<tr>
<td>09-16</td>
<td>$-9.1830$</td>
<td>0.00</td>
</tr>
<tr>
<td>Mean</td>
<td>$-6.0666$</td>
<td></td>
</tr>
<tr>
<td>Std. dev.</td>
<td>$3.6396$</td>
<td></td>
</tr>
</tbody>
</table>

...
## Selected Empirical Results: October 2008

<table>
<thead>
<tr>
<th>Interval</th>
<th>Idiosyncratic volatility premium, $\gamma$</th>
<th>Average idiosyncratic volatility, $\lambda_\sigma/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>P-value</td>
</tr>
<tr>
<td>October 01-08</td>
<td>−8.5104</td>
<td>0.00</td>
</tr>
<tr>
<td>02-09</td>
<td>−8.4123</td>
<td>0.00</td>
</tr>
<tr>
<td>03-10</td>
<td>−8.4921</td>
<td>0.00</td>
</tr>
<tr>
<td>06-13</td>
<td>−7.8321</td>
<td>0.00</td>
</tr>
<tr>
<td>07-14</td>
<td>−5.9003</td>
<td>0.00</td>
</tr>
<tr>
<td>08-15</td>
<td>−0.8830</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mean</td>
<td>−5.5748</td>
<td></td>
</tr>
<tr>
<td>Std. dev.</td>
<td>3.9705</td>
<td></td>
</tr>
</tbody>
</table>
We develop an econometric method to estimate a financial model featuring a common shock:

- the method differs from the two-pass regression approach
- consistent and asymptotically mixed normal estimators are obtained as the number of stocks \((n)\) grows
- estimation is implemented on a single return cross-section

Findings using returns from January and October 2008:

- IV premium is estimated to be negative
- average cross-sectional IV estimates increase by 50% between January and October
Thank you!

Questions?
Large literature on **localized** common shocks:

- general approach: Conley (1999)
- spatial, group, social effects: e.g., Kelejian & Prucha (1999), Lee (2007), Bramoullé et al. (2009)

Sparse literature on **non-localized** common shocks:


We build on Andrews (2003) to develop GMM estimation theory under a non-localized common shock in cross-sectional data
Random variable $Y$ has **mixed normal distribution**:

$$ Y \sim MN \left(0, \eta^2\right) $$

if characteristic function of $Y$ is:

$$ \phi_Y(t) \equiv E \left[\exp(itY)\right] = E \left[\exp \left(-\frac{1}{2} \eta^2 t^2\right)\right] $$

where $\eta$ is random variable

$Y$ can be represented as:

$$ Y = \eta Z $$

where $Z \sim N(0,1)$ and $Z$ is **independent** of $\eta$
Consider testing $r$ parametric restrictions:

$$H_0 : a(\theta_0) = 0$$

Let $A(\cdot)$ be Jacobian of $a(\cdot)$. Under $H_0$, **Wald test** statistic

$$W_n \equiv na(\hat{\theta}_n)' \left[ A(\hat{\theta}_n) V_n A(\hat{\theta}_n)' \right]^{-1} a(\hat{\theta}_n) \rightarrow^d \chi^2(r)$$

**OIR test** can be implemented after two-step estimation. If the model is correctly specified:

$$J_n \equiv n \cdot Q_{2,n}(\hat{\theta}_{2,n}) \rightarrow^d \chi^2(k - p)$$
Monte Carlo Results: 1-Step vs. 2-Step Estimation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>1-step RMSE</th>
<th>2-step RMSE</th>
<th>1-step Median–true value</th>
<th>2-step Median–true value</th>
<th>1-step Mean–true value</th>
<th>2-step Mean–true value</th>
<th>True value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_m$</td>
<td>0.5631</td>
<td>0.7967</td>
<td>0.0916</td>
<td>0.1433</td>
<td>0.2770</td>
<td>0.2643</td>
<td>0.2000</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.5337</td>
<td>0.7546</td>
<td>0.0014</td>
<td>0.0818</td>
<td>0.0223</td>
<td>0.1013</td>
<td>$-2.0000$</td>
</tr>
<tr>
<td>$\kappa_\beta$</td>
<td>1.3163</td>
<td>1.8470</td>
<td>0.0067</td>
<td>0.0514</td>
<td>0.0209</td>
<td>0.0611</td>
<td>0.5000</td>
</tr>
<tr>
<td>$\lambda_\beta$</td>
<td>2.4048</td>
<td>3.3329</td>
<td>0.3979</td>
<td>0.4720</td>
<td>0.0862</td>
<td>0.1034</td>
<td>3.0000</td>
</tr>
<tr>
<td>$\lambda_\sigma$</td>
<td>0.0277</td>
<td>0.0394</td>
<td>0.0039</td>
<td>0.0049</td>
<td>0.0052</td>
<td>0.0060</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

Notes:

- Number of stocks $n = 5,500$
- Number of simulation rounds: 1,000
- Risk-free rate $r = 0.01$
- Moment order vector $\xi = (-2, -1.5, -1, -0.5, 0.5, 1, 1.5, 2)'$
- Moment restrictions of the form: $g_i(\xi; \theta) = (S_T^i / S_0^i)^\xi - E_\theta \left[ (S_T^i / S_0^i)^\xi \mid M_T / M_0 \right]$
Conditional Return Decomposition: January 2008

<table>
<thead>
<tr>
<th>Interval</th>
<th>$E$</th>
<th>$S$</th>
<th>$\mathcal{I}$</th>
<th>$\frac{E-e^{rT}S}{E}$</th>
<th>$e^{rT}(S-I)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 02-09</td>
<td>0.9486</td>
<td>0.9988</td>
<td>0.9492</td>
<td>-0.0535</td>
<td>0.0523</td>
</tr>
<tr>
<td>03-10</td>
<td>0.9647</td>
<td>1.0173</td>
<td>0.9477</td>
<td>-0.0552</td>
<td>0.0722</td>
</tr>
<tr>
<td>04-11</td>
<td>0.9762</td>
<td>1.0512</td>
<td>0.9280</td>
<td>-0.0776</td>
<td>0.1263</td>
</tr>
<tr>
<td>07-14</td>
<td>0.9870</td>
<td>0.9955</td>
<td>0.9909</td>
<td>-0.0092</td>
<td>0.0047</td>
</tr>
<tr>
<td>08-15</td>
<td>0.9833</td>
<td>1.0277</td>
<td>0.9561</td>
<td>-0.0459</td>
<td>0.0729</td>
</tr>
<tr>
<td>09-16</td>
<td>0.9857</td>
<td>1.0741</td>
<td>0.9172</td>
<td>-0.0903</td>
<td>0.1593</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>Mean</td>
<td>0.9890</td>
<td>1.0488</td>
<td>0.9430</td>
<td>-0.0615</td>
<td>0.1071</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.0311</td>
<td>0.0337</td>
<td>0.0311</td>
<td>0.0346</td>
<td>0.0571</td>
</tr>
</tbody>
</table>

Notes:

- Conditional expected gross return $E = \exp(rT) \cdot S \left(M_T/M_0\right) \cdot \mathcal{I}$
- Risk-free component: $\exp(rT)$
- Market risk component: $S \left(M_T/M_0\right)$
- Idiosyncratic volatility component: $\mathcal{I}$
## Conditional Return Decomposition: October 2008

<table>
<thead>
<tr>
<th>Interval</th>
<th>$E$</th>
<th>$S$</th>
<th>$\mathcal{I}$</th>
<th>$\frac{E-e^{rT}S}{E}$</th>
<th>$\frac{e^{rT}(S-\mathcal{I})}{E}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 01-08</td>
<td>0.8211</td>
<td>0.9384</td>
<td>0.8750</td>
<td>-0.1429</td>
<td>0.0773</td>
</tr>
<tr>
<td>02-09</td>
<td>0.8022</td>
<td>0.9267</td>
<td>0.8657</td>
<td>-0.1551</td>
<td>0.0760</td>
</tr>
<tr>
<td>03-10</td>
<td>0.8163</td>
<td>0.9463</td>
<td>0.8626</td>
<td>-0.1593</td>
<td>0.1026</td>
</tr>
<tr>
<td>06-13</td>
<td>0.9455</td>
<td>1.0605</td>
<td>0.8916</td>
<td>-0.1216</td>
<td>0.1787</td>
</tr>
<tr>
<td>07-14</td>
<td>0.9852</td>
<td>1.0797</td>
<td>0.9125</td>
<td>-0.0959</td>
<td>0.1698</td>
</tr>
<tr>
<td>08-15</td>
<td>0.9352</td>
<td>0.9494</td>
<td>0.9851</td>
<td>-0.0151</td>
<td>-0.0382</td>
</tr>
<tr>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
<td>:</td>
</tr>
<tr>
<td>Mean</td>
<td>0.9438</td>
<td>1.0270</td>
<td>0.9184</td>
<td>-0.0929</td>
<td>0.1162</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.0834</td>
<td>0.0564</td>
<td>0.0574</td>
<td>0.0654</td>
<td>0.0778</td>
</tr>
</tbody>
</table>

### Notes:
- Conditional expected gross return $E = \exp (rT) \cdot S (M_T/M_0) \cdot \mathcal{I}$
- Risk-free component: $\exp (rT)$
- Market risk component: $S (M_T/M_0)$
- Idiosyncratic volatility component: $\mathcal{I}$