IOWA STATE UNIVERSITY

Symmetries of Imperfect Competition on a Circle

David A. Hennessy, Harvey E. Lapan

March 2006

Working Paper # 06010

Department of Economics Working Papers Series

Ames, Iowa 50011

Iowa State University does not discriminate on the basis of race, color, age, religion, national origin, sexual orientation, gender identity, sex, marital status, disability, or status as a U.S. veteran. Inquiries can be directed to the Director of Equal Opportunity and Diversity, 3680 Beardshear Hall, (515) 294-7612.

Symmetries of Imperfect Competition on a Circle

July 2, 2005

by

David A. Hennessy
Professor
Department of Economics
578C Heady Hall
Iowa State University
Ames, IA 50011-1070

Harvey E. Lapan
University Professor
Department of Economics
283 Heady Hall
Iowa State University
Ames, IA 50011-1070

Abstract

Taking location as given, we study imperfect competition on a circular city. In Bertrand oligopoly, we identify price harmonics as a function of firm unit costs and locations. The sum of oligopoly profits is larger when costs and/or locations are more dispersed in the ‘dihedral majorization’ sense. This also tends to be the case in which prices are more variable. We study how phase shifts between cost parameters and inter-firm distance parameters change production and oligopoly profits. An exact characterization of production patterns is developed in terms of the eigenvalues for what we call the price harmonics matrix. The same techniques are applied to Cournot oligopoly with spatial externalities on Circular City. Solutions are compared with monopoly and with first-best. Production patterns can differ markedly when cost spillovers are negative.

JEL Classification Numbers: E2, C6, M2.

KEYWORDS: cyclic and dihedral symmetries, harmonic analysis, price and quantity instruments, majorization, spatial differentiation.

Senior authorship is not assigned. Information for correspondence with contact author

David A. Hennessy

Ph: US 515-294-6171, Fax: US 515-294-6336, e-mail: hennessy@iastate.edu

1. Introduction

■ Much of economic analysis is concerned with external effects. These effects can be very diffuse to the extent that impacts on affected parties do not depend on location, as in an homogeneous good oligopoly model. Or there can be differential impacts, where the extent of direct impacts could be ordered by proximity, as with ingestion of second-hand smoke, damage from acid rain, and damage to product reputation due to publicized failure of a similar product. Our intent in this paper is to provide an approach to modeling externalities when distance between agents influences the extent of the externality. We will model firms competing for customers on Circular City where the strategic behavior of a firm most directly affects adjacent firms. The general issue of modeling proximity is important because many, if not most, externalities impact other agents in a manner that depends on how similar an agent is to the externality originator. Circular City was chosen because, as recognized by Papageorgiou and Smith (1983), Salop (1979), and many others, model results are not driven by topological discontinuities that would obscure economic insights.

Salop's study of price competition on Circular City stimulated further inquiries into strategizing when differentiation arises due to location on a seamless loop, such as Eaton and Holtz Wooders (1985), and Steinmetz (1998). All of these articles held location to be exogenous. Kats (1995), for price competition, Gupta *et al.* (2004), for quantity competition, and others have sought to rationalize equilibrium location in a locate-then-compete game. Results point to spatial differentiation under price competition and a large variety of subgame perfect location equilibria under quantity competition. Taking location as exogenous, we seek to characterize the roles of location and cost on firm strategies. Our model comes in two variants, both involving firms with constant marginal costs competing on Circular City. The first is a shopping model, involving Bertrand pricing where non-discriminatory pricing strategies place arbitrage constraints on adjacent firms. The second involves the incidence of a production externality on an adjacent firm when firms compete over quantity strategies to produce an

homogenous good.

The paper commences with a brief explanation of some relevant tools. As with Dixit (1986), Salant and Shaffer (1999), and others, we will exploit symmetries that naturally arise in oligopoly models. In our case, though, the symmetries are not as straightforward and we need to develop the ideas of cyclic and dihedral symmetries. The tools we use have also been employed in economics by Hennessy and Lapan (2003, 2005) to understand portfolio choice and preferences over prices for a price-taking agent.

For Bertrand competition, we establish equilibrium and use harmonic analysis to study some of the symmetries in equilibrium. In particular, a characterization of cost/location parameter dispersions that is tailored to the symmetries of how firms are located is shown to increase price variance as well as aggregate oligopoly profit. A stronger alignment of cost and location parameters in the phase-shift sense has the same effect. We show it is always the case that price variance is smaller than variance in exogenous parameters. This is due to price smoothing arising from competition at the margin. The difference depends on the harmonics of exogenous parameters; firm costs and inter-firm distances in our model. Higher frequency harmonics on exogenous parameters around the circle have less influence on price variance than do lower frequency harmonics. This is because prices are strategic complements, and high heterogeneity vs. adjacent firms impedes the capacity of less competitive firms to price high while increasing the latitude for more competitive firms to do so.

Nonetheless, we show that prices and exogenous parameters always exhibit positive covariance. How sales and exogenous parameters covary depends on dispersion in locations, dispersion in costs, and how costs covary with locations. The spatial pattern of firm sales has a more involved relationship with exogenous parameters than does the spatial pattern of firm prices. Firms price high when costs are high and competitors are remote. Firm sales are low when costs are high and high when competitors are remote. When cost and remoteness characteristics correlate positively across firms, then firm prices will tend to correlate positively

with both but firm sales may express little if any pattern. A comparison with the first-best solution provides theoretical confirmation of a statistical tendency for firms with comparatively remote competitors to serve a customer base that is too small.

The quantity-setting variant of our framework is somewhat different in the modeling of spatial spillovers. We choose spillovers to be one-sided in direction along the circle and they can be strategic complements or substitutes, whereas in Bertrand competition they must be two-sided and must be strategic complements. Consequently the solution characteristics generally differ, and are generally less symmetric in properties. Surprisingly in one regard, concerning the effects of dispersion in cost parameters on oligopoly profits, the less structured symmetries of our Cournot model do not weaken our results. In a linear model we establish that efficient outputs correlate negatively with Cournot outputs when spatial cost spillovers are negative. A brief discussion concludes.

2. Preliminaries

■ Write $\Omega_N = \{0, 1, \dots, N-1\}$. With superscript T as the transpose operation, the vector $u = (u_0, u_1, \dots, u_{N-1})^T$ may be rotated as follows. Specify the $j+k+1$ th ordinate, modulus N , as $u_{[j+k]}$ so that values $N=6, j=3$, and $k=5$ provide $u_{[j+k]} = u_2$ while $u_{[j-4]} = u_5$. We intend to compare the welfare implications of different cost and demand parameter vectors. To this end, some definitions are required.

Definition 1. Vector $u' \in \mathbb{R}^N$ is *cyclic majorized* by $u'' \in \mathbb{R}^N$ (written as $u' \leq_{cyc} u''$) if there exist simplex weights $(\lambda_0, \lambda_1, \dots, \lambda_{N-1}), \lambda_j \geq 0 \forall j \in \Omega_N, \sum_{j \in \Omega_N} \lambda_j = 1$, such that¹

$$u' = \sum_{j \in \Omega_N} \lambda_j L^j(u''); \quad L^j(u'') = (u''_j, u''_{[j+1]}, u''_{[j+2]}, \dots, u''_{[j-1]})^T. \quad (2.1)$$

¹ See Giovagnoli and Wynn (1996) on cyclic majorization.

Definition 2. Vector $u' \in \mathbb{R}^N$ is *dihedral majorized* by $u'' \in \mathbb{R}^N$ (written as $u' \leq_{dih} u''$) if

there exist simplex weights $(\lambda_0, \lambda_1, \dots, \lambda_{2N-1}), \lambda_j \geq 0 \forall j \in \Omega_{2N}, \sum_{j \in \Omega_{2N}} \lambda_j = 1$, such that²

$$\begin{aligned} u' &= \sum_{j=0}^{N-1} \lambda_j L^j(u'') + \sum_{j=N}^{2N-1} \lambda_j L^{j-N}(u''); \\ u'' &= (u''_0, u''_{N-1}, u''_{N-2}, \dots, u''_1)^T; \quad L^j(u'') = (u''_j, u''_{[j+1]}, u''_{[j+2]}, \dots, u''_{[j-1]})^T. \end{aligned} \quad (2.2)$$

Example 1. If $(u'')^T = (1, 3, 4, 2)$ then $(u''_-)^T = (1, 2, 4, 3)$, $L^0[(u'')^T] = (1, 3, 4, 2)$, $L^1[(u'')^T] = (3, 4, 2, 1)$, $L^2[(u'')^T] = (4, 2, 1, 3)$, $L^3[(u'')^T] = (2, 1, 3, 4)$, $L^0[(u''_-)^T] = (1, 2, 4, 3)$, $L^1[(u''_-)^T] = (2, 4, 3, 1)$, $L^2[(u''_-)^T] = (4, 3, 1, 2)$, $L^3[(u''_-)^T] = (3, 1, 2, 4)$, and $(u')^T = \lambda_0(1, 3, 4, 2) + \lambda_1(3, 4, 2, 1) + \lambda_2(4, 2, 1, 3) + \lambda_3(2, 1, 3, 4) + \lambda_4(1, 2, 4, 3) + \lambda_5(2, 4, 3, 1) + \lambda_6(4, 3, 1, 2) + \lambda_7(3, 1, 2, 4)$ satisfies $u' \leq_{dih} u''$ for any unit simplex entries $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \bar{\mathbb{R}}_+^8, \lambda_j \geq 0 \forall j \in \Omega_8, \sum_{j \in \Omega_8} \lambda_j = 1$. Cyclic majorization is clearly a special case. Even for $u' \leq_{dih} u''$, not all permutations of $(u'')^T = (1, 3, 4, 2)$ are used in the mixing; only 8 of 24 are, and $(1, 2, 3, 4)$ is not. Note the symmetrizing nature of hull formation in the definitions 1 and 2. If instead, say, $(u'')^T = (1, 2, 4, 3)$ or $(u'')^T = (2, 1, 3, 4)$ then the generated convex hull would be the same under the dihedral ordering.

Example 2. To provide a sense of the $\leq_{dih}$ order, view the equilateral triangle in Figure 1. It represents a simplex with vertices v_1 , v_2 , and v_3 , which may be read as the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Choose an arbitrary point $\mathcal{G}''$ on the simplex and generate five other

² If $\lambda_j = 0 \forall j \in \{N, N+1, \dots, 2N-1\}$, then dihedral majorization reduces to cyclic majorization. Dihedral majorization is more discerning in that $\lambda_i \neq 0 \forall i \in \{N, N+1, \dots, 2N-1\}$ allows for comparison between vectors that are not pre-ordered under cyclic majorization. Marshall and Olkin (1979) and Niezgoda (1999) provide references to work on the more general concept of group majorization.

points from it, all also on the simplex. Rotate $\mathcal{G}''$ through $2\pi/3$ and $4\pi/3$ radians around the triangle's center (given by $*$ on line $L1$) to obtain two of these points. Then reflect each of these three points orthogonally through line $L1$ to obtain six points in total. Note the two operations of rotation and reflection through a single line are those applied in Definition 2 for dihedral majorization. Form a hexagon from the six points with vertices as the black spots on the simplex. Any point in the hexagon's convex hull is (weakly) dihedral majorized by $\mathcal{G}''$, e.g., $\mathcal{G}' \leq_{dih} \mathcal{G}''$ where $\mathcal{G}'$ is the point identified by the symbol $\otimes$.³

Definition 3. Vector $u' \in \mathbb{R}^N$ is *cyclic supermajorized* by $u'' \in \mathbb{R}^N$ (written as $u' \leq_{cyc}^w u''$) if there exists a $u''' \in \mathbb{R}^N$ such that $u''' \leq_{cyc} u''$ and $u' \geq u'''$ in the coordinate-wise order. Vector $u' \in \mathbb{R}^N$ is *dihedral supermajorized* by $u'' \in \mathbb{R}^N$ ($u' \leq_{dih}^w u''$) if there exists a $u''' \in \mathbb{R}^N$ such that $u''' \leq_{dih} u''$ and $u' \geq u'''$.

Example 3. In example 1, let $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) = (0.1, 0.2, 0.3, 0, 0.1, 0, 0.3, 0)$ so that the value of $(u''')^T$ (not $(u')^T$) in Definition 2 is $(u''')^T = (3.2, 2.8, 1.8, 2.2)$. If $(u')^T = (3.5, 3, 2, 2.5)$, then $u' \leq_{dih}^w u''$.

When exploiting symmetries in models, we will have use for circulant matrices.

Definition 4. An $N \times N$ circulant matrix U has form⁴

³ Dihedral majorization is the same as ordinary majorization (to be explained later) when vectors considered are in $\mathbb{R}^3$. The equivalence is not true when vectors are in $\mathbb{R}^N$, $N \geq 4$, as was the case in Example 1. Then ordinary majorization becomes a stronger pre-ordering of vectors with $N!$ vector permutations used to construct the convex hull of dominated points. Dihedral majorization constructs a convex hull using only a subset, cardinality $2N$, of these vector permutations and $N!/(2N) = (N-1)!/2 > 1$ whenever $N > 3$.

⁴ Davis (1994) and Geller *et al.* (2004) are useful references on the properties of circulants.

$$U = \begin{pmatrix} u_0 & u_1 & u_2 & \cdots & u_{N-2} & u_{N-1} \\ u_{N-1} & u_0 & u_1 & \cdots & u_{N-3} & u_{N-2} \\ u_{N-2} & u_{N-1} & u_0 & \cdots & u_{N-4} & u_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u_2 & u_3 & u_4 & \cdots & u_0 & u_1 \\ u_1 & u_2 & u_3 & \cdots & u_{N-1} & u_0 \end{pmatrix}. \quad (2.3)$$

It may also be written as $U = [u_0, u_1, \dots, u_{N-1}]$. A symmetric circulant is a circulant that is also symmetric around the main diagonal, i.e., $u_j = u_{N-j} \forall j \in \Omega_N$.

3. Circular city in Bertrand oligopoly

■ On Circular City, let $x_0, x_1, \dots, x_{N-1}$ identify the point locations of the firms when measured along the circumference. Points x_0 and x_N describe the same location, $x_0 = x_N$, and circle circumference is normalized to 1. The j th firm's unit cost of production is c_j . The population of consumers is continuously and uniformly distributed on the circumference. All have identical demands with demand at point x being

$$D(x) = \begin{cases} 1, & p(x) \leq K; \\ 0, & p(x) > K; \end{cases} \quad (3.1)$$

where $p(x)$ is the effective cost (not price) to the point x consumer. Transportation costs are τ per unit distance to be traveled. We will assume that costs and firm locations are such that the only relevant competitors are adjacent firms.⁵ Also, the value of K is sufficiently large that a consumer will never be priced out of the market.

Demand facing the firm at location x_j is $q_j(x_j) = \eta |x_U(x_j) - x_L(x_j)|$, where $x_U(x_j)$ and $x_L(x_j)$ are consumer indifference points while $\eta > 0$ is a demand intensity parameter reflecting

⁵ This means that costs and distances should not be too dispersed. When firm fixed costs are strictly positive, then all operating firms must have a customer base of strictly positive length and the only relevant competitors will be adjacent firms.

the depth of population at each point in Circular City and $|x_U(x_j) - x_L(x_j)|$ is the true minimum distance between these points. The consumer indifference points on $j \in \Omega_N$ satisfy⁶

$$\begin{aligned} x_U(x_j) &: p_{[j+1]} + \tau |x_{[j+1]} - x_U| = p_j + \tau |x_U - x_j|; \\ x_L(x_j) &: p_j + \tau |x_j - x_L| = p_{[j-1]} + \tau |x_L - x_{[j-1]}|; \end{aligned} \quad (3.2)$$

and resolve as

$$x_U = \frac{p_{[j+1]} - p_j}{2\tau} + \frac{x_j + x_{[j+1]}}{2}; \quad x_L = \frac{p_j - p_{[j-1]}}{2\tau} + \frac{x_j + x_{[j-1]}}{2}. \quad (3.3)$$

The values of $x_{[j-1]}$ and $x_{[j+1]}$ are such that the minimum distance between x_j and $x_{[j-1]}$ is

$x_j - x_{[j-1]}$ while the minimum distance between x_j and $x_{[j+1]}$ is $x_{[j+1]} - x_j$. Write $\delta_j =$

$x_{[j+1]} - x_{[j-1]}$ as the j th firm location parameter (the distance between adjacent competitors),

where $\bar{\delta} = N^{-1} \sum_{j \in \Omega_N} \delta_j = 2N^{-1}$. Demand can be written as

$$q_j(x_j) = \frac{\eta}{2\tau} (p_{[j+1]} + p_{[j-1]} - 2p_j + \tau\delta_j). \quad (3.4)$$

The j th firm's profit function, $j \in \Omega_N$, is given as

$$V^j = \text{Max}_{p_j} \frac{\eta}{2\tau} (p_{[j+1]} + p_{[j-1]} - 2p_j + \tau\delta_j) (p_j - c_j), \quad (3.5)$$

with first-order condition

$$p_{[j+1]}^* + p_{[j-1]}^* - 4p_j^* + \tau\delta_j + 2c_j = 0. \quad (3.6)$$

Here, superscripted stars indicate privately optimal values under Nash conjectures. Summing (3.4) and (3.6) over firms, we obtain

$$\sum_{j \in \Omega_N} q_j^*(x_j) = \eta; \quad \sum_{j \in \Omega_N} p_j^* = \tau + \sum_{j \in \Omega_N} c_j. \quad (3.7)$$

Aggregate output, $\bar{q}^* = N^{-1} \sum_{j \in \Omega_N} q_j^*(x_j)$, is just the cumulation of demand over the

⁶ Indicator $x_{[j+1]}$ is the location label for a firm clockwise to the right of location x_j , i.e., firm $[j+1]$ is clockwise to the right of firm j . If $x_j = 0.9$ and $x_{[j+1]} = 0.05$ (and not 1.05), then

circumference since all customers are served. Average price, $\bar{p}^* = N^{-1} \sum_{j \in \Omega_N} p_j^*$, is determined in part by transport costs and in part by unit costs of production. In particular, $\bar{c} = N^{-1} \sum_{j \in \Omega_N} c_j$ is a sufficient statistic on the cost vector for average price.⁷

Posing (3.6) in matrix form,

$$\begin{pmatrix} 2 & -0.5 & 0 & \cdots & 0 & -0.5 \\ -0.5 & 2 & -0.5 & \cdots & 0 & 0 \\ 0 & -0.5 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -0.5 \\ -0.5 & 0 & 0 & \cdots & -0.5 & 2 \end{pmatrix} \begin{pmatrix} p_0^* \\ p_1^* \\ p_2^* \\ \vdots \\ p_{N-2}^* \\ p_{N-1}^* \end{pmatrix} = \begin{pmatrix} \mathcal{G}_0 \\ \mathcal{G}_1 \\ \mathcal{G}_2 \\ \vdots \\ \mathcal{G}_{N-2} \\ \mathcal{G}_{N-1} \end{pmatrix}; \quad \mathcal{G}_j = c_j + \frac{\tau \delta_j}{2}; \quad (3.8)$$

our interest in circulant matrices becomes transparent. For future reference, the square matrix in (3.8) is labeled the price harmonics matrix while the $\mathcal{G}_j$ are labeled the raw exogenous parameters. The solution on $j \in \Omega_N$ is⁸

$$p_j^* = \sum_{m \in \Omega_N} u_{[m-j]} \mathcal{G}_m; \quad u_k = \frac{(2 + \sqrt{3})^k + (2 + \sqrt{3})^{N-k}}{\left[(2 + \sqrt{3})^N - 1 \right] \sqrt{3}}; \quad \sum_{j \in \Omega_N} u_j = 1. \quad (3.9)$$

Notice the useful symmetry $u_j \equiv u_{[-j]}$. Insert (3.6) into (3.4) and apply (3.9) to obtain

$$q_j^* = \frac{\eta}{\tau} (p_j^* - c_j) = \frac{\eta}{\tau} \left(\sum_{m \in \Omega_N} u_{[m-j]} \mathcal{G}_m - c_j \right) \quad \forall j \in \Omega_N; \quad (3.10)$$

written in matrix form as $q^* = \eta \tau^{-1} (U \mathcal{G} - c)$, $U = [u_0, u_1, \dots, u_{N-1}]$, a symmetric circulant.

Detailed inspection of (3.9) and (3.10) supports some comparative statics to provide intuition on the nature of firm interactions. We consider only firm 0 profit, $V^0 [\equiv (\eta / \tau) (p_0 - c_0)^2]$,

$|x_{[j+1]} - x_j| = 0.15$.

⁷ Recall that in the standard Cournot model under constant, but firm-specific, unit costs, price in equilibrium depends on the average of unit costs across firms and not on other cost statistics.

⁸ A generalization of this price harmonics matrix was inverted in Hennessy and Lapan's (2004) study of market equilibrium under adjustment cost frictions and perfect competition.

because straightforward symmetry operations establish the qualitative results for other firms.⁹

Proposition 1. **a)** $dV^0/dc_j \geq 0 \forall j \in \Omega_N, j \neq 0$, and $dV^0/dc_0 \leq 0$.

b) $dV^0/dc_j \geq dV^0/dc_{j+1}$ on $\{j \in \Omega_N : (N-1)/2 > j > 0\}$.

c) $dp_0^*/dc_j \geq dp_0^*/dc_{j+1}$ on $\{j \in \Omega_N : (N-1)/2 > j \geq 0\}$ while $dq_0^*/dc_j \geq dq_0^*/dc_{j+1}$ on $\{j \in \Omega_N : (N-1)/2 > j > 0\}$.

d) $dV^0/dc_j \equiv dV^0/dc_{N-j} \forall j \in \Omega_N$.

e) Let $c \rightarrow c + \varepsilon 1_N$ where $\varepsilon \in \mathbb{R}$. Then firm profits and firm outputs do not change provided that solutions remain interior. Firm prices all change by ε provided that solutions remain interior.

f) $dV^j/dc_0 > dV^{N-j}/dc_0$, on $\{j \in \Omega_N : j > 0\}$ if and only if $V^j > V^{N-j}$ and this is true if and only if $p_j^* - c_j > p_{N-j}^* - c_j$.

g) $dV^j/dc_0 > dV^{j+1}/dc_0$ on $\{j \in \Omega_N : (N-1)/2 > j > 0\}$ if $V^j > V^{j+1}$, i.e., if

$$p_j^* - c_j > p_{[j+1]}^* - c_{[j+1]}.$$

h) $d\sum_{j \in \Omega_N} V^j/dc_0$ can be of either sign. In particular, $d\sum_{j \in \Omega_N} V^j/dc_0 \geq 0$ when $p_0^* - c_0 =$

$$\text{Min}_{j \in \Omega_N} \{p_j^* - c_j\} \text{ and } d\sum_{j \in \Omega_N} V^j/dc_0 \leq 0 \text{ when } p_0^* - c_0 = \text{Max}_{j \in \Omega_N} \{p_j^* - c_j\}.$$

Part a) asserts that all other competitors gain when one firm only is subject to higher unit costs, a result that is in common with the standard Bertrand model (Dixit, 1986). The firm itself sees lower equilibrium profits. The reasoning is that firm 0 is obliged to increase price somewhat in response to an increase in c_0 [$dp_0^*/dc_0 \in (0,1)$] and the other firms take advantage of the substitute's higher price. On b) and c), which say almost the same thing, the closer the 0th firm is to a firm that incurs marginally higher unit costs then the more profitable the subsequent equilibrium is for firm 0. Price effects ripple around the circle, but damping occurs. If the j th

⁹ Proofs, where not immediate, are in Appendix B.

firm sees a cost increase, then firms $[j-1]$ and $[j+1]$ can increase their respective prices somewhat in the face of weaker competition at the margin. But the increase is tempered by resulting losses at the other sales boundary for each of firms $[j-1]$ and $[j+1]$. Firms $[j-2]$ and $[j+2]$ then see reduced competition at the margin, endowing them with the opportunity to increase respective prices a little.

Part d) merely recognizes the diagonal symmetries in the square matrix of (3.8), and so in its multiplicative inverse U . Part e) exploits a different sort of symmetry, the invariance of average price to cost simplex location that is evident in equation (3.7). Part f) identifies a limitation to those symmetries, $dV^j / dc_0 \neq dV^{N-j} / dc_0$ for $j \neq 0$ even though $dV^0 / dc_j \equiv dV^0 / dc_{N-j} \forall j \in \Omega_N$ (see part d)). In part g), profit effects tend to be most marked for profitable firms that are near a cost impacted firm. This feature is a consequence of profit function convexity, and bears comparison with the less qualified assertion in b).

Part h) shows that an increase in some firm's cost will only increase the sum of oligopoly profits if the firm is not among the larger firms, as $d \sum_{j \in \Omega_N} V^j / dc_0 = 2 \sum_{j \in \Omega_N} q_j^* u_j - 2q_0^*$. Small firms are further marginalized, effectively increasing firm concentration in this differentiated consumer oligopoly.¹⁰ This means there is no prospect that an otherwise arbitrary cost shift that is ordered by $\leq_{dih}^w$ must increase aggregate profit. Part h) may be viewed as a consequence of e). If, by contrast with e), costs change by different amounts then the extent of differentiation across firms changes. If the firm with the lowest margin suffers a cost increase then other firms as a whole enjoy enhanced market power and so total profits increase. If, on the other hand, the firm with the highest margin suffers a cost increase then firms become more similar in pricing power and so total profits decrease.

We turn now to the role of heterogeneity over the firm dispersion and cost parameters in

¹⁰ For the standard Cournot oligopoly model, Février and Linnemer (2004) analyze in detail the interactions between firm costs, firm competitiveness, industry profits, and social welfare.

determining the industry's profitability, equilibrium industry price dispersion, and equilibrium industry sales dispersion. From (3.5) and (3.6), oligopoly profits sum to

$$\frac{\eta}{\tau}(U\mathcal{G}-c)^T(U\mathcal{G}-c); \quad (3.11)$$

a detailed inspection of which supports¹¹

Proposition 2. a) Let $c = \bar{c}1_N$ where $\bar{c} \in \bar{\mathbb{R}}_+$ and $1_N \in \mathbb{R}^N$ is the unit vector. If $\delta'' \leq_{dih} \delta'$, then the sum of oligopoly profits is larger under δ' than under δ'' while the variance of prices $\text{Var}(p^*)$ and the variance of firm outputs $\text{Var}(q^*)$ are also both larger under δ' than under δ'' .

b) Let $\delta = \bar{\delta}1_N$ where $\bar{\delta} \in \bar{\mathbb{R}}_+$. If $c'' \leq_{dih} c'$, then the sum of oligopoly profits is larger under c' than under c'' while $\text{Var}(p^*)$ and $\text{Var}(q^*)$ are also both larger under c' than under c'' .

Parts a) and b) convey essentially the same insight, and we will interpret only part b).

Dispersion in c with $\delta = \bar{\delta}1_N$, will not be to the benefit of all firms (see Proposition 1). But a qualified form of dispersion in c will in general be to the oligopolists' benefit because the lower cost producers have *de-facto* monopoly power over an enlarged set of captive consumers close to their production location. The qualified form of dispersion, $\leq_{dih}$, requires that the unit costs of neighboring firms be considered. The characterization of dispersion allows for pricing that extracts more surplus from consumers, and so the variance of prices increases. The dispersion, in favoring some firms over others, will ensure that equilibrium outputs are also more dispersed. It is important to observe that the proof of Proposition 2 uses just two properties; the symmetries that the circle topology and oligopoly setting endow the problem with, and also convexity due indirectly to private optimization. These two properties fit the $\leq_{dih}$ pre-ordering very well, so it

¹¹ In part a), if $\delta''' = \kappa\delta'$, $\kappa > 1$, then all of $\text{Var}(p^*)$, $\text{Var}(q^*)$, and summed profits will be still larger under δ''' than under δ' . Thus, if $\delta''' = \kappa\delta'$, $\kappa > 1$, and $\delta'_{dih} \geq \delta''$ then profits are larger under δ''' than under δ'' where κ may be viewed as an increase in population density, all else

is hard to envision a stronger result.¹²

Proposition 2 does not develop the implications for profits of an increase in correlation between c and δ . In general, little can be said because correlation is too crude a statistic when spatial relations matter. For a special case that works with the circular topology, we can identify sufficient conditions such that a better alignment between δ and c has determinate effects on equilibrium.

Proposition 3. Let $\delta_{[j+r]} - \bar{\delta} = K_1(c_j - \bar{c})$, $K_1 > 0$, $c_j - \bar{c} = \text{Cos}(2j\pi/N)$, $j \in \Omega_N$. Then

- a) price variance is (weakly) increasing in r for r an integer in the interval $[-N/2, 0)$;
- b) quantity variance is decreasing in r whenever price variance is increasing in r ;
- c) the sum of oligopoly profits is decreasing in r whenever price variance is increasing in r .

The key to understanding this result is to observe that¹³

$$\text{Cov}(\delta, c) = K_1 N^{-1} \sum_{j \in \Omega_N} \text{Cos}(2j\pi/N) \text{Cos}(2(j-r)\pi/N) = 0.5 K_1 \text{Cos}(2r\pi/N), \quad (3.12)$$

so that $d\text{Cov}(\delta, c)/dr \stackrel{\text{sign}}{=} -\text{Sin}(2r\pi/N)$ and is positive whenever r is an integer in $[-N/2, 0)$.

While $\text{Cov}(\delta, c)$ may be positive or negative on this integer interval, the interval does ensure that spatial location parameters and cost parameters become better aligned with an increase in the value of r .

Price variance increases when cost parameters better match inter-firm distance parameters so that firms with high costs have the market power to remedy their problem. Quantity variance relates inversely with price variance. This is because firms with location advantage raise price to cover (more strongly aligned) costs so that the firms one might expect to dominate the market due to location may not in fact do so. The better matching of costs with location parameters

equal. The observation also applies to a scalar expansion of exogenous parameters in part b).

¹² Pre-order $\leq_{dih}$ also ranks quasi-convex functions that are appropriately symmetric.

¹³ Use identities (A1), (A7), (A5), and (A4) in Appendix A.

reduces the extent of market power for the more profitable firms, so it is not surprising that the sum of oligopoly profits falls.

It is clear that solutions to (3.6) require correlation structure on how p^* responds to $\mathcal{G}$. Intuition might suggest that p^* and $\mathcal{G}$ be positively correlated. The interactions between q^* and $\mathcal{G}$ are not as readily conjectured. Higher costs induce higher prices, and so indirectly increase production incentives. A technical lemma will prove helpful in understanding correlation effects.

Lemma 1. The explicit formula for the k th eigenvalue (starting at $k = 0$) of U is

$$\lambda_k = \sum_{m \in \Omega_N} u_m \text{Cos}(2\pi mk / N) = \frac{1}{2 - \text{Cos}(2\pi k / N)}, \quad k \in \Omega_N; \quad (3.13)$$

so that $\lambda_0 = 1$, $\lambda_k = \lambda_{N-k}$, $\lambda_k \in [1/3, 1]$, and $\lambda_k \geq \lambda_{k+1}$ on $\{k : k \in \Omega_N, k \leq (N-2)/2\}$.

It follows that U is positive definite. The property $\lambda_k \geq \lambda_{k+1}$ on $\{k : k \in \Omega_N, k \leq (N-2)/2\}$ should not be surprising in light of the pattern in u_k from (3.9). Equation (3.13), solution (3.9), and Fourier's famous observation that trigonometric functions are an excellent basis for approximation, suggests that eigenvalue λ_k should be associated with the price and quantity response harmonics of period $2\pi k / N$. Proposition 4 confirms this conjecture. Define $\Omega_N^0 = \{1, 2, \dots, N-1\}$.

Proposition 4. Choose $a_k, b_k, k \in \Omega_N$ to make the exact fit¹⁴

$$\mathcal{G}_j - \bar{\mathcal{G}} = \sum_{k \in \Omega_N^0} a_k \text{Cos}(2\pi jk / N) + \sum_{k \in \Omega_N^0} b_k \text{Sin}(2\pi jk / N). \quad (3.14)$$

Then **a)** with $\hat{a}_k = a_k \lambda_k; \hat{b}_k = b_k \lambda_k$, price harmonics follow

$$p_j^* - \bar{p}^* = \sum_{k \in \Omega_N^0} \left(\hat{a}_k \text{Cos}(2\pi jk / N) + \hat{b}_k \text{Sin}(2\pi jk / N) \right); \quad (3.15)$$

¹⁴ A total of N coefficients among the a_k and b_k suffice to span the $\mathcal{G}_j, j \in \Omega_N$. Overfitting is

- b) price variance satisfies $\text{Var}(p^*) = 0.5 \sum_{k \in \Omega_N^0} (\hat{a}_k^2 + \hat{b}_k^2) \leq 0.5 \sum_{k \in \Omega_N^0} (a_k^2 + b_k^2) = \text{Var}(\mathcal{G})$;
- c) covariance between raw parameters and price is non-negative, $\text{Cov}(p^*, \mathcal{G}) \geq 0$;
- d) for costs that are firm invariant, covariance between raw parameters and quantity is $\text{Cov}(q^*, \mathcal{G}) = \eta \tau^{-1} \text{Cov}(p^*, \mathcal{G}) \geq 0$, while quantity variance is $\text{Var}(q^*) = \eta^2 \tau^{-2} \text{Var}(p^*)$;
- e) if firms are located uniformly on the circle, then $\text{Cov}(q^*, \mathcal{G}) \leq 0$.

Example 4. To better understand how economic forces might determine the values in (3.15), suppose that $a_1 = 1$, while $a_k = 0$ and $b_k = 0$ otherwise on $k \in \Omega_N$. Then $p_j^* - \bar{p}^* = \text{Cos}(2\pi j / N) / [2 - \text{Cos}(2\pi / N)]$ and $\text{Var}(p^*) = 0.5 / [2 - \text{Cos}(2\pi / N)]^2 \leq 0.5$ with $\lim_{N \rightarrow \infty} \text{Var}(p^*) = 0.5$. Price inherits the periodic structure of exogenous parameters, but amplitude is attenuated because of competition from adjacent firms. Structure $\mathcal{G}_j - \bar{\mathcal{G}} = \text{Cos}(2j\pi / N)$ places spatial correlation on firm exogenous parameters, with adjacent firms having similar parameters for $N > 2$. This means that firms are not unduly constrained by neighbor pricing behavior, i.e., prices will reflect the low frequency harmonics of $\mathcal{G}_j - \bar{\mathcal{G}} = \text{Cos}(2j\pi / N)$. Price instruments being strategic complements in Bertrand competition, equilibrium prices will tend to rise and fall in a wave-like manner around the circle.¹⁵ Positive spatial correlation in the $\mathcal{G}_j$ parameters seems reasonable given that the cost environment may be determined by local tax and labor laws, as well as local competition in factor markets.

Concerning the general result, part a) confirms that the eigenvalues of U are central to understanding equilibrium pricing/sales decisions. Part b) holds that eigenvalues determine the

not a concern for one can just set N or more coefficients equal to zero.

¹⁵ When $N = 2$ then the higher price matches the higher $\mathcal{G}_j$ value and the circular topology is indistinguishable from the standard assumption of complete symmetry on inter-firm relations in

extent to which external parameter variation is smoothed when mapped into prices. Part c) identifies positive spatial correlation between prices and the raw exogenous parameters. This is reasonable; larger δ_j correspond to stronger market power while larger c_j create a need to trade volume off against a cost-covering price to customers. Part d) asserts that, for an uniform cost structure, a positive correlation exists between equilibrium firm output and the distance between the firm's two nearest neighbors. In the model, firms take advantage of the size of hinterland by not pricing so high (notwithstanding part c)) that sales shrink with larger hinterland size. Even though costs are partially offset by equilibrium pricing, when firms are located uniformly along the circle then part e) asserts that the offset is not so strong as to create a positive correlation between costs and sales. Together, parts d) and e) show that $\text{Cov}(q^*, \mathcal{G}) = \text{Cov}(q^*, c) + 0.5\tau\text{Cov}(q^*, \delta)$ can be of either sign depending on whether the location motive for high sales outweighs the cost motive for low sales.

Part b) provides an interesting consequence that relies on the idea of (ordinary) majorization.

Definition 5. [See Marshall and Olkin (1979, p. 10 and p. 59)] Vector $q' \in \mathbb{R}^N$ is said to be *majorized* by vector $q'' \in \mathbb{R}^N$ ($q' \leq_{maj} q''$) if $\sum_{j=1}^k q'_{(j)} \geq \sum_{j=1}^k q''_{(j)} \forall k \in \Omega_N$ and $\sum_{j \in \Omega_N} q'_j = \sum_{j \in \Omega_N} q''_j$ where $q_{(1)} \leq q_{(2)} \leq \dots \leq q_{(n)}$.

Corollary 4.1. **a)** Let the coefficients in (3.14) change as follows; $(a_k, a_{N-k}) \rightarrow (a'_k, a'_{N-k})$, $(a_k, a_{N-k}) \leq_{maj} (a'_k, a'_{N-k})$. Then price variance increases.

b) For $\{k : k \in \Omega_N, k \leq (N-1)/2\}$, suppose $a_k < a_{k+1}$ and the coefficients in (3.14) change as follows; $(a_k, a_{k+1}) \rightarrow (a'_k, a'_{k+1}) = (a_{k+1}, a_k)$. Then price variance increases.

Example 5. The corollary may appear to be entirely technical, but in fact the economic motivation for change in variance is strong. The economics underlying the distinction between a) and b) are the same, and so we will only consider part b). Suppose that $a_1 = 1$ and $a_2 = 2$, while all other coefficients in (3.14) are zero. Then $\mathcal{G}_j - \bar{\mathcal{G}} = \text{Cos}(2\pi j / N) + 2\text{Cos}(4\pi j / N)$. Amplitude for $\mathcal{G}_j - \bar{\mathcal{G}}$ is 3, period is 2π , and $\text{Var}(\mathcal{G}) = 2.5$. If instead $a_1 = 2$ and $a_2 = 1$, then (again) amplitude is 3, period is 2π , and $\text{Var}(\mathcal{G}) = 2.5$. It is not variance attributes that modify pricing behavior, but rather autocorrelation along $\mathcal{G}$. When autocorrelation is strong, then adjacent competitors with high (low) $\mathcal{G}$ parameters will price high (low) because prices are strategic complements in Bertrand competition. We will return to this point later.

The corollary illustrates how harmonics in $\mathcal{G}_j$ enter price variance. Harmonic $\text{Cos}(2\pi j(k+1) / N)$ is given a lower weighting than harmonic $\text{Cos}(2\pi jk / N)$ because the former changes more rapidly and prices are better able to respond to smoother variation.

Corollary 4.2. Let $c_j = \bar{c} \forall j \in \Omega_N$ and assume $a_k \geq 0 \forall k \in \Omega_N^0$, $b_k \geq 0 \forall k \in \Omega_N^0$ with $\sum_{k \in \Omega_N^0} a_k = A > 0$, $\sum_{k \in \Omega_N^0} b_k = B > 0$. Then industry profits are maximized when $a_1 = A$, $b_1 = B$.

Corollary 4.3. Let $c_j = \bar{c} \forall j \in \Omega_N$ and assume $a_k \geq 0 \forall k \in \Omega_N^0$, $b_k \geq 0 \forall k \in \Omega_N^0$ with $\sum_{k \in \Omega_N^0} a_k = A > 0$, $\sum_{k \in \Omega_N^0} b_k = B > 0$. Then industry profits are maximized when the correlation between adjacent prices, $\sum_{j \in \Omega_N} (p_j^* - \bar{p}^*)(p_{[j+1]}^* - \bar{p}^*) / \sum_{j \in \Omega_N} (p_j^* - \bar{p}^*)^2$, is maximized.

We turn now to the effect of market power on pricing/sales decisions. Intuitively an increase

in the value of unit transportation cost τ should strengthen oligopolist market power, so leading to pricing decisions that are less sensitive to the decisions of other firms. Transportation costs interact directly with firm location parameters. To focus on transportation costs, we control for production costs in the proposition to follow.

Proposition 5. Let $c_j = \bar{c} \forall j \in \Omega_N$ and choose $a_k, b_k, k \in \Omega_N$ to make the exact fit $\delta_j - \bar{\delta} = \sum_{k \in \Omega_N^0} a_k \text{Cos}(2\pi jk / N) + \sum_{k \in \Omega_N^0} b_k \text{Sin}(2\pi jk / N)$. Then $\text{Var}(p^*)$ and the sum of oligopoly profits are increasing in the value of τ , while $\text{Var}(q^*)$ is invariant to the value of τ .

If the c_j were allowed to vary across firms, then circumstances could arise such that $d\text{Var}(p^*)/d\tau < 0$ because the values of $\mathcal{G}_j = c_j + 0.5\tau\delta_j$ may converge with an increase in τ . Then a more competitive environment would result. We turn now to understanding how prices covary around the circle.

Proposition 6. $\text{Cov}(p_j^*, p_{[j+r]}^*) = 0.5 \sum_{k \in \Omega_N^0} (\hat{a}_k^2 + \hat{b}_k^2) \text{Cos}(2\pi rk / N)$ and $\sum_{r \in \Omega_N} \text{Cov}(p_j^*, p_{[j+r]}^*) = 0$.

This result conveys the extent of structure on pricing in Circular City because we are dealing with an arbitrary specification of exogenous parameters. Note that $\sum_{\substack{r \in \Omega_N \\ r \neq 0}} \text{Cov}(p_j^*, p_{[j+r]}^*) < 0$ and at least one lagged correlation between prices must be negative. As a particular instance, let $a_k, b_k = 0$ on $k \in \Omega_N^0, k \neq 1$. Then $\text{Cov}(p_j^*, p_{[j+r]}^*) = 0.5(a_1^2 + b_1^2) \text{Cos}(2\pi r / N) / [2 - \text{Cos}(2\pi / N)]^2$, so that $\text{Cov}(p_j^*, p_{[j+r]}^*) \geq 0$ on prices $p_{[j+r]}^*$ for the 50% of firms nearest the j th firm. When higher harmonics are relevant, $a_k \neq 0$ or $b_k \neq 0$ for $k \in \Omega_N^0, k \geq 2$, then positive price covariance is not likely to persist far along the circumference because higher periodicity $\text{Cos}(2\pi rk / N)$ terms remain.

4. Comparisons with Monopoly and Efficient Solution

■ In our model a monopolist may have incentive to shut down all plants except the least cost plant, forcing consumers to travel up to half the circumference. If this occurs, the price charged will be increased to the point where the reservation utility of the most distant consumer is met. Prices may be increased beyond that, so losing some custom. On the other hand, if costs are similar across plants there may be an incentive for the monopolist to keep plural plants open in order to price closer to the consumer's reservation utility by reducing consumer travel costs. The monopoly solution depends very much on the particulars of the problem's parameters and little can be asserted in the abstract.

To develop the efficient solution, it is easiest to think in terms of the efficient hinterland, i.e., the interval on the circle that a plant should cater to. Write $y_{[j]}$ as the furthestmost point, counterclockwise, from x_j such that customers shop at x_j . Thus, the hinterland for x_j is $[y_j, y_{[j+1]})$, and minimizing total cost across consumers involves efficiently partitioning the circumference. Normalizing the mass of consumers on the circle to unity, $\eta = 1$, and integrating across consumers, total costs may be written as

$$\begin{aligned} C(y) &= \sum_{j \in \Omega_N} c_j |y_{[j+1]} - y_{[j]}| + \tau \sum_{j \in \Omega_N} \left(\int_{[y_{[j]}, x_j)} (x_j - u) du + \int_{[x_j, y_{[j+1]})} (u - x_j) du \right) \\ &= \sum_{j \in \Omega_N} c_j |y_{[j+1]} - y_{[j]}| + 0.5\tau(x_j - y_{[j]})^2 + 0.5\tau(y_{[j+1]} - x_j)^2. \end{aligned} \quad (4.1)$$

Upon minimizing over the $y_{[j]}$, $j \in \Omega_N$, the minimized cost function is found to be

$$\begin{aligned} C^*(y) &= 0.25\tau \sum_{j \in \Omega_N} |x_{[j+1]} - x_j|^2 + 0.5 \sum_{j \in \Omega_N} c_j \delta_j \\ &\quad + 0.25\tau^{-1} \sum_{j \in \Omega_N} (c_{[j-1]} + c_{[j+1]}) c_j - 0.5\tau^{-1} \sum_{j \in \Omega_N} c_j^2. \end{aligned} \quad (4.2)$$

The term $0.25\tau \sum_{j \in \Omega_N} |x_{[j+1]} - x_j|^2$, being symmetric and convex in the $x_{[j+1]} - x_j$, captures the idea that a more uniform distribution of plant locations should reduce efficient costs given

that consumers are uniformly distributed.¹⁶ The term $0.5 \sum_{j \in \Omega_N} c_j \delta_j$ indicates that it would be more efficient if lower cost plants also happened to have larger hinterlands. The remaining two terms identify minimized total cost as concave, though not strictly so, and dihedrally symmetric in unit costs. Thus, if $\delta_j = \bar{\delta} \forall j \in \Omega_N$ and $c'' \leq_{dih} c'$, then total efficient cost would be lower under c' than under c'' .

Some work with the cost minimizing optimality conditions establishes j th firm sales under the efficient solution as

$$|y_{[j+1]} - y_j| = \frac{\delta_j}{2} + \frac{c_{[j+1]} + c_{[j-1]} - 2c_j}{2\tau}. \quad (4.3)$$

On the other hand, from (3.10), j th firm sales under Bertrand oligopoly amount to

$|x_U(x_j) - x_L(x_j)| = (p_j^* - c_j)\tau^{-1}$ where our knowledge of p_j^* confirms that $p_j^* - c_j$ is decreasing in own cost and increasing in all other costs. Hence,

Observation. Suppose that c_j increases. Under the efficient solution, output of the j th firm decreases, outputs of adjacent firms increase, and outputs of all other firms are unaffected. Under oligopoly, outputs of all firms except the j th firm increase.

Hence, cost disturbances ripple further under oligopoly. Turning to a comparison of outputs at given parameter levels, write $m_j^{eff} = |y_{[j+1]}^{eff} - y_j^{eff}|$ as the j th firm hinterland under the efficient solution and m_j^{olig} as the j th firm hinterland under Bertrand oligopoly pricing. Some calculations confirm that, in matrix form,

$$m^{eff} - m^{olig} = 0.5(I_N - U)(\delta - \bar{\delta}1_N) - \tau^{-1}U^{-1}(U - I_N)^2(c - \bar{c}1_N). \quad (4.4)$$

Given that differences between hinterland lengths under two economic structures are at issue, it

¹⁶ See Definition 5 above, where $|x_{[j+1]} - x_j| = N^{-1} \forall j \in \Omega_N$ is majorized by any other partition of the unit circumference.

is difficult to obtain general results. Two observations can be made, however.

Proposition 7. Choose $a_k, b_k, k \in \Omega_N$ to make the exact fit, as in (3.14). If

a) $c_j = \bar{c} \ \forall j \in \Omega_N$, then covariance between raw parameters and efficient hinterland length is more positive than covariance between raw parameters and hinterland length under Bertrand oligopoly, $\text{Cov}(m^{\text{eff}} - m^{\text{olig}}, \mathcal{G}) \geq 0$;

b) $\delta_j = \bar{\delta} \ \forall j \in \Omega_N$, then covariance between Bertrand prices and efficient hinterland length is more negative than covariance between Bertrand prices and hinterland length under Bertrand oligopoly, $\text{Cov}(m^{\text{eff}} - m^{\text{olig}}, p^*) \leq 0$.

Part a) holds that hinterland length tends to be too small under oligopoly when the firm's raw parameter (in this case, distance between the two adjacent plants) is comparatively large. This captures the idea that firms with a comparatively large base of captive consumers tend to charge too much under oligopoly in Circular City. Part b) asserts that when costs only distinguish firms in regard to relative competitiveness, then hinterland length tends to be too large under oligopoly when oligopoly price is high. Here, firms with lower costs exploit their competitiveness by rationing demand at the margin through charging somewhat higher prices.

5. Cournot oligopoly and circulant production externalities

■ In this section we adapt the harmonic approach to address local production externalities in Cournot oligopoly. The model may be viewed as a generalization of the oligopoly welfare problem in Salant and Shaffer (1999) and Van Long and Soubeyran (2001). In that model, the only externality was that of market power effects whereas we will introduce local cost externalities. The fixed set of N firms are located at equal intervals on the circumference of Circular City. We have no particular interest in the firm's exact location apart from requiring that each firm generates an externality for one of its neighbors. The n th firm produces q_n units

of identical product and sells into a common market, with the twice continuously differentiable, decreasing inverse demand function $P(Q) : \bar{\mathbb{R}}_+ \rightarrow \bar{\mathbb{R}}_+$ where $Q = \sum_{n \in \Omega_N} q_n$.

The n th firm has cost amounting to $(c_n - \sigma q_{[n+1]})q_n + 0.5\mu q_n^2$, with $\mu > 0$ and $c_n \geq 0 \forall n \in \Omega_N$. External cost effects $-\sigma q_n q_{[n+1]}$ may be beneficial (i.e., $\sigma > 0$) because larger local production in that industry ensures thicker input and support service markets in that locality. Alternatively, local externalities may be negative (i.e., $\sigma < 0$) due to traffic and pollution.¹⁷ There are two economic issues concerning local spillovers: spillovers are not fully internalized, and the failure to internalize can be asymmetric. Our specification captures both of these features by assuming that each firm is affected only by actions of its neighbor to one side, say clockwise to the right. The n th firm's profit is

$$V^n(q_n) = P(Q)q_n - c_n q_n - 0.5\mu q_n^2 + \sigma q_n q_{[n+1]}. \quad (5.1)$$

Assuming interior solutions, the Nash best-response functions are given by

$$P(Q) - c_n = (\mu - P_Q(Q))q_n - \sigma q_{[n+1]} \quad \forall n \in \Omega_N. \quad (5.2)$$

Sum over firms and differentiate to conclude $dQ^* / d\sigma =$

$-Q^* / [(N+1)P_Q(Q^*) + Q^* P_{QQ}(Q^*) - \mu + \sigma]$. Condition $(N+1)P_Q(Q^*) + Q^* P_{QQ}(Q^*) - \mu + \sigma \leq 0$ suffices to ensure the uniqueness of a solution to (5.2), and we maintain this assumption throughout the section.¹⁸ The sign $dQ^* / d\sigma \geq 0$ is guaranteed under the uniqueness assumption.

It is intuitive in asserting that a strengthening of positive externalities encourage production in the aggregate.

¹⁷ We use externalities here to reflect the situation in which the costs of one firm are affected by the actions of another. This effect could be due to pecuniary effects (e.g., changes in input prices) or technological effects (such as pollution that lowers the productivity of other firms). While the distinction matters when identifying the efficient solution, it does not matter when characterizing a *laissez faire* equilibrium.

¹⁸ Existence is ensured by the requirement $NP(Q=0) - N\bar{c} > 0 >$

$\lim_{Q \rightarrow \infty} \{[P(Q) - \bar{c}]N + [P_Q(Q) - \mu + \sigma]Q\}$ where $\bar{c} = N^{-1} \sum_{n \in \Omega_N} c_n$. If $P(Q=0) > \bar{c}$,

If demand is linear, with $P(Q) = \alpha - \beta Q$, $\beta > 0$, then the matrix form of (5.2) is

$$\begin{pmatrix} 1+\xi & -\xi & 0 & \cdots & 0 & 0 \\ 0 & 1+\xi & -\xi & \cdots & 0 & 0 \\ 0 & 0 & 1+\xi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\xi & -\xi \\ -\xi & 0 & 0 & \cdots & 0 & 1+\xi \end{pmatrix} \begin{pmatrix} q_0^* - \bar{q}^* \\ q_1^* - \bar{q}^* \\ q_2^* - \bar{q}^* \\ \vdots \\ q_{N-2}^* - \bar{q}^* \\ q_{N-1}^* - \bar{q}^* \end{pmatrix} = \begin{pmatrix} \varsigma_0 \\ \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{N-2} \\ \varsigma_{N-1} \end{pmatrix}; \quad (5.3)$$

where $\xi = \sigma / (\mu + \beta - \sigma)$, $\varsigma_n = (\bar{c} - c_n) / (\mu + \beta - \sigma)$, and $\bar{q}^* = (\alpha - \bar{c}) / [(N+1)\beta + \mu - \sigma]$.

Inversion yields

$$\begin{pmatrix} q_0^* - \bar{q}^* \\ q_1^* - \bar{q}^* \\ q_2^* - \bar{q}^* \\ \vdots \\ q_{N-2}^* - \bar{q}^* \\ q_{N-1}^* - \bar{q}^* \end{pmatrix} = \frac{\rho - 1}{\rho^N - 1} \begin{pmatrix} \rho^{N-1} & \rho^{N-2} & \rho^{N-3} & \cdots & \rho & 1 \\ 1 & \rho^{N-1} & \rho^{N-2} & \cdots & \rho^2 & \rho \\ \rho^2 & 1 & \rho^{N-1} & \cdots & \rho^3 & \rho^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho^{N-3} & \rho^{N-4} & \rho^{N-5} & \cdots & \rho^{N-1} & \rho^{N-2} \\ \rho^{N-2} & \rho^{N-3} & \rho^{N-4} & \cdots & 1 & \rho^{N-1} \end{pmatrix} \begin{pmatrix} \varsigma_0 \\ \varsigma_1 \\ \varsigma_2 \\ \vdots \\ \varsigma_{N-2} \\ \varsigma_{N-1} \end{pmatrix}; \quad (5.4)$$

where $\rho = (\mu + \beta) / \sigma = (1 + \xi) / \xi$. With obvious notation, (5.4) is also written as $q^* = \Gamma \varsigma$.

Some analysis then establishes

Proposition 8. Let $P(Q) = \alpha - \beta Q$, $\beta > 0$, $\mu + \beta > |\sigma|$, and $N \geq 2$. If the sum of costs are fixed at $\sum_{n \in \Omega_N} c_n = N\bar{c}$, then

a) n th firm production is given by

$$q_n^* - \bar{q}^* = \frac{1}{\mu + \beta - \sigma} \sum_{j \in \Omega_N} u_{[j-n]} (\bar{c} - c_n); \quad u_j = \frac{(\rho - 1) \rho^{N-j-1}}{\rho^N - 1}; \quad (5.5)$$

b) $\lim_{\sigma \rightarrow 0} u_j = 0 \forall j \in \Omega_N^0$, and $\lim_{\sigma \rightarrow 0} u_0 = 1$;

c) the weights are positive if $\sigma > 0$. If $\sigma < 0$, then $u_0 > 1$ and $(-1)^{j+1} u_j < 0 \forall j \in \Omega_N$;

d) $|u_n / u_{n+1}| \geq 1$, while u_{n+1} / u_n is increasing in σ ;

$\lim_{Q \rightarrow \infty} P(Q) = 0$, $\lim_{Q \rightarrow \infty} QP_Q(Q) = 0$, and $\mu > \sigma$, then existence is guaranteed.

e) for $\phi = 1/(\mu + \beta - \sigma)$, if $c_n - \bar{c} = \text{Cos}(2\pi nk / N)$ then

$$\begin{aligned} q_0^* - \bar{q}^* &= -\frac{\phi(\rho - 1)(\rho - \text{Cos}(2\pi k / N))}{\rho^2 - 2\rho\text{Cos}(2\pi k / N) + 1}, \\ q_n^* - \bar{q}^* &= \text{Cos}(2\pi nk / N)(q_0^* - \bar{q}^*) - \frac{\text{Sin}(2\pi nk / N)\text{Sin}(2\pi k / N)}{\rho - \text{Cos}(2\pi k / N)}(q_0^* - \bar{q}^*), \end{aligned} \quad (5.6)$$

so that $\sigma > 0$ ($\sigma < 0$) implies $q_0^* < (>) \bar{q}^*$ while $dq_0^* / dk \stackrel{\text{sign}}{=} -(\rho - 1)(1 - \rho^2)\text{Sin}(2\pi k / N) \stackrel{\text{sign}}{=} (\rho - 1)\text{Sin}(2\pi k / N)$, i.e., positive if $\sigma > 0$ and $k \in [0, N/2)$ or $\sigma < 0$ and $k \in [N/2, N)$, and negative otherwise. In addition, if $c_n - \bar{c} = \text{Cos}(2\pi nk / N)$, then

$$\text{Var}(q^*) = \frac{0.5\phi^2(\rho - 1)^2}{\rho^2 - 2\rho\text{Cos}(2\pi k / N) + 1}, \quad (5.7)$$

which is larger (smaller) than $0.5\phi^2$ whenever $k \in (-N/4, N/4)$, while $\lim_{N \rightarrow \infty} \text{Var}(q^*) =$

$0.5\phi^2$, $d\text{Var}(q^*) / dk \stackrel{\text{sign}}{=} -\rho\text{Sin}(2\pi k / N)$, and $d\text{Var}(q^*) / d\sigma > 0$ if $\sigma > 0$;

f) if $c_n - \bar{c} = \text{Cos}(2\pi nk / N)$, then $\text{Cov}(q^*, c) = 0.5(q_0^* - \bar{q}^*) / (\mu + \beta - \sigma)$ so that $\sigma > 0$ ($\sigma < 0$) implies $\text{Cov}(q^*, c) < (>) 0$;

g) if $\sigma > 0$, then $\text{Var}(q^*) \leq \text{Var}(c) / (\mu + \beta - \sigma)^2$.

It is clear that model behavior is not quite as pleasant as under the Bertrand model. Issues arising here, and not in the earlier Bertrand model, are the sign and size of weightings. Whereas in the Bertrand model, the weightings arose naturally from arbitrage conditions facing the indifferent consumer, and were fixed for given N , now they can vary with σ . Parts b) and c) establish that weightings do concentrate smoothly as $\sigma \rightarrow 0$, but the second and even weightings are negative whenever $\sigma < 0$. Part d) identifies a further regularity when $\sigma > 0$, the (positive) weightings tend to flatten out as σ increases. When $\sigma < 0$ then the negative externality becomes less significant as σ increases, allowing the coefficients to concentrate more smoothly toward the case where spatial spillovers disappear. In e), it is shown that equilibrium production

responds to external parameters of form $c_n - \bar{c} = \text{Cos}(2\pi nk / N)$ in a rather involved manner.

This is because, unlike for our earlier U , in Γ the weights do not decline gradually and rise gradually. They decline gradually and rise abruptly from $(\rho - 1)/(\rho^N - 1)$ to

$(\rho - 1)\rho^{N-1}/(\rho^N - 1)$.¹⁹ We will develop intuition for parts f) and g) through an example.

Example 6. Suppose that $N = 2$, $\mu = \beta = 10$, and $\sigma = -2$ so that $\rho = -10$, $u_0 = 10/9$, and $u_1 = -1/9$. If $\bar{c} = 1$, and $c_0 - \bar{c} = 1 = \bar{c} - c_1$ then $q_0^* - \bar{q}^* = 1/18 = \bar{q}^* - q_1^*$ and $\text{Var}(q^*) = 1/(18)^2 > \text{Var}(c)/(\mu + \beta - \sigma)^2 = 1/(22)^2$. When externalities are positive (complementary), then they act to smooth the variance of outputs whereas when they are negative (substitutive) then they act to accentuate variance. The distinction has to do with matrix eigenvalues because eigenvalues describe damping attributes of difference equations. Circulant $U = [10/9, -1/9]$ has eigenvalues $\lambda_0 = 1$ and $\lambda_1 = 11/9$ whereas the moduli of the eigenvalues for the square matrix in (5.4) would all be inside the unit circle whenever $\sigma > 0$. That one of the eigenvalues is outside the unit circle ensures that the u_j weightings in (5.5) are not in the unit simplex.²⁰

The effects of cost dispersion on oligopoly profits can also be ascertained. From the sum of first-order conditions, $(\alpha - \bar{c})/[(N + 1)\beta + \mu - \sigma] = \bar{q}^*$, it is clear that market output is invariant to location on the cost simplex $\sum_{n \in \Omega_N} c_n = N\bar{c}$. For cost vector conditioned outputs $q_n^*(c)$, with $Q^*(c) = \sum_{n \in \Omega_N} q_n^*(c)$, define the Herfindahl concentration index as $\mathcal{H}(c) = [Q^*(c)]^{-2} \sum_{n \in \Omega_N} [q_n^*(c)]^2$. We will show that this is proportional to industry profits and that both increase under a cost increase in the $c' \leq_{dih} c''$ sense. Insert first-order condition (5.2) into (5.1)

¹⁹ We do not insert an exact trigonometric series in part f) because, when compared with (3.15), the comparative inelegance under one-sided spillovers is observed just as well with a single harmonic.

²⁰ See pp. 131-134 in Davis for references and further developments on eigenvalue

and sum to obtain

$$\sum_{n \in \Omega_N} V^n(q_n^*) = (0.5\mu + \beta) \zeta^T \Gamma^T \Gamma \zeta = (0.5\mu + \beta) [Q^*(c)]^2 \mathcal{H}(c). \quad (5.8)$$

As with Bertrand pricing on Circular City, the sum of oligopoly profits is directly proportional to dispersion. Define social surplus as the sum of consumer surplus and industry profits.²¹

Proposition 9. Under the conditions of Proposition 8, let $c' \rightarrow c''$ with $c' \leq_{dih} c''$ where the set of firms active under both cost structures is the same. Then

- a) industry profits and the Herfindahl index are larger under c'' than under c' , and
- b) if the externalities arise from technological spillovers (and not pecuniary effects) then social surplus is larger under c'' than under c' .²²

Notice that we do not require $\sigma > 0$ in the proposition. The u_j coefficients can be off the unit simplex and these results still apply. A more dispersed cost vector, in our dihedral majorization sense, leads to a more dispersed output vector regardless of whether the externality is positive or negative.

□ **Contrasts with monopoly and efficient solutions.** By way of contrast, consider monopoly behavior under which the objective is to $\text{Max}_q \sum_{n \in \Omega_N} V^n(q_n)$ with solution $V(c^T) =$

$V(c_0, c_1, \dots, c_{N-1})$ where

$$\begin{aligned} V(c_0, c_1, \dots, c_{N-1}) = \text{Max}_q \quad & \alpha \sum_{n \in \Omega_N} q_n - \beta \left(\sum_{n \in \Omega_N} q_n \right)^2 - \sum_{n \in \Omega_N} c_n q_n \\ & - (0.5\mu - \sigma) \sum_{n \in \Omega_N} q_n^2 - 0.5\sigma \sum_{n \in \Omega_N} (q_n - q_{[n+1]})^2. \end{aligned} \quad (5.9)$$

The first-order conditions for problem (5.9) are

characterization of smoothing linear operations.

²¹ It is straightforward to show that these results apply under any demand function such that our oligopoly regularity conditions apply.

²² On pecuniary effects, see footnote 17.

$$\begin{aligned} \mu(q_n^{mon} - \bar{q}^{mon}) - \sigma(q_{[n-1]}^{mon} - \bar{q}^{mon}) - \sigma(q_{[n+1]}^{mon} - \bar{q}^{mon}) &= \bar{c} - c_n \quad \forall n \in \Omega_N; \\ \bar{q}^{mon} &= \frac{\alpha - \bar{c}}{2N\beta + \mu - 2\sigma}; \end{aligned} \quad (5.10)$$

where the superscripted *mon* indicates monopoly. To better understand how different the harmonics are likely to be, view the system at solution values in matrix form:

$$\begin{pmatrix} 1+2\gamma & -\gamma & 0 & \cdots & 0 & -\gamma \\ -\gamma & 1+2\gamma & -\gamma & \cdots & 0 & 0 \\ 0 & -\gamma & 1+2\gamma & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\gamma & -\gamma \\ -\gamma & 0 & 0 & \cdots & -\gamma & 1+2\gamma \end{pmatrix} \begin{pmatrix} q_0^{mon} - \bar{q}^{mon} \\ q_1^{mon} - \bar{q}^{mon} \\ q_2^{mon} - \bar{q}^{mon} \\ \vdots \\ q_{N-2}^{mon} - \bar{q}^{mon} \\ q_{N-1}^{mon} - \bar{q}^{mon} \end{pmatrix} = \begin{pmatrix} \theta_0 \\ \theta_1 \\ \theta_2 \\ \vdots \\ \theta_{N-2} \\ \theta_{N-1} \end{pmatrix}; \quad (5.11)$$

where $\gamma = \sigma / (\mu - 2\sigma)$ and $\theta_n = (\bar{c} - c_n) / (\mu - 2\sigma)$. Relative to (5.3), matrix symmetry around the main diagonal ensures that the properties of equilibrium quantity decisions will be very different from those under oligopoly. This is because the monopolist fully internalizes cost spillovers.

The efficient solution on the other hand solves

$$\begin{aligned} \text{Max}_q \quad & \alpha \sum_{n \in \Omega_N} q_n - 0.5\beta \left(\sum_{n \in \Omega_N} q_n \right)^2 - \sum_{n \in \Omega_N} c_n q_n \\ & - (0.5\mu - \sigma) \sum_{n \in \Omega_N} q_n^2 - 0.5\sigma \sum_{n \in \Omega_N} (q_n - q_{[n+1]})^2. \end{aligned} \quad (5.12)$$

First-order conditions are

$$\begin{aligned} \mu(q_n^{eff} - \bar{q}^{eff}) - \sigma(q_{[n-1]}^{eff} - \bar{q}^{eff}) - \sigma(q_{[n+1]}^{eff} - \bar{q}^{eff}) &= \bar{c} - c_n \quad \forall n \in \Omega_N; \\ \bar{q}^{eff} &= \frac{\alpha - \bar{c}}{N\beta + \mu - 2\sigma}; \end{aligned} \quad (5.13)$$

where the superscripted *eff* indicates efficient. It is clear that quantity deviations from the mean are the same as under monopoly. This is due in part to the fact that outputs are perfect substitutes in demand, and in part to the fact that cost spillovers are completely internalized in both the efficient and monopoly solutions. Of course, $\bar{q}^{eff} > \bar{q}^{mon}$. Since q^{eff} and q^{mon} are perfectly correlated, a comparison of q^* with q^{mon} suffices to relate how all correlate.

Proposition 10. Suppose that $\mu > |2\sigma|$. If $c_n - \bar{c} = \text{Cos}(2\pi nk / N)$ and $\sigma > (<) 0$, then $\text{Cov}(q^*, q^{mon}) > (<) 0$.

Thus, if cost spillovers are positive then the monopoly (and efficient) quantity solution vector correlates positively with the Cournot solution vector for this specification of cost parameters.²³ The correlation is negative if cost spillovers are adverse, as would be the case with a pollutant that hindered manufacture. The reason is due to the nature of strategic response to proximate costs that emerges in Cournot oligopoly, see Example 6 above. If a plant's cost is high, then own production tends to be low under oligopoly or monopoly. With negative spillovers, this gives an incentive for the affected neighboring firm to expand relative to the monopoly level of output. With positive spillovers there is an additional incentive to contract relative to the monopoly level of output.

6. Conclusion

■ This paper has studied equilibrium for imperfect competition when producers are discretely located on a circle and consumers are uniformly distributed on the circle. The analysis readily extends to the case where population density is not uniform, while we could almost as readily have modeled competition on a globe or torus. Although the paper's main goal has been to provide insights on behavior and surpluses, several testable hypotheses emerge in propositions 1, 4, 5, and elsewhere. The analysis is also relevant for Linear City, at least at the limit, because Circular City and Linear City are effectively the same when the circumference becomes infinitely large.

The paper's larger goal has been to demonstrate the utility of mathematical tools that can accommodate neighborhood effects in supply and demand. We have shown that symmetries and

²³ Or indeed for any specification of form $c_n - \bar{c} = \sum_{n \in \Omega_N} a_n \text{Cos}(2\pi nk / N)$. Revisit the proof

harmonic analysis can represent local spillovers with parsimony and transparency. Possible alternative applications include information dissemination through a population and how it might affect bid/share pricing. In d'Aspremont and Jacquemin's (1988) model of R&D, duopolists collaborate to fix the extent of cost-reducing private research efforts, and then compete when there are cost spillovers. More realistically, firms will version their research to gain private comparative advantage so that other firms producing similar product lines will benefit most from R&D spillovers. Our approach could also be applied to understand the economic effects of R&D collaboration in an environment where firms are neighbors in product attribute space. In network economics, congestion and other local externalities can also be important. It may be possible to better understand equilibrium in large networks if the symmetries of a network's topology are exploited, as we have done with the circle.

and use identities in Appendix A.

Appendix A: Some trigonometric identities

Let $\mathbb{Z}_0 = \{ \dots, -3, -2, -1, 1, 2, 3, \dots \}$.

$$(A1) \quad \text{Cos}(2\pi(m+j)k/N) = \text{Cos}(2\pi mk/N)\text{Cos}(2\pi jk/N) - \text{Sin}(2\pi mk/N)\text{Sin}(2\pi jk/N);$$

$$(A2) \quad \text{Sin}(2\pi(m+j)k/N) = \text{Sin}(2\pi mk/N)\text{Cos}(2\pi jk/N) + \text{Sin}(2\pi jk/N)\text{Cos}(2\pi mk/N);$$

$$(A3) \quad \text{Cos}^2(2j\pi/N) + \text{Sin}^2(2j\pi/N) = 1;$$

$$(A4) \quad \sum_{j \in \Omega_N} \text{Cos}^2(2jk\pi/N) = 0.5N = \sum_{j \in \Omega_N} \text{Sin}^2(2jk\pi/N) \quad \forall k \in \mathbb{Z}_0, \text{ (Parseval's Theorem);}$$

$$(A5) \quad \left. \begin{aligned} \sum_{j \in \Omega_N} \text{Cos}(2jk\pi/N)\text{Sin}(2jl\pi/N) &= 0 \text{ for } k \text{ and } l \text{ integers,} \\ \sum_{j \in \Omega_N} \text{Cos}(2jk\pi/N)\text{Cos}(2jl\pi/N) &= 0 \text{ for } k \text{ and } l \neq k \text{ integers,} \\ \sum_{j \in \Omega_N} \text{Sin}(2jk\pi/N)\text{Sin}(2jl\pi/N) &= 0 \text{ for } k \text{ and } l \neq k \text{ integers,} \end{aligned} \right\} \text{ (orthogonality property);}$$

$$(A6) \quad \sum_{j \in \Omega_N} u_j \text{Sin}(2jk\pi/N) = 0 \text{ if } u_j = u_{N-j} \text{ } j \in \Omega_N \text{ and } k \geq 1 \text{ an integer, (odd function property);}$$

$$(A7) \quad \text{Cos}(x) = \text{Cos}(-x), \quad \text{Sin}(x) = -\text{Sin}(-x);$$

$$(A8) \quad \sum_{j \in \Omega_N} \text{Cos}(2jk\pi/N) = 0 = \sum_{j \in \Omega_N} \text{Sin}(2jk\pi/N) \quad \forall k \in \mathbb{Z}_0.$$

Appendix B

Proof of Proposition 1. Notice first that $p_j^* \geq c_j \quad \forall j \in \Omega_N$ if the firms are to operate.

Part a): From (3.10), $V^0 = \eta\tau^{-1}(p_0^* - c_0)^2$. From (3.9), $dV^0/dc_j = 2\eta\tau^{-1}(p_0^* - c_0)u_j \geq 0$, and

$$dV^0/dc_0 = 2\eta\tau^{-1}(p_0^* - c_0)(u_0 - 1) \leq 0.$$

Part b): It is only necessary to show that $u_j \geq u_{j+1}$ whenever $(N-1)/2 > j > 0$. From (3.9),

$$u_j \geq u_{j+1} \text{ if and only if } z^j + z^{N-j} \geq z^{j+1} + z^{N-j-1}, z > 1. \text{ Due to convexity of the exponential}$$

function, this follows when the indices on the left are more spread out, i.e., $j < N-j-1$.

Part c): Apply deduction $u_j \geq u_{j+1}$ whenever $(N-1)/2 > j > 0$ to differentials on equations (3.9) and (3.10).

Part d): From $u_j \equiv u_{N-j}$ in (3.9), $dV^0/dc_j = 2\eta\tau^{-1}(p_0^* - c_0)u_j = 2\eta\tau^{-1}(p_0^* - c_0)u_{N-j} = dV^0/dc_{N-j}$

$$\forall j \in \Omega_N.$$

Part e): Insert $c + \varepsilon 1_N$ into equations (3.9), (3.10), and (3.5).

Part f): $dV^j / dc_0 = 2\eta\tau^{-1}(p_j^* - c_j)u_j$ while $dV^{N-j} / dc_0 = 2\eta\tau^{-1}(p_{N-j}^* - c_{N-j})u_j$ so that $dV^j / dc_0 > dV^{N-j} / dc_0$, $j \neq 0$, if and only if $p_j^* - c_j > p_{N-j}^* - c_{N-j}$. This also orders the firm profit functions.

Part g): $dV^j / dc_0 = 2\eta\tau^{-1}(p_j^* - c_j)u_j$ while $dV^{[j+1]} / dc_0 = 2\eta\tau^{-1}(p_{[j+1]}^* - c_{[j+1]})u_{[j+1]}$. Now $p_j^* - c_j > p_{[j+1]}^* - c_{[j+1]}$ if and only if $V^j > V^{[j+1]}$ so it suffices to show that $u_j \geq u_{[j+1]}$ whenever $(N-1)/2 > j > 0$. The latter conclusion was confirmed in part b).

Part h): Observe that $d \sum_{j \in \Omega_N} V^j / dc_0 = 2\eta\tau^{-1} \sum_{j \in \Omega_N} (p_j^* - c_j)u_j - 2\eta\tau^{-1}(p_0^* - c_0)$. The conditions in the part give the derivative sign since $\text{Max}_{j \in \Omega_N} \{p_j^* - c_j\} \geq \sum_{j \in \Omega_N} (p_j^* - c_j)u_j \geq \text{Min}_{j \in \Omega_N} \{p_j^* - c_j\}$. There is always a least value and greatest value in finite set $\{p_j^* - c_j\}$.

Q.E.D.

Proof of Proposition 2. In part a) and for profits, we only need to know how the value of $\delta^T U^T U \delta$ changes with dihedral mixing, $\leq_{dih}$, because the other expressions in (3.11) are invariant. Now $U = U^T$ from (3.9). Since U is a symmetric circulant, all eigenvalues are real (Pettofrezzo, 1966, p. 91) and all eigenvalues of U^2 are positive (Davis, 1994, p. 32).

Therefore, $\delta^T U^2 \delta$ is a convex function of the δ_j . In addition, U^2 is a symmetric matrix. This ensures that $L^j(\delta^T)U^2 L^j(\delta) = \delta^T U^2 \delta \forall j \in \Omega_N$ and $L^j(\delta_-^T)U^2 L^j(\delta_-) = \delta^T U^2 \delta \forall j \in \Omega_N$. Since $\delta^T U^2 \delta$ is convex in the δ_j , Jensen's inequality requires the dihedral majorization consequence

$$\delta^T U^2 \delta \geq \left(\sum_{j=0}^{N-1} \lambda_j L^j(\delta) + \sum_{j=N}^{2N-1} \lambda_j L^j(\delta_-) \right)^T U^2 \left(\sum_{j=0}^{N-1} \lambda_j L^j(\delta) + \sum_{j=N}^{2N-1} \lambda_j L^j(\delta_-) \right). \quad (\text{B1})$$

Price variance depends only on $0.25\tau^2 \delta^T U^2 \delta$, see equations (3.8) and (3.9). The price dispersion result follows from the symmetry and convexity of $\delta^T U^2 \delta$ in δ . The quantity dispersion effect follows similarly given cost invariance and (3.10).

In part b), $(U\vartheta - c)^T (U\vartheta - c)$ reduces to $(Uc + 0.5\tau\bar{\delta}U1_N - c)^T (Uc + 0.5\tau\bar{\delta}U1_N - c)$. Upon

expansion, all terms are constant under the specified conditions except $c^T (U - I_N)^T (U - I_N) c$, I_N the $N \times N$ identity. Now $U - I_N$ is symmetric circulant so that $(U - I_N)^T (U - I_N)$ is symmetric circulant. Matrix $U - I_N$ has real eigenvalues so that $(U - I_N)^T (U - I_N) = (U - I_N)^2$ has non-negative eigenvalues and $c^T (U - I_N)^T (U - I_N) c$ is convex in the arguments of c . Then apply invariances invoked in part a) and Jensen's inequality. On price variance, the relevant quadratic form for prices is $c^T U^2 c$. This is convex and is symmetric to both $c \rightarrow L^j(c)$ and $c \rightarrow L^j(c_-)$. Quantity variance depends only on $\eta^2 \tau^{-2} c^T (U - I_N)^2 c$. See part a) on symmetry and convexity. *Q.E.D.*

Proof of Proposition 3. Part a): Use (A1), and (A7) to write

$$\begin{aligned}
p_j^* - \bar{p}^* &= \sum_{m=0}^{N-1} u_{[m-j]} (c_m - \bar{c} + 0.5\tau(\delta_m - \bar{\delta})) \\
&= \sum_{m=0}^{N-1} u_m \text{Cos}(2(m+j)\pi/N) + 0.5\tau K_1 \sum_{m=0}^{N-1} u_m \text{Cos}(2(m+j-r)\pi/N) \\
&= \sum_{m=0}^{N-1} u_m \text{Cos}(2(m+j)\pi/N) \\
&\quad + 0.5\tau K_1 \sum_{m=0}^{N-1} u_m (\text{Cos}(2(m+j)\pi/N) \text{Cos}(2r\pi/N) + \text{Sin}(2(m+j)\pi/N) \text{Sin}(2r\pi/N)) \\
&= (1 + 0.5\tau K_1 \text{Cos}(2r\pi/N)) \sum_{m=0}^{N-1} u_m \text{Cos}(2(m+j)\pi/N) \\
&\quad + 0.5\tau K_1 \text{Sin}(2r\pi/N) \sum_{m=0}^{N-1} u_m \text{Sin}(2(m+j)\pi/N).
\end{aligned} \tag{B2}$$

Expand $\text{Cos}(2(m+j)\pi/N)$ and $\text{Sin}(2(m+j)\pi/N)$ using (A1) and (A2):

$$\begin{aligned}
p_j^* - \bar{p}^* &= (1 + 0.5\tau K_1 \text{Cos}(2r\pi/N)) \text{Cos}(2j\pi/N) \sum_{m=0}^{N-1} u_m \text{Cos}(2m\pi/N) \\
&\quad - (1 + 0.5\tau K_1 \text{Cos}(2r\pi/N)) \text{Sin}(2j\pi/N) \sum_{m=0}^{N-1} u_m \text{Sin}(2m\pi/N) \\
&\quad + 0.5\tau K_1 \text{Sin}(2r\pi/N) \text{Cos}(2j\pi/N) \sum_{m=0}^{N-1} u_m \text{Sin}(2m\pi/N) \\
&\quad + 0.5\tau K_1 \text{Sin}(2r\pi/N) \text{Sin}(2j\pi/N) \sum_{m=0}^{N-1} u_m \text{Cos}(2m\pi/N).
\end{aligned} \tag{B3}$$

Now (A6) reduces the expression to

$$\begin{aligned}
p_j^* - \bar{p}^* &= (1 + 0.5\tau K_1 \text{Cos}(2r\pi/N)) \text{Cos}(2j\pi/N) \sum_{m=0}^{N-1} u_m \text{Cos}(2m\pi/N) \\
&\quad + 0.5\tau K_1 \text{Sin}(2r\pi/N) \text{Sin}(2j\pi/N) \sum_{m=0}^{N-1} u_m \text{Cos}(2m\pi/N).
\end{aligned} \tag{B4}$$

Write $K_2 = \sum_{m=0}^{N-1} u_m \text{Cos}(2m\pi / N)$, where we will demonstrate in Lemma 1 that $1 \geq K_2 \geq 0$.

From (B4), (A5), (A4), and (A3):

$$\begin{aligned}
\text{Var}(p^*) &= N^{-1} K_2^2 (1 + 0.5\tau K_1 \text{Cos}(2r\pi / N))^2 \sum_{j \in \Omega_N} \text{Cos}^2(2j\pi / N) \\
&+ N^{-1} K_2^2 (0.5\tau K_1 \text{Sin}(2r\pi / N))^2 \sum_{j \in \Omega_N} \text{Sin}^2(2j\pi / N) \\
&= 0.5 K_2^2 (1 + 0.5\tau K_1 \text{Cos}(2r\pi / N))^2 + 0.5 K_2^2 (0.5\tau K_1 \text{Sin}(2r\pi / N))^2 \\
&= 0.5 K_2^2 (1 + \tau K_1 \text{Cos}(2r\pi / N) + 0.25\tau^2 K_1^2 \text{Cos}^2(2r\pi / N)) + 0.5 K_2^2 (0.25\tau^2 K_1^2 \text{Sin}^2(2r\pi / N)) \\
&= 0.5 K_2^2 + 0.5\tau K_1 K_2^2 \text{Cos}(2r\pi / N) + 0.125\tau^2 K_1^2 K_2^2 \geq 0.
\end{aligned} \tag{B5}$$

Differentiate (B5) to obtain $d\text{Var}(p^*)/dr = -\overset{\text{sign}}{\text{Sin}(2r\pi / N)}$, $\text{Sin}(2r\pi / N) \leq 0$ on $r \in [-N/2, 0]$.

Part b): On quantity, use (3.10), (B5), (A4), (B4), (A5), and then (A3):

$$\begin{aligned}
\tau^2 \eta^{-2} \text{Var}(q^*) &= \text{Var}(p^*) + \text{Var}(c) - 2\text{Cov}(p^*, c) \\
&= 0.5 K_2^2 + 0.5\tau K_1 K_2^2 \text{Cos}(2r\pi / N) + 0.125\tau^2 K_1^2 K_2^2 + 0.5 \\
&- 2N^{-1} (1 + 0.5\tau K_1 \text{Cos}(2r\pi / N)) K_2 \sum_{j \in \Omega_N} \text{Cos}^2(2j\pi / N) \\
&= 0.5 K_2^2 (1 + \tau K_1 \text{Cos}(2r\pi / N)) + 0.125\tau^2 K_1^2 K_2^2 + 0.5 - (1 + 0.5\tau K_1 \text{Cos}(2r\pi / N)) K_2.
\end{aligned} \tag{B6}$$

Differentiate the last expression to obtain $d\tau^2 \eta^{-2} \text{Var}(q^*)/dr = K_2(1 - K_2)\overset{\text{sign}}{\text{Sin}(2r\pi / N)}$

$\text{Sin}(2r\pi / N) = -\overset{\text{sign}}{d\text{Var}(p^*)/dr}$, where $1 \geq K_2 \geq 0$ will be shown in Lemma 1.

Part c): This follows from part b) and the fact that $\sum_{j \in \Omega_N} V^j = \tau \eta^{-1} \sum_{j \in \Omega_N} (q_j^*)^2$. *Q.E.D.*

Proof of Lemma 1. In Davis (p. 73 and p. 68) it is shown that

$$\lambda_k = \sum_{m \in \Omega_N} u_m [\text{Cos}(2\pi mk / N) + i \text{Sin}(2\pi mk / N)]; \tag{B7}$$

where $i = \sqrt{-1}$. By (A6), this can be written as

$$\lambda_k = \sum_{m \in \Omega_N} u_m \text{Cos}(2\pi mk / N), \tag{B8}$$

with $K_2 \equiv \lambda_1$ in the proof of Proposition 3. To confirm eigenvalue positivity, insert the weighting values from (3.9):

$$\lambda_k = \sum_{m \in \Omega_N} \left(\frac{(2 + \sqrt{3})^m + (2 + \sqrt{3})^{N-m}}{\left[(2 + \sqrt{3})^N - 1 \right] \sqrt{3}} \right) \text{Cos}(2\pi mk / N) = \frac{L_1 + L_2}{\left[(2 + \sqrt{3})^N - 1 \right] \sqrt{3}}; \quad (\text{B9})$$

$$L_1 = \sum_{m \in \Omega_N} (2 + \sqrt{3})^m \text{Cos}(2\pi mk / N), \quad L_2 = \sum_{m \in \Omega_N} (2 + \sqrt{3})^{N-m} \text{Cos}(2\pi mk / N).$$

With $z = (2 + \sqrt{3})[\text{Cos}(2\pi k / N) + i\text{Sin}(2\pi k / N)]$, De Moivre's theorem asserts that $z^m =$

$(2 + \sqrt{3})^m [\text{Cos}(2\pi mk / N) + i\text{Sin}(2\pi mk / N)]$ and the real component is $\text{Re}(z^m) =$

$(2 + \sqrt{3})^m \text{Cos}(2\pi mk / N)$ so that L_1 has value

$$\begin{aligned} \sum_{m \in \Omega_N} \text{Re}(z^m) &= \text{Re} \left(\sum_{m \in \Omega_N} z^m \right) = \text{Re} \left(\frac{(2 + \sqrt{3})^N - 1}{(2 + \sqrt{3})[\text{Cos}(2\pi k / N) + i\text{Sin}(2\pi k / N)] - 1} \right) \\ &= \left((2 + \sqrt{3})^N - 1 \right) \text{Re} \left(\frac{(2 + \sqrt{3})[\text{Cos}(2\pi k / N) - i\text{Sin}(2\pi k / N)] - 1}{8 + 4\sqrt{3} - 2(2 + \sqrt{3})\text{Cos}(2\pi k / N)} \right) \\ &= \left((2 + \sqrt{3})^N - 1 \right) \left(\frac{(2 + \sqrt{3})\text{Cos}(2\pi k / N) - 1}{8 + 4\sqrt{3} - 2(2 + \sqrt{3})\text{Cos}(2\pi k / N)} \right). \end{aligned} \quad (\text{B10})$$

Likewise,

$$L_2 = \left((2 + \sqrt{3})^N - 1 \right) \left(\frac{(2 + \sqrt{3})(2 + \sqrt{3} - \text{Cos}(2\pi k / N))}{8 + 4\sqrt{3} - 2(2 + \sqrt{3})\text{Cos}(2\pi k / N)} \right), \quad (\text{B11})$$

so that

$$L_1 + L_2 = \left((2 + \sqrt{3})^N - 1 \right) \left(\frac{(3 + \sqrt{3})(1 + \sqrt{3})}{8 + 4\sqrt{3} - 2(2 + \sqrt{3})\text{Cos}(2\pi k / N)} \right), \quad (\text{B12})$$

and arithmetic confirms

$$\lambda_k = \frac{1}{2 - \text{Cos}(2\pi k / N)}. \quad (\text{B13})$$

The lower bound is when $2\pi k / N = \pi$, i.e., $k = N / 2$ for even N , so that $\text{Cos}(2\pi k / N) = -1$

and $\lambda_k = 1/3$. The upper bound is when $k = 0$, $\text{Cos}(0) = 1$, and $\lambda_k = 1$. Eigenvalue symmetry,

$\lambda_k = \lambda_{N-k}$, follows from the symmetry of the cosine function. And $\lambda_k \geq \lambda_{k+1}$ if $\text{Cos}(2\pi k / N) \geq \text{Cos}(2\pi(k+1) / N)$, true whenever $0 \leq k \leq (N-2)/2$. *Q.E.D.*

Proof of Proposition 4. Part a): Insert (3.14) into (3.9), then use (A1), (A2), a switch in order of summations, and (A6):

$$\begin{aligned}
p_j^* - \bar{p}^* &= \sum_{m \in \Omega_N} u_{[m-j]} \left(\sum_{k \in \Omega_N^0} a_k \text{Cos}(2\pi mk / N) + \sum_{k \in \Omega_N^0} b_k \text{Sin}(2\pi mk / N) \right) \\
&= \sum_{m \in \Omega_N} u_m \left(\sum_{k \in \Omega_N^0} a_k \text{Cos}(2\pi(m+j)k / N) + \sum_{k \in \Omega_N^0} b_k \text{Sin}(2\pi(m+j)k / N) \right) \\
&= \sum_{m \in \Omega_N} u_m \left(\sum_{k \in \Omega_N^0} a_k [\text{Cos}(2\pi mk / N) \text{Cos}(2\pi jk / N) - \text{Sin}(2\pi mk / N) \text{Sin}(2\pi jk / N)] \right) \\
&\quad + \sum_{m \in \Omega_N} u_m \left(\sum_{k \in \Omega_N^0} b_k [\text{Sin}(2\pi mk / N) \text{Cos}(2\pi jk / N) + \text{Sin}(2\pi jk / N) \text{Cos}(2\pi mk / N)] \right) \quad (\text{B14}) \\
&= \sum_{k \in \Omega_N^0} a_k \left(\sum_{m \in \Omega_N} u_m [\text{Cos}(2\pi mk / N) \text{Cos}(2\pi jk / N) - \text{Sin}(2\pi mk / N) \text{Sin}(2\pi jk / N)] \right) \\
&\quad + \sum_{k \in \Omega_N^0} b_k \left(\sum_{m \in \Omega_N} u_m [\text{Sin}(2\pi mk / N) \text{Cos}(2\pi jk / N) + \text{Sin}(2\pi jk / N) \text{Cos}(2\pi mk / N)] \right) \\
&= \sum_{k \in \Omega_N^0} a_k \text{Cos}(2\pi jk / N) \sum_{m \in \Omega_N} u_m \text{Cos}(2\pi mk / N) \\
&\quad + \sum_{k \in \Omega_N^0} b_k \text{Sin}(2\pi jk / N) \sum_{m \in \Omega_N} u_m \text{Cos}(2\pi mk / N).
\end{aligned}$$

Now insert (3.13) into the last line of (B14).

Part b): Expand the variance expression, then use (A5), switch order of summation, and (A4):

$$\begin{aligned}
N^{-1} \sum_{j \in \Omega_N} (p_j^* - \bar{p}^*)^2 &= N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} (\hat{a}_k \text{Cos}(2\pi jk / N) + \hat{b}_k \text{Sin}(2\pi jk / N)) \right)^2 \\
&= N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \sum_{l \in \Omega_N^0} \hat{a}_k \hat{a}_l \text{Cos}(2\pi jk / N) \text{Cos}(2\pi jl / N) \\
&\quad + 2N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \sum_{l \in \Omega_N^0} \hat{a}_k \hat{b}_l \text{Cos}(2\pi jk / N) \text{Sin}(2\pi jl / N) \quad (\text{B15}) \\
&\quad + N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \sum_{l \in \Omega_N^0} \hat{b}_k \hat{b}_l \text{Sin}(2\pi jk / N) \text{Sin}(2\pi jl / N) \\
&= N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \hat{a}_k^2 \text{Cos}^2(2\pi jk / N) + N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \hat{b}_k^2 \text{Sin}^2(2\pi jk / N) \\
&= N^{-1} \sum_{k \in \Omega_N^0} 0.5 \hat{a}_k^2 N + N^{-1} \sum_{k \in \Omega_N^0} 0.5 \hat{b}_k^2 N = 0.5 \sum_{k \in \Omega_N^0} (\hat{a}_k^2 + \hat{b}_k^2).
\end{aligned}$$

The value of $\text{Var}(\mathcal{G})$ is calculated similarly. Then use the fact that $\lambda_k \in [1/3, 1]$, as established in Lemma 1.

Part c): Apply part a), (A5), and (A4) to write

$$\begin{aligned}
& N^{-1} \sum_{j \in \Omega_N} (p_j^* - \bar{p}^*) (\mathcal{G}_j - \bar{\mathcal{G}}) \\
&= N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \lambda_k (a_k \cos(2\pi jk / N) + b_k \sin(2\pi jk / N)) \right) \\
&\quad \times \left(\sum_{k \in \Omega_N^0} (a_k \cos(2\pi jk / N) + b_k \sin(2\pi jk / N)) \right) \\
&= N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \lambda_k a_k^2 \cos^2(2\pi jk / N) + N^{-1} \sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} \lambda_k b_k^2 \sin^2(2\pi jk / N) \\
&= N^{-1} \sum_{k \in \Omega_N^0} \lambda_k a_k^2 \sum_{j \in \Omega_N} \cos^2(2\pi jk / N) + N^{-1} \sum_{k \in \Omega_N^0} \lambda_k b_k^2 \sum_{j \in \Omega_N} \sin^2(2\pi jk / N) \\
&= N^{-1} \sum_{k \in \Omega_N^0} 0.5 \lambda_k a_k^2 N + N^{-1} \sum_{k \in \Omega_N^0} 0.5 \lambda_k b_k^2 N = 0.5 \sum_{k \in \Omega_N^0} \lambda_k (a_k^2 + b_k^2) \geq 0.
\end{aligned} \tag{B16}$$

Part d): In this case, insert (3.10) into the covariance formula:

$$\begin{aligned}
& N^{-1} \sum_{j \in \Omega_N} (q_j^* - \bar{q}^*) (\mathcal{G}_j - \bar{\mathcal{G}}) = \eta \tau^{-1} N^{-1} \sum_{j \in \Omega_N} (p_j^* - \bar{p}^*) (\mathcal{G}_j - \bar{\mathcal{G}}) = \eta \tau^{-1} \text{Cov}(p^*, \mathcal{G}); \\
& N^{-1} \sum_{j \in \Omega_N} (q_j^* - \bar{q}^*)^2 = \eta^2 \tau^{-2} N^{-1} \sum_{j \in \Omega_N} (p_j^* - \bar{p}^*)^2 = \eta^2 \tau^{-2} \text{Var}(p^*).
\end{aligned} \tag{B17}$$

Part e): In this case, insert (3.10) into the covariance formula and use (3.9):

$$\begin{aligned}
& N^{-1} \sum_{j \in \Omega_N} (q_j^* - \bar{q}^*) (\mathcal{G}_j - \bar{\mathcal{G}}) = \eta \tau^{-1} N^{-1} \sum_{j \in \Omega_N} (p_j^* - c_j - \bar{p}^* + \bar{c}) (c_j - \bar{c}) \\
&= \eta \tau^{-1} N^{-1} \sum_{j \in \Omega_N} \left(\sum_{m \in \Omega_N} u_{[m-j]} c_m - c_j \right) c_j \\
&= \eta \tau^{-1} N^{-1} \sum_{j \in \Omega_N} \left(\sum_{m \in \Omega_N} u_{[m-j]} c_m \right) c_j - \eta \tau^{-1} N^{-1} \sum_{j \in \Omega_N} c_j^2 \leq 0.
\end{aligned} \tag{B18}$$

The final inequality is true because $\sum_{j \in \Omega_N} c_m c_j \leq \sum_{j \in \Omega_N} c_j^2 \forall m \in \Omega_N$, by the Cauchy-Schwartz inequality, so that a positive weighted average of the inequalities will also satisfy the inequality relation, $\sum_{m \in \Omega_N} \sum_{j \in \Omega_N} u_{[m-j]} c_m c_j \leq \sum_{j \in \Omega_N} c_j^2$. *Q.E.D.*

Proof of Corollary 4.1. Part a): $\lambda_k = \lambda_{N-k}$, while elementary computations show that $a_k^2 + a_{N-k}^2 \leq (a'_k)^2 + (a'_{N-k})^2$ for $(a_k, a_{N-k}) \leq_{maj} (a'_k, a'_{N-k})$.

Part b): From (3.13), $\lambda_k > \lambda_{k+1}$ when $\{k : k \in \Omega_N, k < (N-2)/2\}$ so that $\lambda_k^2 (a'_k)^2 + \lambda_{k+1}^2 (a'_{k+1})^2 > \lambda_k^2 a_k^2 + \lambda_{k+1}^2 a_{k+1}^2$. *Q.E.D.*

Proof of Corollary 4.2. From $\sum_{j \in \Omega_N} V^j = \eta \tau^{-1} \sum_{j \in \Omega_N} (p_j^* - c_j)^2 =$

$$\eta\tau^{-1}\sum_{j\in\Omega_N}\left(\bar{p}^*-\bar{c}+\sum_{k\in\Omega_N^0}\left(a_k\lambda_k\cos(2\pi jk/N)+b_k\lambda_k\sin(2\pi jk/N)\right)\right)^2=\eta\tau^{-1}N(\bar{p}^*-\bar{c})^2$$

+0.5N\eta\tau^{-1}\sum_{k\in\Omega_N^0}\left(a_k^2\lambda_k^2+b_k^2\lambda_k^2\right), it is clear that the function is convex in the a and b

coefficients. The problem resolves to one of choosing $\{a_1, a_2, \dots, a_{N-1}\}$ to maximize

$\sum_{k\in\Omega_N^0}a_k^2\lambda_k^2$ over the closed, bounded set $a_k \geq 0 \forall k \in \Omega_N^0$, $\sum_{k\in\Omega_N^0}a_k = A > 0$, and the separate

problem of choosing $\{b_1, b_2, \dots, b_{N-1}\}$ to maximize $\sum_{k\in\Omega_N^0}b_k^2\lambda_k^2$ over the closed, bounded set

$b_k \geq 0 \forall k \in \Omega_N^0$, $\sum_{k\in\Omega_N^0}b_k = B > 0$. With optimum on the boundary, and $\lambda_0^2 = \text{Max}_{k\in\Omega_N^0}\{\lambda_k^2\}$

from Lemma 1, the solutions must be $a_1 = A$ and $b_1 = B$ with other coefficients at zero. *Q.E.D.*

Proof of Corollary 4.3. From Corollary 4.2 we know that parameter specification $\mathcal{G}_j - \bar{\mathcal{G}} =$

$A\cos(2\pi jk/N) + B\sin(2\pi jk/N)$ maximizes profits over the considered coefficient set. From

Wirtinger's inequality (Davis, p. 118), it is also the configuration of coefficients on the choice set

that minimizes the value of $\sum_{j\in\Omega_N}(p_j^* - p_{[j+1]}^*)^2 / \sum_{j\in\Omega_N}(p_j^* - \bar{p}^*)^2 = 2$

$-2\sum_{j\in\Omega_N}(p_j^* - \bar{p}^*)(p_{[j+1]}^* - \bar{p}^*) / \sum_{j\in\Omega_N}(p_j^* - \bar{p}^*)^2$. This is equivalent to maximizing the value of

$\sum_{j\in\Omega_N}(p_j^* - \bar{p}^*)(p_{[j+1]}^* - \bar{p}^*) / \sum_{j\in\Omega_N}(p_j^* - \bar{p}^*)^2$. *Q.E.D.*

Proof of Proposition 5. From proposition assumptions and from part a) of Proposition 4,

$$p_j^* - \bar{p}^* = 0.5\tau \sum_{k\in\Omega_N^0} \left(\hat{a}_k \cos(2\pi jk/N) + \hat{b}_k \sin(2\pi jk/N) \right). \quad (\text{B19})$$

Price variance is $\text{Var}(p^*) = 0.125\tau^2 \sum_{j\in\Omega_N} (\hat{a}_k^2 + \hat{b}_k^2)$ and $d\text{Var}(p^*)/d\tau \geq 0$. From part d) of

Proposition 4, $\text{Var}(q^*) = 0.125\eta^2 \sum_{j\in\Omega_N^0} (\hat{a}_k^2 + \hat{b}_k^2)$ so that $d\text{Var}(q^*)/d\tau = 0$. For $c_j = \bar{c} \forall j \in \Omega_N$,

then $\sum_{j\in\Omega_N} V^j = \eta\tau^{-1} \sum_{j\in\Omega_N} (p_j^* - \bar{c})^2 = 0.25\eta\tau \sum_{j\in\Omega_N} \left(\sum_{m\in\Omega_N} u_{[m-j]} \delta_m \right)^2$, so that

$d\sum_{j\in\Omega_N} V^j / d\tau > 0$. *Q.E.D.*

Proof of Proposition 6. Insert (3.15) into the covariance formula and expand:

$$\begin{aligned}
\text{Cov}(p_j^*, p_{[j+r]}^*) &= N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi(j+r)k / N) \right) \\
&+ N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi(j+r)k / N) \right) \\
&+ N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi(j+r)k / N) \right) \\
&+ N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi(j+r)k / N) \right).
\end{aligned} \tag{B20}$$

Use (A1) to expand $\text{Cos}(2\pi(j+r)k / N)$, interchange order of summation and obtain

$$\begin{aligned}
&N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi jk / N) \sum_{l \in \Omega_N^0} \hat{a}_l \text{Cos}(2\pi(j+r)l / N) \right) \\
&= N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi jk / N) \sum_{l \in \Omega_N^0} \hat{a}_l \text{Cos}(2\pi jl / N) \text{Cos}(2\pi rl / N) \right) \\
&- N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi jk / N) \sum_{l \in \Omega_N^0} \hat{a}_l \text{Sin}(2\pi jl / N) \text{Sin}(2\pi rl / N) \right) \\
&= N^{-1} \sum_{k \in \Omega_N^0} \hat{a}_k^2 \text{Cos}(2\pi rk / N) \sum_{j \in \Omega_N} \text{Cos}^2(2\pi jk / N) = 0.5 \sum_{k \in \Omega_N^0} \hat{a}_k^2 \text{Cos}(2\pi rk / N),
\end{aligned} \tag{B21}$$

where (A5) and (A4) have simplified the expression. A similar analysis of the remaining three expanded terms in (B21) provides

$$\begin{aligned}
&N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi(j+r)k / N) \right) \\
&= 0.5 \sum_{k \in \Omega_N^0} \hat{a}_k \hat{b}_k \text{Sin}(2\pi rk / N); \\
&N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{a}_k \text{Cos}(2\pi(j+r)k / N) \right) \\
&= -0.5 \sum_{k \in \Omega_N^0} \hat{a}_k \hat{b}_k \text{Sin}(2\pi rk / N); \\
&N^{-1} \sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi jk / N) \sum_{k \in \Omega_N^0} \hat{b}_k \text{Sin}(2\pi(j+r)k / N) \right) \\
&= 0.5 \sum_{k \in \Omega_N^0} \hat{b}_k^2 \text{Cos}(2\pi rk / N).
\end{aligned} \tag{B22}$$

Sum the four terms in (B21) and (B22) to obtain $\text{Cov}(p_j^*, p_{[j+r]}^*) =$

$$0.5 \sum_{k \in \Omega_N^0} (\hat{a}_k^2 + \hat{b}_k^2) \text{Cos}(2\pi rk / N). \text{ As for } \sum_{r \in \Omega_N} \text{Cov}(p_j^*, p_{[j+r]}^*) = 0, \text{ sum the covariances,}$$

interchange the order of summation, and apply (A8). *Q.E.D.*

Proof of Proposition 7. Part a): With firm-invariant costs and using the proof of part a) in

Proposition 4, (4.4) may be written as

$$\begin{aligned}
m_j^{eff} - m_j^{olig} &= 0.5\delta_j - 0.5\sum_{m \in \Omega_N} u_{[m-j]} \delta_j. \\
&= 0.5\sum_{k \in \Omega_N^0} a_k \text{Cos}(2\pi jk / N) + 0.5\sum_{k \in \Omega_N^0} b_k \text{Sin}(2\pi jk / N) \\
&\quad - 0.5\sum_{k \in \Omega_N^0} a_k \lambda_k \text{Cos}(2\pi jk / N) - 0.5\sum_{k \in \Omega_N^0} b_k \lambda_k \text{Sin}(2\pi jk / N) \\
&= 0.5\sum_{k \in \Omega_N^0} a_k (1 - \lambda_k) \text{Cos}(2\pi jk / N) + 0.5\sum_{k \in \Omega_N^0} b_k (1 - \lambda_k) \text{Sin}(2\pi jk / N).
\end{aligned} \tag{B23}$$

Bearing in mind that $1 \geq \lambda_k \ \forall k \in \Omega_N^0$ (from Lemma 1), the covariance at issue is then

$$\begin{aligned}
&N^{-1}\sum_{j \in \Omega_N} (m_j^{eff} - m_j^{olig})(\vartheta_j - \bar{\vartheta}) \\
&= 0.5N^{-1}\sum_{j \in \Omega_N} \left(\sum_{k \in \Omega_N^0} (1 - \lambda_k) (a_k \text{Cos}(2\pi jk / N) + b_k \text{Sin}(2\pi jk / N)) \right) \\
&\quad \times \left(\sum_{k \in \Omega_N^0} (a_k \text{Cos}(2\pi jk / N) + b_k \text{Sin}(2\pi jk / N)) \right) \\
&= 0.5N^{-1}\sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} (1 - \lambda_k) a_k^2 \text{Cos}^2(2\pi jk / N) \\
&\quad + 0.5N^{-1}\sum_{j \in \Omega_N} \sum_{k \in \Omega_N^0} (1 - \lambda_k) b_k^2 \text{Sin}^2(2\pi jk / N) \\
&= 0.5N^{-1}\sum_{k \in \Omega_N^0} (1 - \lambda_k) a_k^2 \sum_{j \in \Omega_N} \text{Cos}^2(2\pi jk / N) \\
&\quad + 0.5N^{-1}\sum_{k \in \Omega_N^0} (1 - \lambda_k) b_k^2 \sum_{j \in \Omega_N} \text{Sin}^2(2\pi jk / N) \\
&= 0.25\sum_{k \in \Omega_N^0} (1 - \lambda_k) (a_k^2 + b_k^2) \geq 0.
\end{aligned} \tag{B24}$$

Identities in Appendix A have been used in this calculation.

Part b): With firm-invariant inter-firm distances, the inner product of (4.3) and prices may be written as

$$\begin{aligned}
(m^{eff} - m^{olig})^T (p^*) &= -\tau^{-1} (c - \bar{c}1_N)^T \left((U - I_N)^2 \right)^T (U^{-1})^T U (c - \bar{c}1_N) \\
&= -\tau^{-1} (c - \bar{c}1_N)^T (U - I_N)^2 (c - \bar{c}1_N).
\end{aligned} \tag{B25}$$

Now it was already shown in the proof of Proposition 2 that $(U - I_N)^2$ is positive semi-definite.

Therefore, the last term in (B25) is negative. *Q.E.D.*

Proof of Proposition 8. Part a): Obvious from (5.4).

Part b): Note $u_j / u_0 = 1 / \rho^j = \sigma^j / (\mu + \beta)^j$ where $|1 / \rho| < 1$. Clearly, $(u_j / u_0)|_{\sigma=0} = 0, j \in \Omega_N^0$,

while $u_0|_{\sigma=0} = 1$ from (5.2). The result follows from continuity of the u_j in σ .

Part c): From $u_j = (\rho - 1)\rho^{N-j-1} / (\rho^N - 1)$, if $\sigma \geq 0$ then $\mu + \beta > |\sigma|$ implies $\rho \geq 1$ and

$u_j \geq 0 \forall j \in \Omega_N$. Write $u_0 = (\rho - 1)\rho^{N-1}/(\rho^N - 1) = 1 - (\rho^{N-1} - 1)/(\rho^N - 1)$. If $\sigma < 0$ and N is even, then $(\rho^{N-1} - 1)/(\rho^N - 1) < 0$, implying $u_0 > 1$. If $\sigma < 0$ and N is odd, then

$(\rho^{N-1} - 1)/(\rho^N - 1) < 0$ also, implying $u_0 > 1$. Since $u_{j+1}/u_j = 1/\rho < 0$, it follows that

$$u_j(-1)^{j+1} < 0 \forall j \in \Omega_N.$$

Part d): $|u_n/u_{n+1}| = |\rho| > 1$, while $d(u_{n+1}/u_n)/d\sigma = 1/(\mu + \beta) > 0$.

Part e): Use De Moivre's theorem in (5.5), see proof of Lemma 1 above, to show that

$$q_0^* - \bar{q}^* = -\frac{\phi(\rho - 1)}{\rho(\rho^N - 1)} \sum_{j \in \Omega_N} \rho^{N-j} \text{Cos}(2\pi jk/N) = -\frac{\phi(\rho - 1)(\rho - \text{Cos}(2\pi k/N))}{\rho^2 - 2\rho \text{Cos}(2\pi k/N) + 1}. \quad (\text{B26})$$

Turning to q_n^* , use (A1) to write

$$\begin{aligned} q_n^* - \bar{q}^* &= -\sum_{j \in \Omega_N} \frac{\phi(\rho - 1)\rho^{N-j-1}}{\rho^N - 1} \text{Cos}(2\pi(n+j)k/N) \\ &= -\frac{\phi(\rho - 1)\rho^{N-1}}{\rho^N - 1} \sum_{j \in \Omega_N} \rho^{-j} (\text{Cos}(2\pi nk/N)\text{Cos}(2\pi jk/N) - \text{Sin}(2\pi nk/N)\text{Sin}(2\pi jk/N)) \quad (\text{B27}) \\ &= \text{Cos}(2\pi nk/N)(q_0^* - \bar{q}^*) + \frac{\phi(\rho - 1)\rho^{N-1}\text{Sin}(2\pi nk/N)}{\rho^N - 1} \sum_{j \in \Omega_N} \rho^{-j} \text{Sin}(2\pi jk/N). \end{aligned}$$

Now set $z = \rho^{-1}[\text{Cos}(2\pi k/N) + i\text{Sin}(2\pi k/N)]$, with $\text{Im}(z)$ the imaginary part, so that

$$\begin{aligned} \sum_{j \in \Omega_N} \rho^{-j} \text{Sin}(2\pi jk/N) &= \text{Im}\left(\sum_{j \in \Omega_N} z^j\right) = \text{Im}\left(\frac{z^N - 1}{z - 1}\right) \\ &= \text{Im}\left(\frac{\rho^{-N} - 1}{\rho^{-1}[\text{Cos}(2\pi k/N) + i\text{Sin}(2\pi k/N)] - 1}\right) = \frac{1 - \rho^N}{\rho^N} \left(\frac{-\rho^{-1}\text{Sin}(2\pi k/N)}{\rho^{-2} - 2\rho^{-1}\text{Cos}(2\pi k/N) + 1}\right) \quad (\text{B28}) \\ &= \frac{\rho^N - 1}{\rho^{N-1}} \left(\frac{\text{Sin}(2\pi k/N)}{\rho^2 - 2\rho \text{Cos}(2\pi k/N) + 1}\right). \end{aligned}$$

Insert into (B27) and use (B26):

$$\begin{aligned} q_n^* - \bar{q}^* &= \text{Cos}(2\pi nk/N)(q_0^* - \bar{q}^*) + \frac{\phi(\rho - 1)\text{Sin}(2\pi nk/N)\text{Sin}(2\pi k/N)}{\rho^2 - 2\rho \text{Cos}(2\pi k/N) + 1} \\ &= \text{Cos}(2\pi nk/N)(q_0^* - \bar{q}^*) - \frac{\text{Sin}(2\pi nk/N)\text{Sin}(2\pi k/N)}{\rho - \text{Cos}(2\pi k/N)}(q_0^* - \bar{q}^*), \quad (\text{B29}) \end{aligned}$$

as stipulated in (5.6). For variance, take the mean sum of squares on (5.6), use Appendix A relations and (B26):

$$\begin{aligned}
& N^{-1} \sum_{n \in \Omega_N} (q_n^* - \bar{q}^*)^2 \\
&= (q_0^* - \bar{q}^*)^2 \left(N^{-1} \sum_{n \in \Omega_N} \text{Cos}^2(2\pi nk / N) + N^{-1} \frac{\text{Sin}^2(2\pi k / N)}{[\rho - \text{Cos}(2\pi k / N)]^2} \sum_{n \in \Omega_N} \text{Sin}^2(2\pi nk / N) \right) \\
&= 0.5(q_0^* - \bar{q}^*)^2 \left(\frac{\rho^2 - 2\rho \text{Cos}(2\pi k / N) + \text{Cos}^2(2\pi k / N) + \text{Sin}^2(2\pi k / N)}{[\rho - \text{Cos}(2\pi k / N)]^2} \right) \quad (\text{B30}) \\
&= \frac{0.5\phi^2(\rho - 1)^2}{\rho^2 - 2\rho \text{Cos}(2\pi k / N) + 1}.
\end{aligned}$$

The other parts follow immediately from inspection and differentiation.

Part f): From (5.6), (A5), and (A4);

$$N^{-1} \sum_{j \in \Omega_N} (q_n^* - \bar{q}^*)(c_n - \bar{c}) = (q_0^* - \bar{q}^*) N^{-1} \sum_{j \in \Omega_N} \text{Cos}^2(2\pi nk / N) = 0.5(q_0^* - \bar{q}^*). \quad (\text{B31})$$

The rest follows from comments in part e).

Part g): Square (5.5) and sum over firms. Recognize that the $u_j, j \in \Omega_N$, are positive and sum to one when $\sigma > 0$. Then use Jensen's inequality. *Q.E.D.*

Proof of Proposition 9. Consumer surplus is invariant under $c' \rightarrow c''$, so it is sufficient to confirm the effect on the sum of oligopolists' profits. In (5.8), $0.5\mu + \beta > 0$ while $\Gamma^T \Gamma$ is a symmetric circulant. This means that all eigenvalues are real. In addition, $L^j(c^T) \Gamma^T \Gamma L^j(c) = c^T \Gamma^T \Gamma c = L^j(c_-^T) \Gamma^T \Gamma L^j(c_-) \forall j \in \Omega_N$. We need only show that the eigenvalues are positive in order to apply the analysis that established (B1). But, as is well-known (e.g., p. 431 in Hohn (1973)) a real matrix A is positive definite if and only if there exists a non-singular matrix B such that $A = B^T B$. In our case, Γ is non-singular since it was arrived at from inversion. *Q.E.D.*

Proof of Proposition 10. After some work, using computations analogous to those used in the inversion of U in (3.8)-(3.9), it can be shown that

$$q_n^{mon} - \bar{q}^{mon} = \sum_{m=0}^{N-1} v_{[m-n]} \theta_m; \quad v_j = \frac{\omega^j + \omega^{N-j}}{\Delta \times (\omega^N - 1)}; \quad (B32)$$

$$\Delta = \sqrt{1 + 4\gamma}; \quad \omega = (2\gamma + 1 + \Delta)/(2\gamma).$$

Therefore

$$\begin{aligned} q_n^{mon} - \bar{q}^{mon} &= (\mu - 2\sigma)^{-1} \sum_{m=0}^{N-1} v_{[m-n]} (\bar{c} - c_m) = -(\mu - 2\sigma)^{-1} \sum_{m=0}^{N-1} v_{[m-n]} \text{Cos}(2\pi mk / N) \\ &= -\frac{1}{(\mu - 2\sigma)\Delta \times (\omega^N - 1)} \sum_{j=0}^{N-1} (\omega^j + \omega^{N-j}) \text{Cos}(2\pi(n+j)k / N) \\ &= -\frac{(\omega - 1)(\omega + 1) \text{Cos}(2\pi nk / N)}{(\mu - 2\sigma)\Delta \times [\omega^2 - 2\omega \text{Cos}(2\pi k / N) + 1]}, \end{aligned} \quad (B33)$$

where summation methods in Lemma 1 have been used. From (B33), (5.6), and (A5),

$$\begin{aligned} \text{Cov}(q^*, q^{mon}) &= N^{-1} \sum_{n \in \Omega_N} (q_n^* - \bar{q}^*) (q_n^{mon} - \bar{q}^{mon}) \\ &= -\frac{(\omega^2 - 1)(q_0^* - \bar{q}^*)}{(\mu - 2\sigma)\Delta \times [\omega^2 - 2\omega \text{Cos}(2\pi k / N) + 1]} N^{-1} \sum_{n \in \Omega_N} \text{Cos}^2(2\pi nk / N) \\ &= -\frac{(\omega^2 - 1)(q_0^* - \bar{q}^*)}{2(\mu - 2\sigma)\Delta \times [\omega^2 - 2\omega \text{Cos}(2\pi k / N) + 1]}. \end{aligned} \quad (B34)$$

Notice that $\omega^2 - 1 = [1 + 4\gamma + (1 + 2\gamma)(1 + 4\gamma)^{0.5}] / (2\gamma^2)$. If $\sigma > 0$, then expression $\omega^2 - 1$ is positive. If $\sigma < 0$, then $1 + 4\gamma < 1 + 2\gamma$ so that $\omega^2 - 1$ is positive provided that $1 + 4\gamma > 0$, i.e., provided that $(\mu + 2\sigma) / (\mu - 2\sigma) > 0$. This condition is satisfied if and only if $\mu > |2\sigma|$, as assumed in the proposition. Since $\omega^2 - 2\omega \text{Cos}(2\pi k / N) + 1 = [\omega - \text{Cos}^2(2\pi k / N)]^2 + \text{Sin}^2(2\pi k / N) > 0$, therefore

$$\text{Cov}(q^*, q^{mon}) \stackrel{\text{sign}}{=} -\frac{(q_0^* - \bar{q}^*)}{\omega^2 - 2\omega \text{Cos}(2\pi k / N) + 1} \stackrel{\text{sign}}{=} -(q_0^* - \bar{q}^*). \quad (B35)$$

We know from Proposition 8 that $\sigma > (<) 0$ implies $q_0^* < (>) \bar{q}^*$, and it follows that $\sigma > (<) 0$ implies $\text{Cov}(q^*, q^{mon}) > (<) 0$. *Q.E.D.*

References

- D'ASPREMONT, C., AND JACQUEMIN, A. "Cooperative and Noncooperative R&D in Duopoly with Spillovers." *American Economic Review*, Vol. 78 (December, 1988), pp. 1133–1137.
- DAVIS, P.J. *Circulant Matrices*. New York, NY: Chelsea Publishing, 1994.
- DIXIT, A. "Comparative Statics for Oligopoly." *International Economic Review*, Vol. 27 (February, 1986), pp. 107–122.
- EATON, B.C., AND HOLTZ WOODERS, M. "Sophisticated Entry in a Model of Spatial Competition." *RAND Journal of Economics*, Vol. 16 (Summer, 1985), pp. 282–297.
- FÉVRIER, P., AND LINNEMER, L. "Idiosyncratic Shocks in an Asymmetric Cournot Oligopoly." *International Journal of Industrial Organization*, Vol. 22 (June, 2004), pp. 835–848.
- GELLER, D., KRA, I., POPESCU, S., AND SIMANCA, S. "On Circulant Matrices." Preprint Department of Mathematics, State University of New York at Stonybrook, 2004.
- GIOVAGNOLI, A., AND WYNN, H.P. "Cyclic Majorization and Smoothing Operators." *Linear Algebra and Its Applications*, 239 (June, 1996), pp. 215–225.
- GUPTA, B., LAI, F.-C., PAL, D., SARKAR, J., AND YU, C.-M. "Where to Locate in a Circular City." *International Journal of Industrial Organization*, Vol. 22 (June, 2004), pp. 759–782.
- HENNESSY, D. A., AND LAPAN, H.E. "An Algebraic Theory of Portfolio Allocation." *Economic Theory*, Vol. 22 (August, 2003), pp. 193–210.
- HENNESSY, D. A., AND LAPAN, H.E. "An Algebraic Theory of Multi-Product Decisions." *Economic Theory*, Vol. 25 (June, 2005), pp. 819–829.
- HENNESSY, D.A., AND LAPAN, H.E. "Market Cycles for a Non-Storable Product under Adjustment Costs." Unpublished Manuscript, Department of Economics, Iowa State University, February 2004.
- HOHN, F.E. *Elementary Matrix Algebra*, 3rd edn. New York, NY: Macmillan, 1973.
- KATS, A. "More on Hotelling's Stability in Competition." *International Journal of Industrial Organization*, Vol. 13 (March, 1995), pp. 89–93.

- MARSHALL, A.W., AND OLKIN, I. *Inequalities: Theory of Majorization and Its Applications*. San Diego, CA: Academic Press, 1979.
- NIEZGODA, M. “G-majorization Inequalities for Linear Maps.” *Linear Algebra and its Applications*, Vol. 292 (May, 1999), pp. 207–231.
- PAPAGEORGIOU, Y.Y., AND SMITH, T.R. “Agglomeration as Local Instability of Spatially Uniform Steady-States.” *Econometrica*, Vol. 51 (July, 1983), pp. 1109–1119.
- PETTIFREZZO, A.J. *Matrices and Transformations*. New York, NY: Dover Publications, 1966.
- SALANT, S.W., AND SHAFFER, G. “Unequal Treatment of Identical Agents in Cournot Equilibrium.” *American Economic Review*, Vol. 89 (June, 1999), pp. 585–604.
- SALOP, S.C. “Monopolistic Competition with Outside Goods.” *Bell Journal of Economics*, Vol. 10 (Spring, 1979), pp. 141–156.
- STEINMETZ, S. “Spatial Preemption with Finitely Lived Equipments.” *International Journal of Industrial Organization*, Vol. 16 (March, 1998), pp. 253–270.
- VAN LONG, N., AND SOUBEYRAN, A. “Cost Manipulation Games in Oligopoly with Costs of Manipulating.” *International Economic Review*, Vol. 42 (May, 2001), pp. 505–533.

FIG. 1. Generating a dihedral convex hull.