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On the nature of certainty equivalent functionals

By David A. Hennessy and Harvey E. Lapan^{*}

Abstract

We explore connections between the certainty equivalent return (CER) functional and the underlying utility function. Curvature properties of the functional depend upon how utility function attributes relate to Hyperbolic Absolute Risk Aversion (HARA) type utility functions. If the CER functional is concave, i.e., if risk tolerance is concave in wealth, then preferences are standard. The CER functional is linear in lotteries if utility is HARA and lottery payoffs are on a line in state space. Implications for the optimality of portfolio diversification are given. When utility is concave and Non-increasing Relative Risk Averse, then the CER functional is superadditive in lotteries. Depending upon the nature of covariation among lottery payoffs, CERs for Constant Absolute Risk Averse utility functions may be subadditive or superadditive in lotteries. Our approach lends itself to straightforward experiments to elicit higher order attributes on risk preferences.

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1. Introduction

The implications of utility function structure for preferences over lotteries are at the foundations of the economics of uncertainty. While stochastic dominance methods have identified major univariate results concerning the interactions between preferences and lotteries, much remains unknown about preferences when there are two or more sources of risk. Kimball (1993), Eeckhoudt, Gollier, and Schlesinger (1996), Gollier and Pratt (1996) and others have identified inferences pertaining to bivariate sources of randomness when high-order conditions on risk preferences are satisfied. In the main these papers study sums of, usually independent, random variables. The purpose of this paper is to compare arbitrary lotteries with sums and convex combinations of these lotteries. Our belief is that a more comprehensive understanding of the nature of the CER functional will facilitate deeper insights into choice and welfare problems when there are multiple sources of randomness. Our basic approach is to flesh out the implications of the quasi-arithmetic mean structure of the CER functional.

After briefly identifying the role of CERs in diversification for risk averters in Section 2, Section 3 studies conditions on the utility function for the CER functional to be concave in lotteries. Conditions for linearity are also identified. In order to understand what these conditions deliver, it is necessary to separate the determinants of the value of a CER. To this end, in Section 4 we study the impact of covariance among lotteries on the CER.

A further structural attribute, studied in Section 5, is lottery superadditivity (SA). That is, whether the CER of a sum of lotteries exceeds the sum of the CERs of the constituent lotteries. We show that Non-increasing Relative Risk Aversion (NIRRA) is sufficient for SA. But SA may or may not hold under constant absolute risk aversion (CARA), and the conditions under which it holds are determined by attributes of lottery covariation that are presented in Section 4. Relations between the Non-increasing Absolute Risk Aversion (NIARA) property and SA are also identified. The paper concludes with a discussion of empirical issues.

2. Preliminaries

For von Neumann-Morgenstern (vNM) preferences, the CER functional is a quasi-arithmetic mean. Let the vector of returns over n states be $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ with associated probability weight vector $\mathbf{q} = (q_1, q_2, \dots, q_n) \in \mathbb{R}^n$, $q_i > 0 \forall i \in \Omega_n$, $\Omega_n = \{1, 2, \dots, n\}$, $\sum_{i \in \Omega_n} q_i = 1$.

Our notation will script bold all vectors. With strictly increasing, continuous utility function $U(\cdot)$ and inverse function $U^{-1}(\cdot)$, the CER functional is

$$\mathbb{C}(\mathbf{a}) = U^{-1} \left[\sum_{i \in \Omega_n} q_i U(a_i) \right], \quad (2.1)$$

where we have suppressed $\mathbf{q}$ because the same probability weights are maintained throughout.

Defining the set of all twice continuously differentiable, strictly increasing, strictly concave utility functions by $U \in U_2$, suppose that two lotteries, $\mathbf{a}$ and $\mathbf{b}$, are available to a $U \in U_2$ decision maker. A linear convex combination of $\mathbf{a}$ and $\mathbf{b}$, i.e., $v\mathbf{a} + (1-v)\mathbf{b}$ for $v \in [0, 1]$, is also available with $\mathbb{C}(v\mathbf{a} + (1-v)\mathbf{b}) = U^{-1} \left[\sum_{i \in \Omega_n} q_i U[v a_i + (1-v)b_i] \right]$. Then $U \in U_2$ ensures

$$\begin{aligned} U[\mathbb{C}(v\mathbf{a} + (1-v)\mathbf{b})] &= \sum_{i \in \Omega_n} q_i U[v a_i + (1-v)b_i] \geq \\ &\sum_{i \in \Omega_n} q_i v U(a_i) + \sum_{i \in \Omega_n} q_i (1-v) U(b_i) = v U[\mathbb{C}(\mathbf{a})] + (1-v) U[\mathbb{C}(\mathbf{b})]. \end{aligned} \quad (2.2)$$

That is, the composite function $U[\mathbb{C}(\cdot)]$ is concave in lotteries. Certainty equivalent $\mathbb{C}(\cdot)$ need not be strictly concave in v for this to be true. If $\mathbb{C}(\cdot)$ is concave in v , then

$$U[\mathbb{C}(v\mathbf{a} + (1-v)\mathbf{b})] \geq U[v\mathbb{C}(\mathbf{a}) + (1-v)\mathbb{C}(\mathbf{b})] \geq v \sum_{i \in \Omega_n} q_i U(a_i) + (1-v) \sum_{i \in \Omega_n} q_i U(b_i). \quad (2.3)$$

Here, the convex combination of lotteries is preferable to a two-stage lottery over lotteries for two reasons; because $U \in U_2$ and because the CER functional is concave. Thus it is likely, when considering risk averter choices among lotteries, that stronger conclusions can be drawn when the CER functional is concave.

3. Convex combinations of lotteries

The properties of quasi-arithmetic means have been studied since before the work of Hardy, Littlewood and Pólya (1934). In this section we confine our interests to the curvature of the CER functional. Where the number of primes on a function indicate the order of differentiation, the following is a result proved in Hardy, Littlewood and Pólya:

Proposition A. *Let $V(\cdot)$ be a four times continuously differentiable, strictly increasing and strictly convex function. Then $V^{-1}\left[\sum_{i \in \Omega_n} q_i V(a_i)\right]$ is convex as a function of α if and only if $V'(\cdot)/V''(\cdot)$ is concave.*

Denote the coefficient of absolute risk aversion for utility function by $U(\cdot)$ as $r(\cdot) = -U''(\cdot)/U'(\cdot)$ and say that a function for which the absolute risk tolerance $Tol(\cdot) = 1/r(\cdot)$ is concave is in class $HARA^{cv}$, $r(\cdot) \in HARA^{cv}$. If $U(\cdot)$ is a strictly increasing, strictly concave, four times continuously differentiable function, or $U(\cdot) \in U_4$, then the necessary and sufficient condition for $r(\cdot) \in HARA^{cv}$ is $r''(\cdot) \geq 2[r'(\cdot)]^2 / r(\cdot)$. It is necessary that $r''(\cdot) \geq 0$ for $r(\cdot) \in HARA^{cv}$. With respective definitions for the coefficients of absolute prudence and absolute temperance as $p(\cdot) = -U'''(\cdot)/U''(\cdot)$ and $t(\cdot) = -U''''(\cdot)/U'''(\cdot)$, Kimball (1993) defined a utility function satisfying $t(\cdot) \geq p(\cdot) \geq r(\cdot)$ as standard [see Table 1 of Gollier and Pratt (1996)]. Write

$$\pi(\alpha, \ell; v) = \mathbb{C}(v\alpha + (1-v)\ell) - v\mathbb{C}(\alpha) - (1-v)\mathbb{C}(\ell). \quad (3.1)$$

Define $U(\cdot) \in HARA$ if

$$U(w) = \frac{1-\phi}{\phi} \left[\frac{\beta w}{1-\phi} + \eta \right]^\phi, \quad (3.2)$$

where β , η , and ϕ are constants such that $\phi \neq 1$, $\beta > 0$, $\eta > -\beta w/[1-\phi]$, and $\eta = 1$ when

$\phi \rightarrow -\infty$. Proof of the following is in the appendix

Proposition 1. *Suppose that $U(\cdot) \in U_4$. Then*

- i) $\pi(\mathbf{a}, \mathbf{\ell}; \nu) \geq 0 \forall \mathbf{a} \in \mathbb{R}^n, \forall \mathbf{\ell} \in \mathbb{R}^n, \forall \nu \in [0, 1]$, if and only if $U(\cdot) \in HARA^{cv}$.
- ii) $\pi(\mathbf{a}, \mathbf{\ell}; \nu) = 0 \forall \mathbf{a} \in \mathbb{R}^n, \forall \mathbf{\ell} \in \mathbb{R}^n, \forall \nu \in [0, 1], \forall U(\cdot) \in HARA$ if and only if $\mathbf{\ell} = \mathbf{a}k_0 + k_1$ where $k_0 = [\beta \mathbb{C}(\mathbf{\ell}) + \tau] / [\beta \mathbb{C}(\mathbf{a}) + \tau]$, $k_1 = [\mathbb{C}(\mathbf{\ell}) - \mathbb{C}(\mathbf{a})]\tau / [\beta \mathbb{C}(\mathbf{a}) + \tau]$, and $\tau = (1 - \phi)\eta$.
- iii) if $U(\cdot) \in NIARA \cap HARA^{cv}$, then $U(\cdot)$ is standard.

The strength of part i) is that no structure has been placed on how $\mathbf{a}$ and $\mathbf{\ell}$ relate.^{1,2} On part iii), standardness implies properness and vulnerability and so carries strong comparative statics implications. To illustrate part ii), suppose that $\mathbb{C}(\mathbf{a}) = \mathbb{C}(\mathbf{\ell})$. Then $\pi(\mathbf{a}, \mathbf{\ell}; \nu) = 0$ requires $\mathbf{\ell} = \mathbf{a}$, so that not even scaling is permitted. Suppose instead that $\mathbb{C}(\mathbf{a}) = 0$ and $\mathbb{C}(\mathbf{\ell}) = 1$. Then $b_i - 1 = (\beta + \tau)\tau^{-1}a_i \forall i \in \Omega_n$, where utility function parameters determine the scaling factor. In all cases, the unweighted correlation between $\mathbf{a}$ and $\mathbf{\ell}$ is required to be 1. It should be intuitive that positive covariation between the lotteries will drive $\pi(\mathbf{a}, \mathbf{\ell}; \nu)$ down toward zero. In the next section we will identify covariation as one of the issues that motivates $\pi(\mathbf{a}, \mathbf{\ell}; \nu) \geq 0$ even under $HARA$ preferences.

4. AI order and portfolio diversification

¹ The inference in part i) was made in Hennessey and Lapan (1998). Unbeknownst to us, it was also made in Polak's (1996) unpublished notes and used in Gollier (1998).

² Gollier (2001) finds implications for the condition in understanding equilibrium asset pricing. Gollier and Zeckhauser (2002), and also Gollier (2002), establish the central role of $HARA^{cv}$ preferences in determining how various aspects of financial risk taking changes with age in the vNM model. Hara and Kuzmics (2004) study the aggregation issues that heterogeneous risk tolerances raise when studying risk sharing in representative consumer models of general equilibrium. Ghiglino and Olszak-Duquenne (2005) find implications of uniformly curved risk

Proposition 1 relies, in part on an order involving arrangements of vector pairs. Considering α and β , if the coordinates of vector β are re-permuted so that they more closely correspond to the rank order of vector α , then the re-permuted pair is in a sense an improved arrangement. To formalize, denote ω as some permutation of the natural arrangement $\{1, 2, \dots, n\}$ and let $\omega(i)$ be the i^{th} element of ω . Then, following Boland, Proschan, and Tong (1988), the AI (partial) order (symbolized by $\leq_a$) can be defined as:

Definition 1. (*Arrangement increasing order*) For a given vector, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, let $\mathbf{x}^\uparrow = (x_{(1)}, x_{(2)}, \dots, x_{(n)}) \in \mathbb{R}^n$ be the vector that has the components of $\mathbf{x}$ arranged in increasing order. For any permutation ω of $\{1, 2, \dots, n\}$, let $\mathbf{x}_\omega = (x_{\omega(1)}, x_{\omega(2)}, \dots, x_{\omega(n)})$.³ For vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, and $\mathbf{v}$, assign $\{\mathbf{x}, \mathbf{y}\} =_a \{\mathbf{u}, \mathbf{v}\}$ if there exists a permutation ω such that $\mathbf{x}_\omega = \mathbf{u}$ and $\mathbf{y}_\omega = \mathbf{v}$. Assign $\{\mathbf{x}, \mathbf{y}\} \leq_a \{\mathbf{u}, \mathbf{v}\}$ if there exists a finite number of transposition vectors $\{\mathbf{z}^1, \mathbf{z}^2, \dots, \mathbf{z}^k\}$ such that

(i) $\{\mathbf{x}, \mathbf{y}\} =_a \{\mathbf{x}^\uparrow, \mathbf{z}^1\}$ and $\{\mathbf{x}^\uparrow, \mathbf{z}^k\} =_a \{\mathbf{u}, \mathbf{v}\}$, and

(ii) $\mathbf{z}^i$ can be obtained from $\mathbf{z}^{i-1}$ by an interchange of two components of $\mathbf{z}^{i-1}$, the first of which is larger than the second.

If $\{\mathbf{x}, \mathbf{y}\} \leq_a \{\mathbf{u}, \mathbf{v}\}$ then $\{\mathbf{u}, \mathbf{v}\}$ is said to be larger than $\{\mathbf{x}, \mathbf{y}\}$ in the AI order.

Thus, for the vector pair $\{\mathbf{x}, \mathbf{y}\} = \{(3, 2, 6, 1), (0, 4, 2, 3)\}$ we have $\{\mathbf{x}, \mathbf{y}\} =_a \{\mathbf{x}^\uparrow, \mathbf{z}^1\} = \{(1, 2, 3, 6), (3, 4, 0, 2)\} \leq_a \{(1, 2, 3, 6), (3, 0, 4, 2)\} \leq_a \{(1, 2, 3, 6), (0, 3, 2, 4)\} \leq_a \{(1, 2, 3, 6), (0, 2, 3, 4)\}$.

We will say that the best ordered arrangement of a vector pair is the maximal arrangement. The worst, or minimal, arrangement is where the vectors are in reverse order, i.e., for $\{\mathbf{x}, \mathbf{y}\}$ it is

tolerance for determinacy of equilibrium in general equilibrium models with externalities.

$\{(1,2,3,6),(4,3,2,0)\}$.

Decision makers are held to optimize over whatever their objective function is, and so the order on state space must be converted to an order on functions. From Hollander, Proschan, and Sethuraman (1977):⁴

Definition 2. (*Arrangement increasing function*) Function $g(\mathbf{a}, \mathbf{x})$ is said to be arrangement increasing (AI) if $g(\mathbf{a}, \mathbf{x}) \leq g(\mathbf{c}, \mathbf{u})$ whenever $\{\mathbf{a}, \mathbf{x}\} \leq_a \{\mathbf{c}, \mathbf{u}\}$.

The order is a discrete variant of supermodularity orders studied in Athey (2000) and has also been considered by Epstein and Tanny (1980). A multivariate generalization has been applied to a variety of welfare and optimal choice problems in Hennessy and Lapan (2003). If the AI ordering is to represent a discrete distribution in state space, then the technical issue of accommodating replications must be addressed. To do so we re-present the state space as follows. Suppose that the elements of $\mathcal{Q}$ are all rational. Nothing is then lost by adjusting the discrete probability measure to be uniform if we accommodate unequal weights through repeated observations. For example, let $\mathbf{a} = (1,4,2)$, $\mathbf{b} = (3,5,1)$, and $\mathbf{q} = (1/8, 5/8, 1/4)$. Then the adjustments would substitute triple $\{\mathbf{a}^*, \mathbf{b}^*, \mathbf{q}^*\} \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ for $\{\mathbf{a}, \mathbf{b}, \mathbf{q}\} \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ where $\mathbf{a}^* = (1,4,4,4,4,2,2)$, $\mathbf{b}^* = (3,5,5,5,5,5,1,1)$, $\mathbf{q}^* = (1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8, 1/8)$, $n = 3$, and $N = 8$. From Boland, Proschan, and Tong (1988), the function $-N^{-1} \sum_{i \in \Omega_N} U(p_1 a_i + p_2 b_i)$ is AI when $U(\cdot) \in U_2$ and $p_i \geq 0$, $i \in \{1,2\}$. For $U \in U_2$, a positive rational weighting vector $\mathbf{q}$ and positive coefficients p_i , $i \in \{1,2\}$, CER functional $\mathbb{C}(p_1 \mathbf{a} + p_2 \mathbf{b}) = U^{-1} \left[\sum_{i \in \Omega_n} q_i U(p_1 a_i + p_2 b_i) \right]$ decreases with an increase in the AI order between $\mathbf{a}^*$ and $\mathbf{b}^*$.

³ $\omega(i)$ is the inverse map, i.e., ‘takes to the i^{th} ordinate’ and not ‘originates from the i^{th} ordinate.’

⁴ They describe the concept as ‘decreasing in transposition.’ ‘Arrangement increasing’ is now

Returning to Proposition 1, $\pi(\mathbf{a}, \mathbf{b}; v) \geq 0$ even under $U(\cdot) \in HARA$ and the maximal arrangement between $\mathbf{a}^*$ and $\mathbf{b}^*$. Thus, while part ii) of Proposition 1 requires ordinal alignment between the two random vectors, i.e., across marginals, ordinal alignment is not enough for $\pi(\mathbf{a}, \mathbf{b}; v) = 0$. HARA utility functions are quasi-homogeneous of degree 1, i.e., there exist lines such that the function is linear along them. Otherwise, though, the function is strictly concave. The pair of payoff vectors $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^n \times \mathbb{R}^n$ must be on one of these lines.

Another way of confirming the strength of proposition 1 is to apply it to a straightforward and important problem for which little is actually known. Samuelson (1967) has shown that if assets are independent and identically distributed then diversification is optimal for all $U(\cdot) \in U_2$. Brummelle (1974) with negatively interdependent asset returns, McEntire (1984) with independent returns, and Kijima (1997) with dependent returns identify conditions involving the interaction between utility functions and the distribution such that diversification is optimal. McMinn (1984), and also Hong and Herk (1996), place dependence structures on joint returns. None of these works deal with arbitrary distributions. We have, however,

Proposition 2. *Suppose $U(\cdot) \in HARA^{cv}$ for a portfolio investor that is constrained never to short an asset. If $\mathbf{a}^{*,j} \in \mathbb{R}^n$, $j \in \Omega_m$, are permutations of $\mathbf{a}^{*,1} \in \mathbb{R}^n$, then $\mathbb{C}\left(\sum_{j \in \Omega_m} v_j \mathbf{a}^{*,j}\right) \geq \mathbb{C}(\mathbf{a}^{*,j}) \forall j \in \Omega_m$ for all $(v_1, v_2, \dots, v_m)$ such that $v_j \geq 0 \forall j \in \Omega_m$ and $\sum_{j \in \Omega_m} v_j = 1$.*

The proof is straightforward. Since $\mathbf{a}^{*,j} \in \mathbb{R}^n$, $j \in \Omega_m$, are permutations of $\mathbf{a}^{*,1} \in \mathbb{R}^n$, then $\mathbb{C}(\mathbf{a}^{*,j}) \equiv \mathbb{C}(\mathbf{a}^{*,1}) \forall j \in \Omega_m$ so that $\sum_{j \in \Omega_m} v_j \mathbb{C}(\mathbf{a}^{*,j}) = \mathbb{C}(\mathbf{a}^{*,1}) \forall j \in \Omega_m$. Then apply part i) of Proposition 1. This means that an allocation of all funds to one $\mathbf{a}^{*,j} \in \mathbb{R}^n$, $j \in \Omega_m$, is always

the more common term.

weakly dominated by an allocation that diversifies into other assets. No conditions are imposed on the assets other than that they have the same marginals and that they can be presented as random vectors.

5. Superadditivity

It is also of interest to ascertain the conditions under which a summed lottery is, in some sense, preferable to the individual lotteries. Lottery superadditivity (SA) occurs when

$$\theta(\alpha, \ell) = \mathbb{C}(\alpha + \ell) - \mathbb{C}(\alpha) - \mathbb{C}(\ell) \geq 0. \quad (5.1)$$

Intuitively, SA should bear relation to the constant relative risk aversion (CRRA) property because $\mathbb{C}(2\alpha) = 2\mathbb{C}(\alpha)$ under CRRA. When $\theta(\alpha, \ell) \leq 0$ on the domain of $\alpha \times \ell$, then the functional is said to be subadditive on that domain. A result due to Mulholland (1950, p. 300) is of interest in the study of the SA property.

Proposition B. *For $\alpha \geq 0$ and $\ell \geq 0$, let i) $U(\cdot) \in U_2$ on $[0, \infty)$, with $U(0) = 0$, and ii) $F(t) = -\ln |U'(e^t)|$ be concave on $t \in (-\infty, \infty)$. Then $\theta(\alpha, \ell) \geq 0$.*

Notice that the concavity of $F(t) = -\ln |U'(e^t)|$ on $t \in (-\infty, \infty)$ requires $-e^t U''(e^t)/U'(e^t)$ to be non-increasing in t on $t \in (-\infty, \infty)$. Substituting $w = e^t$, the condition implies that $R(w) = -wU''(w)/U'(w)$ is non-increasing, i.e., non-increasing RRA or NIRRA.⁵ Write $U(\cdot) \in \text{NIRRA}$, $U(\cdot) \in \text{CRRA}$, and $U(\cdot) \in \text{IRRA}$ if the utility function has, respectively non-increasing, constant, or increasing RRA over the relevant domain.

Proposition 3. *Let $\alpha \geq 0$ and $\ell \geq 0$. Then*

⁵ We thank an anonymous reviewer of our 1998 draft for demonstrating the consequence of IRRA in this proposition. Part ii) is an application of the Minkowski inequality, see p. 55 in Mitrinović (1970).

- i) $\theta(\mathbf{a}, \mathbf{b}) \geq 0$ if $U(\cdot) \in \text{NIRRA}$ on $w \in (0, \infty)$. Alternatively, if $U(\cdot) \in \text{IRRA}$ on $w \in (0, \infty)$ then there exists an $\mathbf{a} \geq 0$ and a $\mathbf{b} \geq 0$ such that $\theta(\mathbf{a}, \mathbf{b}) < 0$.
- ii) for $U(\cdot) \in \text{CRRA}$ and $p_1, p_2 \in \mathbb{R}_+$, then $\mathbb{C}(p_1\mathbf{a} + p_2\mathbf{b}) = p_1\mathbb{C}(\mathbf{a}) + p_2\mathbb{C}(\mathbf{b})$ if and only if there exists a scalar λ such that $\mathbf{a} = \lambda\mathbf{b}$.

Here again, no structure has been placed on how $\mathbf{a}$ and $\mathbf{b}$ inter-relate. Notice too that AI rearrangements in the vector returns depress $\mathbb{C}(\mathbf{a} + \mathbf{b})$ without affecting either $\mathbb{C}(\mathbf{a})$ or $\mathbb{C}(\mathbf{b})$.⁶ NIRRA precludes CARA-type preferences because if $wr(w)$ is non-increasing then CARA implies $wr'(w) + r(w) = r(w) < 0$, which is impossible given $U(\cdot) \in U_2$. Therefore, it is natural to inquire into the SA status of the CER functional for CARA and NIARA preferences. Pratt (1964) showed that for the constant lottery $\mathbf{t} = t\mathbf{1}_n$, $\mathbf{1}_n = (1, 1, \dots, 1)$, and random lottery $\mathbf{a}$, then CARA risk preferences imply $\mathbb{C}(\mathbf{a} + \mathbf{t}) = \mathbb{C}(\mathbf{a}) + t$. He also showed that $\mathbb{C}(\mathbf{a} + \mathbf{t}) \geq \mathbb{C}(\mathbf{a}) + t$ for $t \geq 0$ under NIARA, i.e., SA prevails if one lottery is a sure thing. The following definition will allow us to identify conditions on lottery interdependence under NIARA such that $\theta(\mathbf{a}, \mathbf{b}) \geq 0$.⁷

Definition 3. (*Statistical association*) Random vector $X = (X_1, X_2, \dots, X_m) \in \mathbb{R}^m$ is associated if $E[g(X)]E[h(X)] \leq E[g(X)h(X)]$ for all non-decreasing functions $g(X): \mathbb{R}^m \rightarrow \mathbb{R}$ and $h(X): \mathbb{R}^m \rightarrow \mathbb{R}$ such that the expectation exists, where $E[\cdot]$ is the expectation with respect to the distribution of X .

The notion of affiliated random variables, where the density function is log-supermodular, is

⁶ NIRRA is of interest when seeking to understand risk sharing mechanisms because, according to Wilson's (1968) framework, it determines how consumption shares vary with aggregate consumption.

⁷ Association and related conditions are presented in detail in Shaked and Shanthikumar (1994).

a concept of positive dependence that has proven useful in auction theory (Milgrom and Weber, 1982). All vectors of affiliated random variables are associated. Association is also a stronger concept of dependence than positive covariance.⁸

Proposition 4. i) *Assuming NIARA, denote by $\alpha^*, \ell^* \in \mathbb{R}^N$ the expanded equal-weight vectors of $\alpha, \ell \in \mathbb{R}^n$. Let $c_{(i)}^*$ be the i^{th} least element of arbitrary vector c^* , and denote the mean of ℓ by $\mu_b = \sum_{i \in \Omega_n} q_i b_i = N^{-1} \sum_{i \in \Omega_N} b_i^*$ where it is assumed that $\mu_b \geq 0$. Then a sufficient condition for SA is $\sum_{i=1}^k (a^* + b^*)_{(i)} \geq \sum_{i=1}^k (a_{(i)}^* + \mu_b) \forall k \in \Omega_N$. Where $(b^* | a_{(i)}^*)$ is the element in ℓ^* associated with the i^{th} least element in α^* , then a simpler sufficient condition is that*

$$k^{-1} \sum_{i=1}^k (b^* | a_{(i)}^*) \geq \mu_b \forall k \in \Omega_N.$$

ii) *Assume CARA preferences. If α and ℓ are associated then $\theta(\alpha, \ell) \leq 0$, while if α and $-\ell$ are associated then $\theta(\alpha, \ell) \geq 0$.*

It was not necessary to require that $\alpha \geq 0$ or $\ell \geq 0$ in Proposition 4. Also, both conditions in part i) impose a stochastic structure akin to negative covariation between α and ℓ .

Condition $k^{-1} \sum_{i=1}^k (b^* | a_{(i)}^*) \geq \mu_b \forall k \in \Omega_N$ is most clear. The mean value of the ℓ^* when conditioned on the lower sets $a_i \in \{a_{(1)}, a_{(2)}, \dots, a_{(k)}\}$ are, for all $k \in \Omega_N$, bounded from below by μ_b . Of course, $N^{-1} \sum_{i=1}^N (b^* | a_{(i)}^*) = \mu_b$. The additional structure on preference in part ii) allows for a more explicit representation on stochastic interactions.

6. Discussion

⁸ Association does not restrict the forms to be $g(X_1, X_2) = g_0 + g_1 X_1$, $g_1 \geq 0$, $h(X_1, X_2) = h_0 + h_1 X_2$, $h_1 \geq 0$.

The results established herein suggest many avenues for empirical investigations of the nature of risk preferences. Guiso and Paiella (2003) have studied a Bank of Italy survey that included a question in which subjects placed a certain value on a lottery. Specifying a utility function, their econometric analysis found evidence in favor of $HARA^{cv}$. Specifying a utility function, Ogaki and Zhang (2001) and Zhang and Ogaki (2004) used ICRISAT data on Pakistani and Indian household resources and decisions to find evidence in favor of DRRA. While working with real-world data, this approach is prone to function mis-specification problems. Furthermore, the information on choices over risk situations is very indirect. To the extent that experimental evidence, through hypothetical or real lotteries, provide insights, our approach allows hypotheses concerning vNM utility functions to be tested directly. For example, $NIARA$ together with $HARA$ or $HARA^{cv}$ are sufficient to establish standardness, properness, and vulnerability. Subjects might be asked to bid (simultaneously) in second-price auctions for each of lotteries a , $0.25a + 0.75\ell$, $0.5a + 0.5\ell$, $0.75a + 0.25\ell$, and ℓ . Then one of these lotteries could be randomly assigned to the high bidder at the second highest bid. Each subject has then provided five points upon which to test for functional linearity/concavity. Similar experimental auctions for lotteries a , $a + \ell$, $2a + \ell$, etc., could reveal evidence on NIRRA, and could be the basis of a test for NIARA.

Appendix

Proof of Proposition 1. i) This is immediate from Proposition A when a map is introduced to change an increasing, convex function into an increasing, concave function. Let $U(w) = -V(-w)$ so that $[-dV(-w)/dw > 0] \Leftrightarrow [dU(w)/dw > 0]$ and $[-d^2V(-w)/dw^2 < 0] \Leftrightarrow [d^2U(w)/dw^2 < 0]$. Thus, $V(x)$ is increasing and convex.

ii) Define $z_i = [\beta a_i + \tau]/[1 - \phi]$, $v_i = [\beta b_i + \tau]/[1 - \phi]$, $w_i = p_1 a_i + p_2 b_i$, and $\sigma_i = [\beta w_i + \tau]/[1 - \phi]$ with $p_1 > 0$ and $p_2 > 0$. Therefore, $\sigma_i = p_1 z_i + p_2 v_i + \delta$ where $\delta = (1 - p_1 - p_2)\eta = (1 - p_1 - p_2)\tau/(1 - \phi)$. Define $a_0 = \mathbb{C}(\boldsymbol{a})$ and $z_0 = [\beta a_0 + \tau]/[1 - \phi]$ where, of course, a_0 solves

$$\left[\frac{\beta a_0}{1 - \phi} + \eta \right]^\phi = \sum_{i \in \Omega_n} q_i \left[\frac{\beta a_i}{1 - \phi} + \eta \right]^\phi. \quad (\text{A1})$$

Similarly, define $b_0 = \mathbb{C}(\boldsymbol{b})$ and $v_0 = [\beta b_0 + \tau]/[1 - \phi]$. And define $w_0(p_1, p_2) = \mathbb{C}(p_1 \boldsymbol{a} + p_2 \boldsymbol{b})$.

From substitutions into expressions $\pi(\boldsymbol{a}, \boldsymbol{b}, p_1, p_2) = w_0(p_1, p_2) - p_1 a_0 - p_2 b_0$, we have

$$\pi(\boldsymbol{a}, \boldsymbol{b}, p_1, p_2) = \frac{1 - \phi}{\beta} (\sigma_0 - p_1 z_0 - p_2 v_0 - \delta). \quad (\text{A2})$$

Now, for $\{\boldsymbol{q} \in \mathbb{R}^n : 0 < q_i, \sum_{i \in \Omega_n} q_i = 1\}$, define $V(v_0) \subset \mathbb{R}^n$ as $\{V(v_0) = \boldsymbol{a} : \sum_{i \in \Omega_n} q_i v_i^\phi = v_0^\phi\}$ for some fixed v_0 . To investigate the relation between lotteries that minimize $\pi(\boldsymbol{a}, \boldsymbol{b}, p_1, p_2)$ subject to requiring that $a_0 = \mathbb{C}(\boldsymbol{a})$ and $b_0 = \mathbb{C}(\boldsymbol{b})$ be held constant, we hold all the z_i fixed and alter the v_i subject to $\boldsymbol{a} \in V(v_0)$. That is, we hold the $\boldsymbol{a}$ fixed and alter the $\boldsymbol{b}$ subject to $b_0 = \mathbb{C}(\boldsymbol{b})$. Adhering to these constraints, the minimization of $\pi(\boldsymbol{a}, \boldsymbol{b}, p_1, p_2)$ over the choices of lottery $\boldsymbol{b}$ is equivalent to minimizing $w_0(p_1, p_2)$. But, because $U(\cdot) \in U_2$, the problem is equivalent to

$$\text{Min}_{v \in V(v_0)} \frac{1-\phi}{\phi} \sum_{i \in \Omega_n} q_i (p_1 z_i + p_2 v_i + \delta)^\phi. \quad (\text{A3})$$

And this task may be re-written as

$$\text{Min}_{v \in V(v_0)} \mathcal{Q}(\mathbf{a}) = \text{Min}_{v \in V(v_0)} \frac{1-\phi}{\phi} \left[q_n (p_1 z_n + p_2 v_n + \delta)^\phi + \sum_{i=1}^{n-1} q_i (p_1 z_i + p_2 v_i + \delta)^\phi \right], \quad (\text{A4})$$

subject to $q_n v_n^{\phi-1} dv_n = -\sum_{i=1}^{n-1} q_i v_i^{\phi-1} dv_i$. The $n-1$ first-order conditions are given by

$$\begin{aligned} \frac{d\mathcal{Q}(\mathbf{a})}{dv_i} &= (1-\phi) p_2 \left[q_i \sigma_i^{\phi-1} + q_n \sigma_n^{\phi-1} \left(\frac{-q_i v_i^{\phi-1}}{q_n v_n^{\phi-1}} \right) \right] \\ &= (1-\phi) p_2 q_i \left[\sigma_i^{\phi-1} - \left(\frac{\sigma_n v_i}{v_n} \right)^{\phi-1} \right] = 0, \quad i \in \Omega_{n-1}. \end{aligned} \quad (\text{A5})$$

A demonstration that the second-order conditions (SOC) are satisfied whenever $p_1 z_i + \delta > 0 \forall i \in \Omega_n$, can be obtained from the authors. For HARA to have meaning, it must be that $z_i > 0 \forall i \in \Omega_n$, so it is sufficient that $\delta > 0$. Thus, $\eta = 0$ or $p_1 + p_2 = 1$ suffices for the SOC to hold. In Proposition 1 we have $p_1 + p_2 = 1$.

Due to (A5), $\sigma_i = \sigma_n v_i / v_n$. Substitution yields $v_i = (p_1 z_i + \delta) v_n / (p_1 z_n + \delta)$. Noting that $(\sigma_0 / v_0)^\phi = \sum_{i \in \Omega_n} q_i \sigma_i^\phi / \sum_{i \in \Omega_n} q_i v_i^\phi$, use of $\sigma_i = \sigma_n v_i / v_n$ reveals that $\sigma_0 / v_0 = \sigma_n / v_n = \sigma_i / v_i$. Substituting in, we obtain $(\beta b_i + \tau) / (\beta b_0 + \tau) = (\beta a_i + \tau) / (\beta a_0 + \tau)$. Upon rearrangement,

$$b_i = a_i \left(\frac{\beta \mathbb{C}(\boldsymbol{\ell}) + \tau}{\beta \mathbb{C}(\mathbf{a}) + \tau} \right) + \tau \left(\frac{\mathbb{C}(\boldsymbol{\ell}) - \mathbb{C}(\mathbf{a})}{\beta \mathbb{C}(\mathbf{a}) + \tau} \right), \quad i \in \Omega_n. \quad (\text{A6})$$

It remains to show that the minimum point requires $\pi(\mathbf{a}, \boldsymbol{\ell}, p_1, p_2) = 0$. By substitution,

$$v_0^\phi = \sum_{i \in \Omega_n} q_i v_i^\phi = \left(\frac{v_n}{p_1 z_n + \delta} \right)^\phi \sum_{i \in \Omega_n} q_i (p_1 z_i + \delta)^\phi, \quad (\text{A7})$$

and consequently

$$v_i = \frac{(p_1 z_i + \delta) v_0}{\left[\sum_{i \in \Omega_n} q_i (p_1 z_i + \delta)^\phi \right]^{1/\phi}} = \frac{(p_1 z_i + \delta) v_0}{p_1 z_0 + \delta} \quad \forall i \in \Omega_n. \quad (\text{A8})$$

Because $\sigma_i = p_1 z_i + \delta + p_2 v_i$, substitution yields,

$$\sigma_0^\phi = \sum_{i \in \Omega_n} q_i \sigma_i^\phi = \left(1 + \frac{p_2 v_0}{p_1 z_0 + \delta} \right)^\phi \sum_{i \in \Omega_n} q_i (p_1 z_i + \delta)^\phi = (p_1 z_0 + p_2 v_0 + \delta)^\phi, \quad (\text{A9})$$

And so (A2) implies $\pi(\mathbf{a}, \mathbf{\ell}, p_1, p_2) = 0$.

iii) If $U(\cdot) \in HARA^{cv}$, then differentiation of $Tol(w)$ establishes that $t(w) + r(w) \geq 2p(w)$.

From NIARA, $p(w) \geq r(w)$. Consequently, $U(\cdot) \in HARA^{cv} \cap NIARA$ implies that $t(w) \geq p(w) \geq r(w)$; i.e., the condition for standardness is satisfied. $\square$

Proof of Proposition 3. i) The ‘if’ part is immediate from Proposition B. On existence, it suffices to demonstrate subhomogeneity, i.e., $\mathbb{C}(k\mathbf{a}) < k\mathbb{C}(\mathbf{a})$ for $U(\cdot) \in IRRA$ whenever $\mathbf{a} \geq 0$ and $k > 1$. Define $\hat{U}(y) = \hat{U}(\text{Ln}(e^y)) = U(e^y)$ so that $\hat{U}(\text{Ln}(w)) = U(w)$. The following calculations

$$\begin{aligned} \frac{dU(w)}{dw} &= \frac{d\hat{U}(\text{Ln}(w))/d\text{Ln}(w)}{w}; \\ \frac{d^2U(w)}{dw^2} &= \frac{d^2\hat{U}(\text{Ln}(w))/d(\text{Ln}(w))^2 - d\hat{U}(\text{Ln}(w))/d\text{Ln}(w)}{w^2}; \\ -\frac{wd^2U(w)/dw^2}{dU(w)/dw} &= -\frac{d^2\hat{U}(\text{Ln}(w))/d(\text{Ln}(w))^2 - d\hat{U}(\text{Ln}(w))/d\text{Ln}(w)}{d\hat{U}(\text{Ln}(w))/d\text{Ln}(w)} \\ &= 1 - \frac{d^2\hat{U}(\text{Ln}(w))/d(\text{Ln}(w))^2}{d\hat{U}(\text{Ln}(w))/d\text{Ln}(w)}; \end{aligned} \quad (\text{A10})$$

demonstrate that $U(w) \in IRRA$ is equivalent to increasing ARA of $\hat{U}(\text{Ln}(w))$, in $\text{Ln}(w)$.

Define the certainty equivalent under function $\hat{U}(\cdot)$ as $\hat{\mathbb{C}}(\cdot)$. From $\text{Ln}(k) > 0$ and $U(w) \in IRRA$, i.e., $\hat{U}(\text{Ln}(w))$ IARA in $\text{Ln}(w)$, it follows that $\hat{\mathbb{C}}(\text{Ln}(k) + \text{Ln}(\mathbf{a})) < \text{Ln}(k) + \hat{\mathbb{C}}(\text{Ln}(\mathbf{a}))$ where

$$\hat{\mathbb{C}}(\text{Ln}(\mathbf{a})) \equiv \text{Ln}(\mathbb{C}(\mathbf{a})) \text{ due to } \sum_{i \in \Omega_n} q_i \hat{U}(\text{Ln}(a_i)) \equiv \sum_{i \in \Omega_n} q_i U(a_i) \equiv \hat{U}(\hat{\mathbb{C}}(\text{Ln}(\mathbf{a}))) \equiv U(\mathbb{C}(\mathbf{a})).$$

Here, we have used the fact $U(\mathbb{C}(\mathbf{a})) \equiv \hat{U}(\hat{\mathbb{C}}(\text{Ln}(\mathbf{a}))) \equiv U \circ \exp[\text{Ln}(\mathbb{C}(\mathbf{a}))]$, i.e.,

$\hat{\mathbb{C}}(\text{Ln}(\mathbf{a})) \equiv \hat{U}^{-1} \circ U(\mathbb{C}(\mathbf{a})) \equiv \text{Ln} \circ U^{-1} \circ U(\mathbb{C}(\mathbf{a})) \equiv \text{Ln}(\mathbb{C}(\mathbf{a}))$. Therefore, with $U(w) \in \text{IRRA}$, we have $\hat{\mathbb{C}}(\text{Ln}(k) + \text{Ln}(\mathbf{a})) < \text{Ln}(k) + \text{Ln}(\mathbb{C}(\mathbf{a}))$, $\hat{\mathbb{C}}(\text{Ln}(k\mathbf{a})) < \text{Ln}(k\mathbb{C}(\mathbf{a}))$, $\text{Ln}(\mathbb{C}(k\mathbf{a})) < \text{Ln}(k\mathbb{C}(\mathbf{a}))$ and $\mathbb{C}(k\mathbf{a}) < k\mathbb{C}(\mathbf{a})$. In particular, $\mathbb{C}(2\mathbf{a}) < 2\mathbb{C}(\mathbf{a})$.

ii) When $U(\cdot) \in \text{CRRA}$, then $\eta = 0$ and so $\delta = 0$ where δ was defined in the proof of Proposition 1. Consequently the SOC for a minimization are satisfied regardless of the value of $p_1 + p_2$, $p_i > 0$, $i \in \{1, 2\}$. The proof then follows as in part ii) of Proposition 1. $\square$

Proof of Proposition 4. i) Under NIARA and $\mu_b \geq 0$ we know that $\mathbb{C}(\mathbf{a} + \mu_b) \geq \mathbb{C}(\mathbf{a}) + \mu_b \geq \mathbb{C}(\mathbf{a}) + \mathbb{C}(\boldsymbol{\ell})$. Because $U(\cdot) \in U_2$, we know that $U^e = \sum_{i \in \Omega_N} U(a_i^* + b_i^*)$ is Schur-concave and so decreases under a majorization of $\mathbf{a}^* + \boldsymbol{\ell}^*$.⁹ Therefore, any increasing function of U^e , e.g., $U^{-1}(U^e)$, is Schur-concave. Thus, $\mathbb{C}(\mathbf{a} + \boldsymbol{\ell})$ falls under a majorization of $\mathbf{a}^* + \boldsymbol{\ell}^*$. Condition $\sum_{i=1}^k (a_i^* + b_i^*)_{(i)} \geq \sum_{i=1}^k (a_{(i)}^* + \mu_b) \forall k \in \Omega_N$ is necessary and sufficient for $\mathbf{a}^* + \mu_b$ to majorize $\mathbf{a}^* + \boldsymbol{\ell}^*$. As the shift from $\mathbf{a}^* + \mu_b$ to $\mathbf{a}^* + \boldsymbol{\ell}^*$ leads to $\mathbb{C}(\mathbf{a} + \boldsymbol{\ell}) \geq \mathbb{C}(\mathbf{a} + \mu_b)$, it follows that $\mathbb{C}(\mathbf{a} + \boldsymbol{\ell}) \geq \mathbb{C}(\mathbf{a}) + \mathbb{C}(\boldsymbol{\ell})$.

Notice that inequality $\sum_{i=1}^k (a_i^* + b_i^*)_{(i)} \geq \sum_{i=1}^k (a_{(i)}^* + \mu_b) \forall k \in \Omega_N$ may involve a re-ordering of $\mathbf{a}^*$ on the left-hand side. That is, the coordinate of $\mathbf{a}^*$ associated with the i^{th} smallest value of $\mathbf{a}^* + \boldsymbol{\ell}^*$ may not be the i^{th} smallest value of $\mathbf{a}^*$. Suppose instead that we do not allow a re-ordering, and impose the stricter requirement $\sum_{i=1}^k (a_{(i)}^* + (b^* | a_{(i)}^*)) \geq \sum_{i=1}^k (a_{(i)}^* + \mu_b) \forall k \in \Omega_N$.

⁹ Vector $\mathbf{v}' \in \mathbb{R}^m$ is said to be weakly majorized by vector $\mathbf{v}'' \in \mathbb{R}^m$ ($\mathbf{v}' \prec \mathbf{v}''$) if $\sum_{i=1}^k v'_{(i)} \geq \sum_{i=1}^k v''_{(i)} \forall k \in \Omega_m$ and $\sum_{i=1}^m v'_{(i)} = \sum_{i=1}^m v''_{(i)}$ where $v_{(1)} \leq v_{(2)} \leq \dots \leq v_{(m)}$. A Schur-concave function reverses this ordering; if $f(\mathbf{v}) : \mathbb{R}^1 \rightarrow \mathbb{R}$ is Schur-concave then $\mathbf{v}' \prec \mathbf{v}'' \Rightarrow f(\mathbf{v}') \geq f(\mathbf{v}'')$.

Then, by cancellation, we only require $k^{-1} \sum_{i=1}^k (b^* | a_{(i)}^*) \geq \mu_b \ \forall k \in \Omega_N$. Relaxing the constraint by allowing re-ordering will only give more ways to ensure majorization, so $k^{-1} \sum_{i=1}^k (b^* | a_{(i)}^*) \geq \mu_b \ \forall k \in \Omega_N$ is sufficient.

ii) Write $U(w) = -e^{-\lambda w}$, $\lambda > 0$ the coefficient of absolute risk aversion. This implies

$$\mathbb{C}(\boldsymbol{a} + \boldsymbol{\ell}) = -\lambda^{-1} \text{Ln} \left[\sum_{i \in \Omega_n} q_i e^{-\lambda(a_i + b_i)} \right] = -\lambda^{-1} \text{Ln} \left[\sum_{i \in \Omega_n} q_i e^{-\lambda a_i} e^{-\lambda b_i} \right]. \quad (\text{A11})$$

Now $\sum_{i \in \Omega_n} q_i e^{-\lambda a_i} e^{-\lambda b_i} = \sum_{i \in \Omega_n} q_i (-e^{-\lambda a_i}) (-e^{-\lambda b_i}) \geq \sum_{i \in \Omega_n} q_i (-e^{-\lambda a_i}) \sum_{i \in \Omega_n} q_i (-e^{-\lambda b_i})$ if the

vectors are associated, in which case the monotonicity of the log function implies

$$\begin{aligned} \mathbb{C}(\boldsymbol{a} + \boldsymbol{\ell}) &= -\lambda^{-1} \text{Ln} \left[\sum_{i \in \Omega_n} q_i e^{-\lambda a_i} e^{-\lambda b_i} \right] \leq -\lambda^{-1} \text{Ln} \left[\sum_{i \in \Omega_n} q_i e^{-\lambda a_i} \sum_{i \in \Omega_n} q_i e^{-\lambda b_i} \right] \\ &= -\lambda^{-1} \text{Ln} \left[\sum_{i \in \Omega_n} q_i e^{-\lambda a_i} \right] - \lambda^{-1} \text{Ln} \left[\sum_{i \in \Omega_n} q_i e^{-\lambda b_i} \right] = \mathbb{C}(\boldsymbol{a}) + \mathbb{C}(\boldsymbol{\ell}). \end{aligned} \quad (\text{A12})$$

The inequality reverses in the other case. □

$f(v'')$. See Marshall and Olkin [1979, p. 10 and p. 59].

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Demonstration that the Second-Order Conditions for Problem (A4) are Satisfied

It will be shown that convexity holds when evaluated at points where the first-order conditions are met. Then there can only be one global minimum for a twice continuously differentiable function because the function turns up whenever the derivatives equal zero. Differentiating (A5), subject to $\sigma_i = p_1 z_i + p_2 v_i + \delta$, $\sigma_i v_n = \sigma_n v_i$, and the conservation of v_0 constraint, yields

$$\begin{aligned} \frac{d^2 \mathcal{L}(\boldsymbol{u})}{dv_i dv_j} &= (1-\phi)^2 q_i p_2 v_i^{\phi-1} \left[\left(\frac{v_i}{\sigma_i} \right)^{1-\phi} \left(\frac{1}{v_i} - \frac{p_2}{\sigma_i} \right) \delta_{ij} - \left(\frac{v_n}{\sigma_n} \right)^{1-\phi} \left(\frac{1}{v_n} - \frac{p_2}{\sigma_n} \right) \left(\frac{-q_j}{q_n} \right) \left(\frac{v_j}{v_n} \right)^{\phi-1} \right] \\ &= (1-\phi)^2 q_i p_2 \left(\frac{v_n}{v_i \sigma_n} \right)^{1-\phi} \left[\left(\frac{\sigma_i - p_2 v_i}{\sigma_i v_i} \right) \delta_{ij} + \left(\frac{q_j}{q_n} \right) \left(\frac{\sigma_n - p_2 v_n}{\sigma_n v_n} \right) \left(\frac{v_j}{v_n} \right)^{\phi-1} \right], \end{aligned} \quad (*)$$

$\forall i \times j \in \Omega_{n-1} \times \Omega_{n-1}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$. Now set $m_i = \sigma_i - p_2 v_i \equiv p_1 z_i + \delta$, $i \in \Omega_n$. Note, m_i is independent of $\boldsymbol{u}$. Re-write expression (*) as

$$\begin{aligned} \frac{d^2 \mathcal{L}(\boldsymbol{u})}{dv_i dv_j} &= (1-\phi)^2 q_i p_2 \left(\frac{v_n}{\sigma_n} \right)^{1-\phi} \left[\left(\frac{v_i^{\phi-2} m_i}{\sigma_i} \right) \delta_{ij} + \left(\frac{q_j}{q_n} \right) \left(\frac{m_n}{\sigma_n v_n} \right) \left(\frac{v_i v_j}{v_n} \right)^{\phi-1} \right] \\ &= \frac{(1-\phi)^2 p_2}{q_n v_n \sigma_n} \left(\frac{v_n^2}{\sigma_n} \right)^{1-\phi} \left[\frac{q_i q_n v_i^{\phi-2} v_n^\phi m_i \sigma_n}{\sigma_i} \delta_{ij} + q_i q_j m_n v_i^{\phi-1} v_j^{\phi-1} \right], \end{aligned} \quad (**)$$

$\forall i \times j \in \Omega_{n-1} \times \Omega_{n-1}$. Clearly, the eigenvalue solutions to the determinant polynomial

$$|d^2 \mathcal{L}(\boldsymbol{u}) / dv_i dv_j - \lambda I_{n-1}| = 0, \quad I_{n-1} \text{ the } (n-1) \times (n-1) \text{ identity, are } \lambda_i = q_i q_n v_i^{\phi-2} v_n^\phi m_i \sigma_n / \sigma_i \quad \forall i \in \Omega_{n-1}.$$

Given $v_i > 0$ and $\sigma_i > 0$, the eigenvalues are all positive whenever $m_i > 0 \quad \forall i \in \Omega_n$.

Positivity of m_i is assured if $\eta \geq 0$, $p_1 + p_2 \leq 1$, and $z_i > 0 \quad \forall i \in \Omega_n$. Strictly positive eigenvalues assure that the $(n-1) \times (n-1)$ matrix $(d^2 \mathcal{L}(\boldsymbol{u}) / dv_i dv_j)$ is positive definite.