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Agent-based dynamics in the general equilibrium model

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We introduce agent-based dynamics in a class of Arrow-Debreu economies with capital accumulation and technological progress. In this framework we confirm the results obtained by Gintis (2006, 2007) in an exchange economy: the micro-behavior of boundedly rational agents can lead to the emergence of equilibrium at the macro-level provided some form of collective optimization takes place via evolutionary mechanisms. Sensitivity analysis hint at further properties of out-of-equilibrium dynamics which can be analyzed using agent-based models.

1. Introduction

This paper starts from a reading of the Sonnenschein-Mantel-Debreu (SMD) theorem (see e.g. Sonnenschein, 1973) as a result characterizing the complexity of the general equilibrium framework. The aggregate excess demand has the property to fit any homogeneous and continuous function satisfying Walras law, while individual excess demand have to satisfy much more stringent conditions such as the weak axiom of revealed preference. Multiplicity of equilibria and the richness of out-of-equilibrium dynamics can be seen as emergent properties at the macro-level.

Much of the subsequent work in general equilibrium theory has tried to cast aside this complexity by establishing general conditions under which the law of demand holds (see e.g. Hildenbrand, 1994) or as far as computable general equilibrium (CGE) models are concerned, by discarding the micro-level through the use of representative agents (see e.g. Kirman, 1992; Mitra-Kahn, 2008). This line of research has yielded useful results for policy analysis through local comparative statics exercises but has left general equilibrium theory mute as far as regime changes or equilibrium transitions, are concerned. The main pitfall in this respect is the lack of models for out-of-equilibrium dynamics.

In this paper, we investigate which insights on the stability of equilibrium and transitions between multiple equilibria, can be gained by equipping the general equilibrium model with out-of-equilibrium agent-based dynamics. We also wish to revisit the duality between the micro and the macro level in general equilibrium theory and in particular

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to show that bounded rationality at the micro-level is not necessarily inconsistent with the emergence of equilibrium at the macro-level. This has been showed for exchange economies in two recent contributions of Gintis (2006, 2007) where strong properties of convergence to equilibrium are obtained in a framework where agents use private prices as conventions in the sense of Peyton-Young (1993). The validity range of these conclusions, obtained in a framework without capital accumulation, were then questioned in Bilancini & Petri (2008).

This questioning is the main driver of the work presented here. We introduce agent-based dynamics for a relatively large class of Arrow-Debreu economies, in particular allowing for capital accumulation and endogenous technological progress. In this framework, we first demonstrate that the micro-behavior of boundedly rational agents can lead to the emergence of equilibrium at the macro-level provided some form of collective optimization takes place through evolutionary/genetic mechanisms. We then investigate which properties of out-of-equilibrium dynamics are crucial for such convergence and stability. This investigation puts forward the role of time-scales: the relative speed of adjustment of prices and quantities, the speed at which expectations evolve, the implicit temporal dimension embedded in agents decision rules. Consistency between a model's notional time and empirical time also appears as a key criteria for validation and the computational treatment of time in agent-based models (usually handled via a central scheduler) as a major area for future research. A second set of issues to be analyzed concern the use of agent-based models for the exploration of the dynamics of economies with multiple equilibria. In our setting, multiple equilibria materializes as a result of Monte-Carlo simulations and we observe endogenous equilibrium transitions. Our results are very preliminary but suggest that agent-based models could also be useful tools for the study of regime changes in economic systems. We conclude by pointing out a few technological and theoretical challenges which ought to be faced by further research in that direction.

The remaining of this paper is organized as follows. In the second section, we present the class of Arrow-Debreu economies under investigation. In the third section, we give a summarized description of the Lagom model (see also Mandel et al., 2009, 2010; Wolf et al., 2012) which is used to equip these economies with agent-based dynamics. The fourth section investigates the emergence of equilibrium in this framework. The fifth and sixth section discuss respectively the role of the time-scales and the occurrence of regime changes in simulations. Section seven concludes by identifying a few challenges which have to be faced by agent-based models if they are to constitute one of the building-blocks of a theory of out-of-equilibrium dynamics.

2. A framework for model intercomparison

We consider an economy with discrete periods of time, indexed by $t \in \mathbb{N}$, and an infinite horizon. There are a finite number N_C of commodities and one kind of labor.

The usage of commodities is not, a priori, specialized: each can potentially be used as fixed capital, as intermediary consumption or for final consumption.

A finite number of firms, indexed by $j = 1, \dots, N_F$, are partitioned into sectors: a firm j produces a single kind of commodity $\gamma_j \in \{1, \dots, N_C\}$ which defines the sector it belongs to. Each firm in sector $\gamma \in \{1, \dots, N_C\}$ has the same technological possibilities described by a production function $f_\gamma : \mathbb{R}_+^{N_c} \times \mathbb{R}_+^{N_c} \times \mathbb{R}_+ \to \mathbb{R}_+$ which gives the quantity of commodity γ, $f_\gamma(i, k, l)$, which can be produced from a vector of intermediary input $i \in \mathbb{R}_+^{N_c}$, fixed capital $k \in \mathbb{R}_+^{N_c}$, and labor $l \in \mathbb{R}_+$. Intermediary inputs and labor are entirely consumed during the production process while fixed capital made of commodity γ depreciates at rate $\delta_\gamma \in [0, 1]$ (so that $\delta \in [0, 1]^{N_c}$ denotes the vector of depreciation rates). As they specify the production possibilities at the sectoral level, we shall consider in the following that production functions have constant returns to scale.

A finite number of households, indexed by $h = 1, \dots, N_H$ consume commodities and provide one unit of labor each period. Their preferences over commodities are represented by a utility function $u : \mathbb{R}_+^{N_c} \to \mathbb{R}_+$. There is no disutility of labor.

Finally, we consider there is labor-augmenting technological progress at the sectoral level. It is represented in an abstract fashion as a function of the history of the economy. Namely, if one denotes by i_j^t, k_j^t, l_j^t the intermediary inputs, capital stock and labor used as inputs by firm j in period t, by s_j^t its stock of output and by c_h^t the consumption of household h, the history of the economy up to time T can then be summed up by $h_T = ((i_j^t, k_j^t, l_j^t, s_j^t)_{j=1,\dots,N_F}, (c_h^t)_{h=1,\dots,N_H})_{t=1,\dots,T}$. In all generality, we shall consider that labor productivity in sector γ in period T is determined as a function of history $\lambda_\gamma^T(h_T)$. This representation embeds exogenous technological progress, e.g. by setting $\lambda_\gamma^T(h_T) = (1 + n_\gamma)^T$ where n_γ is a parameter independent of the history, as well as endogenous one, considering for example that technological progress is embodied in capital goods and hence that investment has an external effect on labor productivity. This abstract representation would certainly not be fit for an in-depth analysis of the sources of economic growth but it seems to us sufficient to analyze the dynamics it induces.

The functions f_γ, δ_γ, λ_γ and u define an economy $\mathcal{E}((f_\gamma), (\delta_\gamma), (\lambda_\gamma), u)$. Given initial values for input and output stocks $(i_j^0, k_j^0, s_j^0)_{j=1,\dots,N_F}$, one can define the feasible paths in this economy, $\mathcal{F}((i_j^0, k_j^0, s_j^0)_{j=1,\dots,N_F})$, as the set of intertemporal consumption, $(c_h^t)_{h=1,\dots,N_H, t\in\mathbb{N}}$, intermediary inputs, capital stocks, labor inputs and output stocks $(i_j^t, k_j^t, l_j^t, s_j^t)_{j=1,\dots N_F, t\in\mathbb{N}}$ which obey the technological constraints, that is satisfy the following inequations[1]:

[1] Note that we slightly abuse notations here, as we write $f_{\gamma_j}^t(i_j^t, k_t^j, \lambda_\gamma^t(h_t) l_j^t)$ instead of the vector whose γ_jth coordinate is $f_{\gamma_j}^t(i_j^t, k_t^j, \lambda_\gamma^t(h_t) l_j^t)$ (and the others zero). Also $\otimes$ stands for the multiplication coordinatewise, so that $(1 - \delta) \otimes k_j^t$ represents the vector of depreciated fixed capital. finally x^+ stands for the vector of positive parts of $x \in \mathbb{R}_+^{N_c}$, underlining here the fact that fixed capital can not be recycled as a commodity once it has been invested.

$$\sum_{h=1}^{N_H} c_h^{t+1} + \sum_{j=1}^{N_F} i_j^{t+1} + \sum_{j=1}^{N} (k_j^{t+1} - (1-\delta) \otimes k_j^t)^+ + \sum_{j=1}^{N_F} s_j^{t+1}$$
$$\leq \sum_{j=1}^{N_F} f_{\gamma_j}^t (i_j^t, k_j t, \lambda_\gamma^t (h_t) l_j^t) + \sum_{j=1}^{N_F} s_j^t, \quad (1)$$
$$\sum_{j=1}^{N_F} l_j^t \leq N_h. \quad (2)$$

We have hence defined a subset of the Arrow-Debreu framework which accounts for capital accumulation and leave room for different assumptions on technological progress, in particular ones involving externalities. It also represents the production structure in a way which is consistent with the availability of data in the form of input-output tables. Most of the existing computable general equilibrium models used for policy assessment can hence be accommodated. It finally embeds the workhorses of optimal growth theory when one considers there is a single commodity and that production does not require intermediary inputs: if labor productivity grows at an exogenously given rate, the framework is this of the neoclassical growth model; if labor productivity grows proportionally to net investment, the framework can easily be shown to be equivalent to this of the AK-model (see Mandel et al., 2009 for details).

Now, the use of the Arrow-Debreu formalism put little constraints on the kind of dynamics which can emerge from agents behavior. A full-fledged general equilibrium approach would entail the introduction of an intertemporal discount rate for each consumer and the assumption that capital is held by consumers and rented out to firms. With a complete set of markets, economic dynamics would then be identified, with the set of equilibrium trajectories. That is consumption and production choices for which there exist commodity prices, capital rental prices and wages such that each household maximizes discounted utility under its budget constraint; each firm makes zero profits (given that there are constant returns to scale) and all markets clear (that is equation 1 hold with equality and the output stocks are identically zero). In this framework, the emergence of complex macro-economic dynamics from micro-behavior is made perfectly clear by the Sonnenschein-Mantel-Debreu (SMD) theorem: although the solution of each agent maximization problem is unique, there may be multiple equilibria, although the influence of a price change at the micro-level is consistent with the law of demand, the dynamics of prices at the macro-level can be arbitrarily complex. In absence of efficient computational methods to tame this complexity (see e.g. Mitra-Kahn, 2008; Velupillai, 2006), the SMD theorem ought to be seen as an "anything goes" result. The insights it was providing about the uncertainty associated to economic policy were discarded in favor of an engineered sense of determinacy with the development of computable general equilibrium (CGE) models: abandoning the micro-level by considering all optimizing choices are made at the aggregate level by a representative household and representative industries, CGE models enforce the uniqueness of equilibrium and provide a framework fit only for local comparative static exercises.

Agent-based dynamics can be another way forward, possibly more faithful to the complexity insights provided by general equilibrium theory, if one recognizes that bounded rationality is not necessarily inconsistent with the emergence of equilibrium (as suggested in particular by recent results in Gintis (2006, 2007)) and more generally that phenomena at the micro and macro levels are dependent rather than similar.

3. Generic agent-based dynamics

Using the Lagom family of models (extensively described in Mandel et al. (2009, 2010); Wolf et al. (2012)) the Arrow-Debreu framework introduced above can be equipped with agent-based dynamics. Namely, we can define agent-based dynamics which associate to every tuple $((f_\gamma), (\delta_\gamma), (\lambda_\gamma), u)$ a feasible path in the sense of equations 1 and 2. We provide here a summarized description of these dynamics (see references above for a full-description) in order to set the scene for further discussions on the relationships between agent-based dynamics and general equilibrium.

Formally, we consider a random dynamical system on a set X, defined as the cartesian product of the state spaces X_F of N_F firms, X_H of N_H households, X_G of a government, X_{FS} of a financial system and of an environment E. The state space of firms and households contain on the one hand stocks of commodities, money and financial assets and on the other hand strategic variables such as prices, wages, technologies. Among the stocks of firms, there are vectors of intermediary inputs and fixed capital, workforce and stock of output while these of households contain commodities consumed. This ensures one can identify within the agents state space the variables which characterize them in the economy $\mathcal{E}((f_\gamma), (\delta_\gamma), (\lambda_\gamma), u)$. The stocks variables are evolved during a core economic cycle consisting of trading, production/consumption, accounting, labor productivity updating and expectations updating. The outcome of agents actions and interactions during this core economic cycle is governed by the strategic variables which themselves evolve according to evolutionary mechanisms leading to some form of collective optimization. More precisely, the timeline of events is as follows:

1. Every period starts with a *Preparatory step* during which firms profits, costs and sales are set to zero.
2. The *Trading* process takes place every period. As in Gintis (2006) and Gintis (2007), it is organized around a random queuing mechanism and trade proceeds in a bilateral fashion, on the basis of private prices. However, in a framework where the main input to production is accumulated capital, rather than “self-reproducing” natural resources, accidental failures of the market process may have drastic consequences on the dynamics. In order to avoid the occurrence of major rationing shocks due to stochastic mismatches between supply and demand, we implement

some form of permanence of commercial relationships between sellers (firms) and buyers (firms and households) through a trade network which specifies which buyer(s) can trade with which seller(s). Competition is then implemented through the evolution of the trade network, buyers replacing progressively their most expensive suppliers.

Agents trade in a random order: buyers first observe the stocks and prices of the sellers to which they are connected and then determine their demand: a firm demands the intermediary inputs necessary to produce its target production given its current production technique (see below) and if its production capacity (determined by its stock of fixed capital) is less than the sales it forecasts for the next period it additionally demands fixed capital so as to increase its production capacity (according to a fixed investment rate). Each household demands commodities so as to spend available monetary holdings proportionally to its consumption technology (given by a vector in $\mathbb{R}_+^{N_C}$). On the supply side, price is considered fixed at this stage (see price updating below) and supply is rationed by firms at the level corresponding to production at full capacity.

It is worth-pointing out that commodities are conserved during the trading process but money isn't. Indeed, a firm demand is not constrained by its monetary holdings. It may run a deficit in the course of the exchange process. If this deficit is not compensated by its sales, the firm will subscribe a debt towards the financial system during the accounting step (see below). On the other hand, the demand of households is bounded by their monetary holdings: households have no access to credit.

3. The *Labor market* takes place periodically. Employment relations are summed up by work contracts which specify a share of a benchmark wage (sector specific and indexed on the labor productivity, see below) to be paid by a firm to an household in exchange of a certain share of its workforce. The interactions on the labor market are determined by firms wage indexes which specify the share of the benchmark wage a firm proposes and by households' fallbacks which specify the share of the benchmark wage they accept (the wage index is evolved during the genetic step, see below). A work offer from a firm with reservation wage-index w_f is acceptable by an household with fallback w_h if $w_h \geq w_f$. At any point in time, the labor demand of a firm is given by the difference between its target employment (corresponding to a level sufficient to produce its target production and to maintain a minimum ratio between production capacity and workforce) and its current workforce (given by its current work-contracts). The labor capacity of an household is normalized to one, its current employment level is given by the sum of its work-contracts, and its available labor supply by the difference between the labor capacity and the employment level. Labor demand and labor supply are then matched during a bilateral search mechanism.
4. *Production and consumption* take place every period. At a given point in time, a firm only uses a single production technology $(\iota, \kappa, \lambda) \in \mathbb{R}_+^{N_C} \times \mathbb{R}_+^{N_C} \times \mathbb{R}_+$ which

specifies the vector of intermediary input, fixed capital and labor required to produce one unit of output. Applying this technology to its stocks of intermediary inputs, fixed capital and workforce, the firm produces up to its target production. In this process, fixed capital is depreciated and intermediary inputs consumed. Similarly, households consume their stock of goods according to their consumption technology. Consistency with equation 1 is ensured as fixed capital is depreciated at rates δ_γ while the technology (ι, κ, λ) of a firm in sector γ is by construction (see genetic evolution of technologies below) such that $f_\gamma(\iota, \kappa, \lambda) = 1$.

5. *Accounting* takes place every period. Firms pay wages to households and interests on their debt to the financial system. If the balance of their current account is negative at this point, they increase their debt towards the financial system, if it is positive they distribute a dividend to the household owning the firm. Apart from wages and dividends paid by firms, households receive interests on their savings from the financial system. The government taxes part of households income in order to pay an unemployment insurance while keeping its budget balanced.
6. *Expectations updating* takes place every period. Using exponential smoothing, firms determine expected sales. The target production is then set equal to expected sales. Using exponential smoothing also, households update their income expectations. On this basis, they allocate their wealth between monetary holdings and savings using Deaton thumb rule (see Deaton, 1992).
7. *Labor productivity updating* takes place every period. The labor productivity in sector γ is updated according to the function λ_γ. The benchmark wage is indexed on labor productivity.
8. *Price updating* takes place periodically. Firms fix their price as a mark-up over production costs (the mark-up itself evolves endogenously, see below).
9. *Interest rate updating* takes place periodically. The financial system updates the interest rate according to a Taylor rule.
10. *Firms entry, exit and bankrupcy* take place periodically on the basis of a sector specific profitability threshold expressed as a risk-premium above the interest rate. Firms are deleted in sectors whose average profit rate is below the threshold. Firms are created/activated (up to the maximal number of firms N_F) in sectors whose average profit is above the threshold. Independently of their sector, firms whose profit rate is below the interest rate are liquidated and start anew with optimal characteristics.
11. *Genetic evolution of firms technologies* takes place periodically. With some probability, each firm observes the technology of a sample of its peers and adopts the one with the lowest unit production cost. Also, each firm mutates its technology with some probability, that is it randomly draws a new technology on the unit level line of its sector's production function.
12. *Genetic evolution of households technologies* takes place periodically. With some probability, each household observes the (consumption) technology of a sample of

its peers and adopts the most efficient one according to its utility. Also, each household mutates its technology with some probability, that is it randomly draws a new technology on the unit level line of the utility function.

13. *Genetic evolution of firms mark-ups* takes place periodically. With some probability, each firm observes the mark-ups of a sample of its peers and adopts the one of the most efficient firm, where efficiency is measured as a convex combination of profit and sales growth rates. Also, each firm mutates its mark-up with some probability.
14. *Genetic evolution of firms reservation wage-indexes* takes place periodically. With some probability, each firm observes the reservation wage-indexes of a sample of its peers and adopts the one of the most efficient firm on the labor market, where efficiency is measured in terms of share of vacancies filled. Also, each firm mutates its reservation wage-index with some probability.

4. Equilibrium as an emerging phenomenon

The main properties of these dynamics already appear in a simple setting with a single commodity, c.e.s production functions and labor productivity growing at the same rate as fixed capital. In the aggregate, this framework is equivalent to this of the AK model where equilibrium trajectories are characterized by exponential growth of output. Results of simulations are consistent with this "AK-like" picture in the long-run but exhibit much variability at short time-scales.

4.1. Short-term dynamics

Non-linearity, bounded rationality and stochasticity of agents' behavior at the micro-level give rise to short-term fluctuations akin to business cycles (see figure 1). The main driver of these cycles is the investment behavior of firms (see figure 2) which is governed by a non-linear rule, firms increasing their capacity in fixed proportion (e.g. of 20%) whenever they approach full capacity usage.

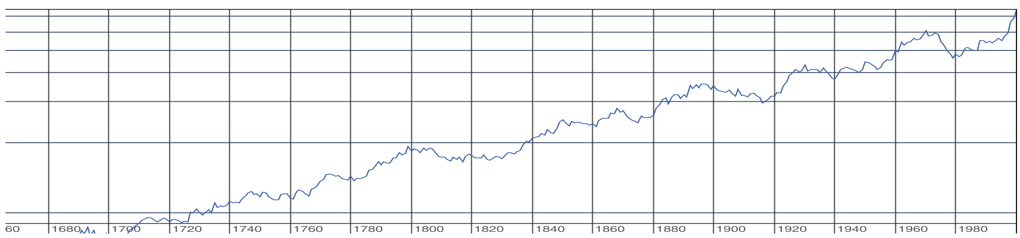

Figure 1. Output (rate of expectations change 0.2), log. scale from 10^7 to 7.10^7.

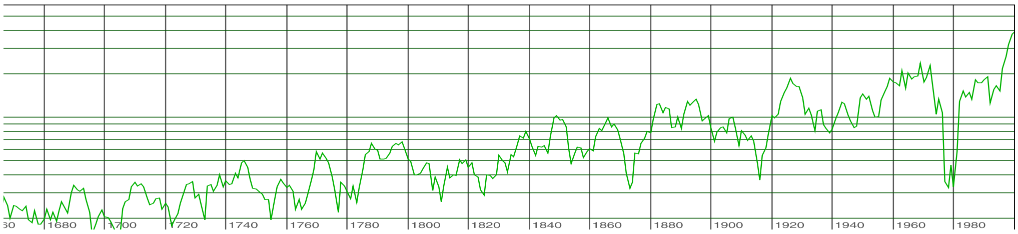

Figure 2. Investment, moving average over 5 periods, log. scale from 2.10^6 to 6.10^7.

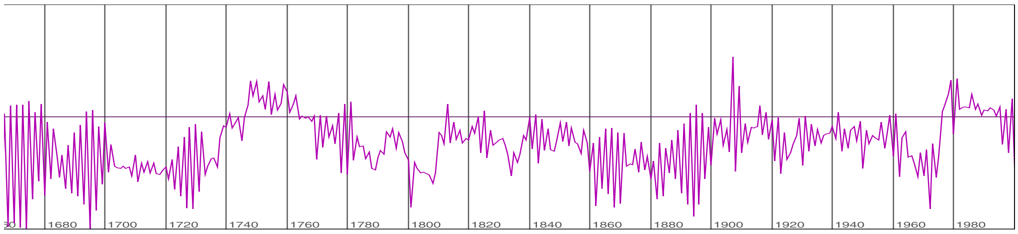

Figure 3. Unemployment, scale from 0 to 0.2.

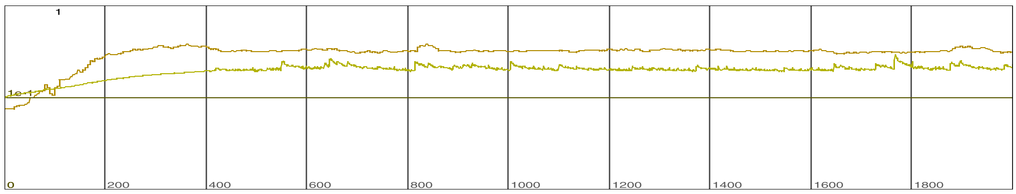

Figure 4. 10*Mark-Up (orange) and Price (yellow).

It is also the case that in absence of a central mechanism, the coordination of boundedly rational agents can not be frictionless. Demand, supply, prices exhibit some volatility at short-time scales as illustrated in the case of unemployment in figure 3.

4.2. *Long-term dynamics*

In the long-run, the effects of the collective optimization of strategies through genetic processes as well as these of competition among firms through entry and exit become dominant. The mark-up tends towards its equilibrium value (the normal rate of profit given by the risk-premium), the price accordingly stabilizes itself (see figure 4) and rationing vanishes (see figure 5). A similar pattern is observed on the labor market (see figure 6). The interactions between the fallback behavior of households and the genetic search for efficient wages lead to a point where there is no involuntary unemployment and wages are stabilized.

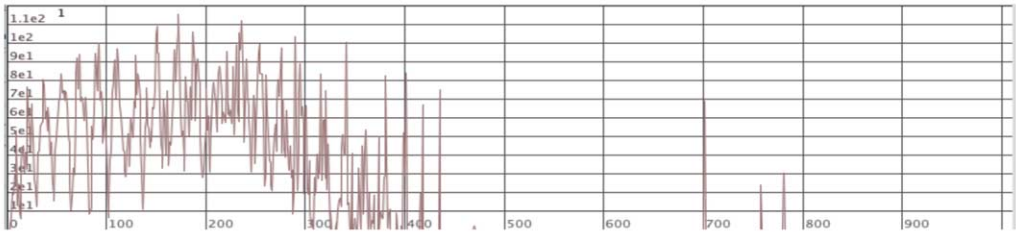

Figure 5. Unfulfilled demand.

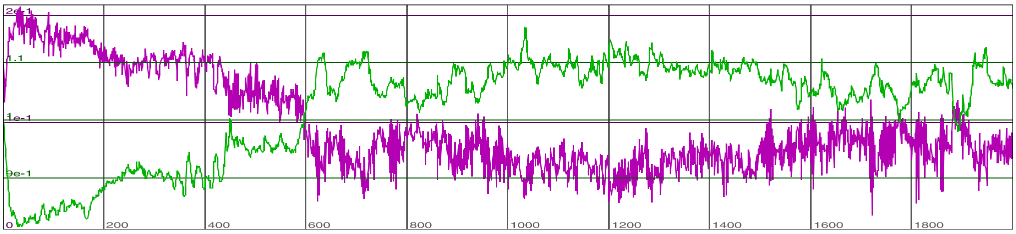

Figure 6. Unemployment (magenta) and average wage reference (green).

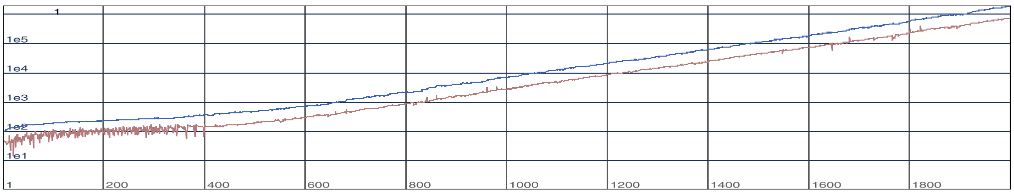

Figure 7. Output (blue) and Consumption (pink), log. scale.

Once, equilibrium has been established on the labor and commoditie markets, the economy enters an exponential growth path (see figure 7) characteristic of growth models with endogenous technological progress.

4.3. Robustness

Those results were systematically reproduced in a number of Monte-Carlo simulations. They are also conserved when the number of commodities is increased and some asymmetry is introduced in the production structure. In this respect, figures 8 and 9 characterize the results of simulations with four sectors: the two first sectors producing commodities used as capital goods only, the two other commodities used for final consumption. One again observes the characteristic movement of prices towards an equilibrium value and the establishment of an exponential growth path (there is here greater volatility in the capital good sector than in the consumption good ones, again in relation with the non-linearities in the firm investment behavior).

Figure 8. 2*Mark-up (orange) and Price (yellow).

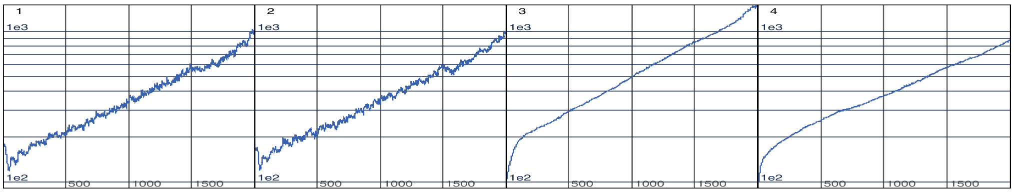

Figure 9. Output (blue), log. scale.

The main insight gained from this initial series of experiments concern the relationships agent-based dynamics can entertain with general equilibrium theory. We indeed provide an answer to the issues raised in Bilancini & Petri (2008) about the extension to a framework with capital accumulation of Gintis' results (Gintis, 2006, 2007) on the attainability and stability of equilibrium by boundedly rational agents improving their behavior via imitation and random innovations. If equilibrium ought to be understood as market-clearance and balancedness of the capital accumulation and growth processes obtained while searching for optimality (which takes place here at the collective level), it is the case that the agent-based dynamics presented here provide some micro-foundations for the use of such solution concepts at the aggregate level.

This answer is nevertheless partial and many more questions can be raised through an analysis of the sensitivity of agent-based dynamics to changes in the relative speeds of adjustment of prices and quantities, to the institutional characteristics of markets or to the introduction of economic policy measures. The remaining of this paper provides a discussion of a sample of these issues focusing on the role of time-scales, the drivers of regime changes, the influence of institutional and computational design.

5. The role of time-scales

5.1. The determinant of growth rates

In an optimal growth setting, the discount rate of the representative agent is the main determinant of the growth rate. In our framework, where none of the agents use exponential discounting, the determinants of the growth rate are of a much more keynesian

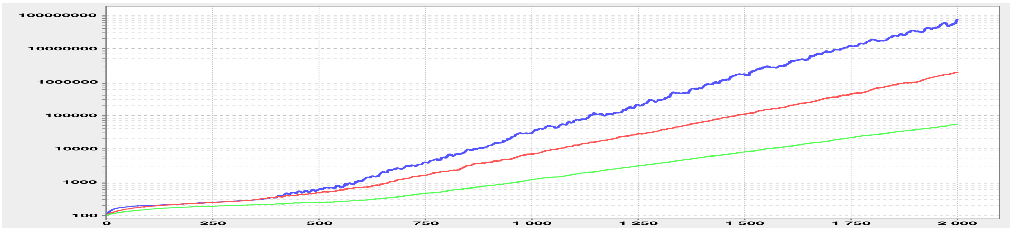

Figure 10. Output for expectations rate of change of 0.05 (green), 0.1 (red), 0.2 (blue), log. scale.

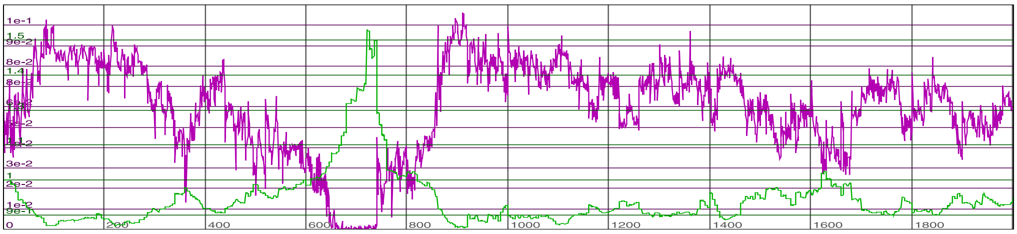

Figure 11. Unemployment (magenta) and average wage reference (green).

nature. Output and investment are determined by the final demand expected by firms, consumption and savings by the revenues anticipated by households. "Animal spirits" are hence embedded in firms and households forecasts. As illustrated by figure 10, the rate of change of these forecasts are key drivers of the growth rate.

The rate of change of variables at the macro-level are hence determined by the temporal dimension of agents' behavior at the micro-level. However, only a narrow range of the possible micro-behavior have yet been explored (here as well as in related literature, e.g. in Deissenberg, van der Hoog & Dawid, 2008; Dosi, Fagiolo & Roventini, 2010). A better understanding of the validity and the limits of the utility discounting approach at the macro-level could certainly be gained through an analysis of a wider range of behavior at the micro-level, in particular through the introduction of forward-looking agents with longer time-horizons.

5.2. *The speed of price adjustment and convergence to equilibrium*

The second major occurrence of time places itself at the structural level in the definition of the relative frequency of the genetic evolution processes and of the core economic cycle. Equilibrium convergence results are conditioned by a rapid evolution of prices (genetic evolution steps shall take place every period). As the speed of prices evolution slows down, it starts interfering with the adjustment of supply and demand. For example, decreasing the speed of adjustment of wages, the labor market might fail to equilibrate (see figure 11).

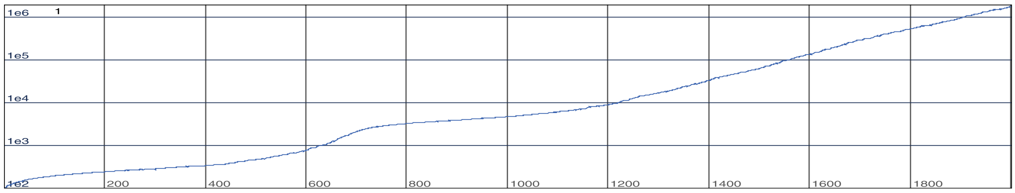

Figure 12. Output, log. scale.

The ensuing rationing shocks might then have lasting consequences on output (see figure 12).

This underlines a major difference with Gintis' models induced by the introduction of capital accumulation and growth. In Gintis (2006) and Gintis (2007), the set of resource is fixed, consequently the evolution of strategies take place in a fixed fitness landscape. The variability of resources change the nature of this part of the dynamics to this of an evolutionary process in a fitness landscape which is changing. The relative speed of evolution of the strategies and the landscape then acquire a crucial influence on the stability properties of the dynamics.

5.3. Temporal consistency and the role of data

The correspondence between the model's notional time and "empirical" time is mainly informed by the production function which establishes a relationship between the capital stock and the production flow. The ensuing ratio can be used to establish endogenous measures of speed for the model, e.g. by considering the time required to reproduce the entire capital stock. The fact that agents micro behavior might not lead to consistent macro-dynamics for every such model speed underlines the implicit notion of time embedded in agents decision rules.

For the speed of price adjustment as well as for these decision rules, temporal consistency shall eventually be judged on the basis of empirical data (the ratio between capital stock and production flow can here also provide a conversion scale). On top of this consistency check, temporal equivalence might provide an additional dimension where to measure aggregation: the smallest the time-scale at which agents behavior is defined, the more disaggregated the model is.

6. Multiple equilibria and regime changes

With a large number of household, the multiplicity of equilibria is the rule in general equilibrium theory. In a framework with two perfectly symmetric sectors (with c.e.s production functions and labor productivity growing at the same rate as fixed capital) and households characterized by linear preferences, the dynamics clearly signal the pres-

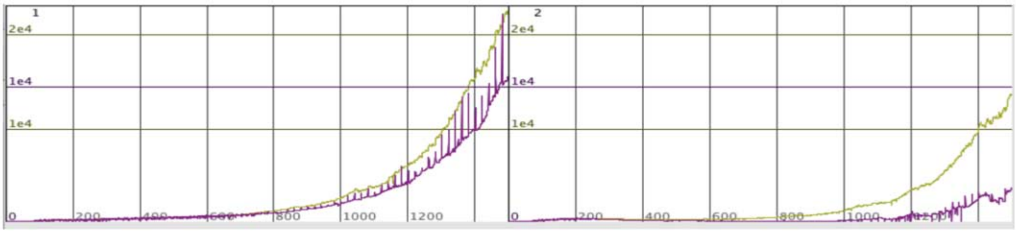

Figure 13. Output (yellow), final consumption (magenta).

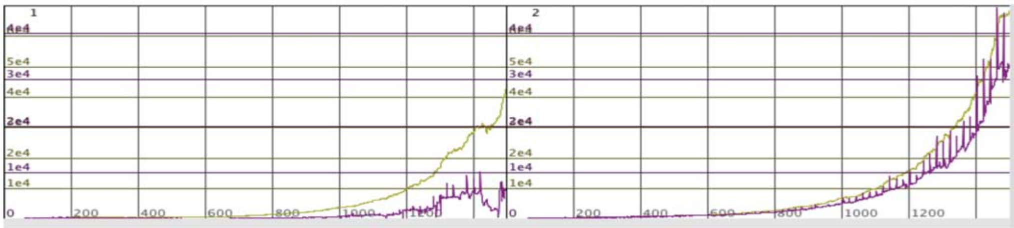

Figure 14. Output (yellow), final consumption (magenta).

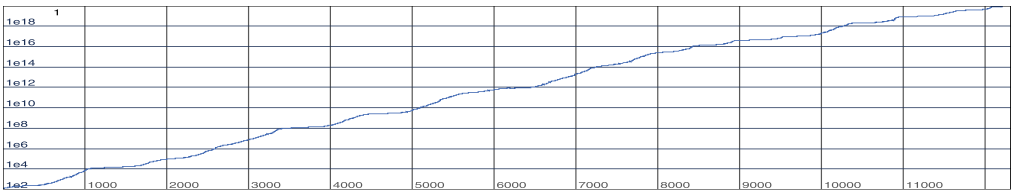

Figure 15. Output, log. scale.

ence of multiple equilibria. Figures 13 and 14 characterize two Monte-Carlo simulations ran using the very same set of parameters. The two simulations exhibit the same pattern: a sector where production and consumption grow fast, another with a much slower growth rate. However, the role of the two sectors is inverted in the two simulations. One of the commodity is in fact endogenously (and randomly) chosen as the main input to final consumption, what then determines the difference growth rates in the two sectors. Randomness hence put to light the existence of multiple equilibria.

In this particular series of experiments, the equilibrium was stable once established. Yet, the possibility of regime change has been demonstrated in another series of experiments: increasing the sensitivity of the Taylor rule in the framework of section 4 lead to changes in the interest rate of larger amplitude which through the effects on final demand (due both to consumption behavior and firms bankruptcy rates) lead to the coexistence of two growth regimes (see figure 15 which represent the dynamics at a much longer time-scale than the previous figures). These results appear very much in line with the literature on the indeterminacy of equilibrium in growth models with Taylor rules (see Benhabib, Schmitt-Grohé & Uribe, 2001). They also indicate that shocks might be necessary to induce changes of dynamical regime. Put in relation with the literature on the evolution of conventions or more generally with the theory of random perturbations in dynamical systems (see e.g. Freidlin & Wentzell, 1984), these result suggest that further analysis of agent-based models as stochastic processes, could help characterize the nature and the amount of randomness or uncertainty necessary to trigger transition phenomena via out-of-equilibrium dynamics in general equilibrium models.

7. Conclusion: Steps towards a theory of out-of-equilibrium dynamics

The Lagom model has been introduced here as a mean to equip Arrow-Debreu economies with agent-based dynamics. Through the presentation of a series of experiments, we have illustrated how the behavior of boundedly rational agents can be consistent with the emergence of equilibrium, hence partly extending Gintis results (Gintis, 2006, 2007) to a framework with capital accumulation. These experiments also suggest the possibility of transitions between equilibria in settings with multiple equilibria. A deeper understanding of the out-of-equilibrium processes still remains to be gained. In the conclusion of this paper, we try to identify some of the challenges which lie ahead.

7.1. Increasing design robustness

The analysis of out-of-equilibrium processes require well-defined dynamics in every point of the state space. This puts much stronger stability requirements on agent-based models than on general equilibrium ones. A local instability of the model, even far away from equilibrium, can trap the dynamics into a rationing cascade within which production eventually vanishes. In the development of the Lagom model, the design of contracts and the role assumed by randomness have been found to play a major role in this respect.

In a framework with constant returns to scale, even at equilibrium, prices convey to firms information about the optimal combination of inputs but not about the optimal level of production. Therefore, the balance between supply and demand require more than price adjustment. The design of contracts appear to be a key determinant of the extent and of the form of rationing. In agent-based models, one often considers spot markets only. Demand being moreover forecasted in a boundedly rational fashion, firms ought to maintain a high level of inventory to prevent dramatic rationing of demand. The introduction of future market, or simply of the possibility for buyers to book future deliveries, seems a promising way to increase stability. This involves however the development of a detailed description of contracts, in particular ways to deal with the possibility of default.

Randomness can be another major source of instability in models with boundedly rational agents. Indeed on top of "essential" randomness which is used to model choices involving some uncertainty, e.g. the choice of a new technology, agent-based models also often involve a form of instrumental randomness. In models with a central scheduler, the sequentialization of certain processes (e.g. trading) introduces a huge dissymmetry between agents. Randomizing this sequentialization scheme allows to restore some uniformity among agents: the expected outcome (before sequentialization) become symmetric. However, this instrumental randomness also introduces an additional source of volatility in the model for mostly technical reasons. New computational designs, in particular involving a different treatment of time, should help dispense of it.

7.2. The mathematics of agent-based models

Together with technological advances in the design and the performance of agent-based models, the development of a theory of out-of-equilibrium dynamics as dynamics of complex systems requires a transition from computer simulations to mathematical analysis. Agent-based models are stochastic processes with little continuity properties. As such, they have mainly been analyzed as discrete Markov chains (see e.g. Peyton-Young, 2006) or approximated using methods of statistical physics involving master equations (see e.g. Aoki & Yoshikawa, 2006). These methods have mainly been used in relatively simple models and provided asymptotic results. Their combination with the accumulated knowledge on the geometry of equilibria in the Arrow-Debreu framework (see e.g. Balasko, 2009) might however prove useful to tame part of the complexity of transitional dynamics in the general equilibrium framework.

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