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COMPLETE COMPARATIVE STATIC DIFFERENTIAL EQUATIONS*

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1. INTRODUCTION

COMPARATIVE static problems in economics typically reduce to determining the response of a vector $x = (x_1, \ldots, x_n)$ to changes in a scalar parameter α, where x and α satisfy a system of one-dimensional equations

$$0 = \Psi(x, \alpha) \equiv \begin{pmatrix} \Psi^1(x, \alpha) \\ \cdot \\ \cdot \\ \cdot \\ \Psi^n(x, \alpha) \end{pmatrix}. \tag{1}$$

Equations having form (1) arise as first-order conditions in microeconomic constrained optimization models, as defining characterizations for general equilibrium and macroeconomic models, and as steady-state solution characterizations for descriptive and optimal growth models. (See Silberberg [12], Hansen [7], Cass and Shell [3], and Brock [1].)

If $\Psi: R^{n+1} \to R^n$ is continuously differentiable over a neighborhood of a point (x^0, α^0) in R^{n+1} for which $\Psi(x^0, \alpha^0) = 0$ and $|\Psi_x(x^0, \alpha^0)| \neq 0$, then the implicit function theorem guarantees the existence of a continuously differentiable function $x: N(\alpha^0) \to R^n$ over some open neighborhood $N(\alpha^0)$ of α^0 in R such that

$$\frac{dx}{d\alpha}(\alpha) = -\Psi_x(x(\alpha), \alpha)^{-1}\Psi_\alpha(x(\alpha), \alpha), \quad \alpha \in N(\alpha^0). \tag{2}$$

Equation (2) is the fundamental relation underlying almost all comparative static studies in economics.† The basic objective of these studies is to obtain determinate signs for the components of $dx(\alpha)/d\alpha$, so that in principle the economic theory underlying the system of equations (1) can be empirically tested. Unfortunately, optimality and stability postulates, which place

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† Silberberg [11] has recently shown that the usual comparative statics relations in maximization models which are obtainable by use of (2) are special cases of the positive semi-definiteness of a matrix of cross partials of the Lagrangean function for a certain primal-dual problem introduced by Samuelson [10].

restrictions on the Jacobian matrix

$$J(\alpha) \equiv \Psi_x(x(\alpha), \alpha), \tag{3}$$

often allow at best a partial signing of the components of $dx(\alpha)/d\alpha$. (See Silberberg [12].) A third type of postulate, specific functional forms for $\Psi(.)$, leads in principle to a complete signing of $dx(\alpha)/d\alpha$, but such postulates are rarely used in theoretical economic studies.

One basic reason for the relatively limited local nature of the comparative static results obtainable by use of equation (2) is that (2) is typically an analytically incomplete differential system, even for known functions $\Psi(.)$. Specifically, closed form representations for the Jacobian inverse $J(\alpha)^{-1}$ as a function of α are often not feasible when $n \geqslant 3$. Thus integration of (2) from initial conditions generally requires the supplementary algebraic determination of $J(\alpha)^{-1}$ at each step in the integration process. This was Davidenko's original proposal in [6], and it has since become a standard numerical procedure. Nevertheless, the need to invert a matrix at each integration step is highly undesirable from a computational viewpoint, and the lack of an explicit representation for $dJ(\alpha)^{-1}/d\alpha$ dictates that qualitative parameter sensitivity inferences obtained by use of (2) must be strictly local in nature.

In Section 2, below, we provide a complete differential system for $x(\alpha)$ and $J(\alpha)^{-1}$. More precisely, recalling that $J(\alpha)^{-1}$ can be represented as the ratio $A(\alpha)/\delta(\alpha)$ of the $n \times n$ adjoint matrix $A(\alpha)$ and determinant $\delta(\alpha)$ of $J(\alpha)$, we validate a differential system for $x(\alpha)$, $A(\alpha)$, and $\delta(\alpha)$ of the form

$$\frac{dx}{d\alpha}(\alpha) = -A(\alpha)\Psi_\alpha(x(\alpha), \alpha)/\delta(\alpha), \tag{4a}$$

$$\frac{dA}{d\alpha}(\alpha) = [A(\alpha)\mathrm{Trace}(A(\alpha)B(\alpha)) - A(\alpha)B(\alpha)A(\alpha)]/\delta(\alpha), \tag{4b}$$

$$\frac{d\delta}{d\alpha}(\alpha) = \mathrm{Trace}(A(\alpha)B(\alpha)), \tag{4c}$$

where $B(\alpha) \equiv dJ(\alpha)/d\alpha$ is expressible as a known function of $x(\alpha)$, $A(\alpha)$, $\delta(\alpha)$, and α. Initial conditions for system (4) must be provided at a parameter point α^0 by specifying values for $x(\alpha^0)$, $A(\alpha^0)$, and $\delta(\alpha^0)$ satisfying $\Psi(x(\alpha^0), \alpha^0) = 0$, $A(\alpha^0) = \mathrm{Adj}(J(\alpha^0))$, and $\delta(\alpha^0) = |J(\alpha^0)| \neq 0$, where $J(\alpha^0)$ is defined as in (3). In general there may be more than one vector $x(\alpha^0)$ satisfying $\Psi(x(\alpha^0), \alpha^0) = 0$ for any given α^0. System (4) tracks the solution branch corresponding to the selected $x(\alpha^0)$. (See the illustrative examples in Section 3, below.)

The basic system of comparative static differential equations (4) can be used in a qualitative manner to investigate sign retention over parameter intervals of interest. The sign of $\delta(\alpha)$ is obviously of critical importance.

Alternatively, system (4) can be numerically integrated on a computer. The ability to obtain explicit solution trajectories for $x(\alpha)$, $\mathrm{Adj}(J(\alpha))$, and $|J(\alpha)|$ over parameter intervals of interest would seem to provide a useful additional tool for examining and suggesting theoretical conjectures about system (1). It is important to stress that initial value problems such as (4), comprising $n^2 + n + 1$ ordinary differential equations, can be solved with great speed and accuracy by present-day computers even if n is of the order 10^2, and this computer capability is steadily being improved. (See [2, 4, 5].) Thus, using (4), a wide variety of specific functional forms for

$\Psi(.)$ can be tested with minimal effort, which should overcome in part the understandable past reluctance of economists to use specific functional forms in theoretical comparative static studies.

A fortran program for solving the comparative static differential equations (4) for $n = 2$ is available from the authors upon request. (A program for arbitrary n is in preparation.) The analytical usefulness and computational accuracy of the program are illustrated in Section 3 in the context of several examples.

The final Section 4 briefly describes extensions of system (4) which will be treated in future papers. For example, system (4) can be supplemented by differential equations for tracking the eigenvalues of the Jacobian matrix $J(\alpha)$, a useful generalization for optimization and equilibrium applications where the definiteness and stability properties of $J(\alpha)$ are of crucial importance.

2. COMPLETE COMPARATIVE STATIC DIFFERENTIAL EQUATIONS FOR $\Psi(x, \alpha) = 0$

Let $\Psi: R^{n+1} \to R^n$ be twice continuously differentiable over an open neighborhood of a point (x^0, α^0), x^0 in R^n and α^0 in R, with component function representation $\Psi = (\Psi^1, \ldots, \Psi^n)^T$. Suppose $\Psi(x^0, \alpha^0) = \mathbf{0}$ and the determinant $|\Psi_x(x^0, \alpha^0)|$ of the $n \times n$ Jacobian matrix

$$\Psi_x(x, \alpha) \equiv \begin{bmatrix} \frac{\partial \Psi^1}{\partial x_1}(x, \alpha) & \cdots & \frac{\partial \Psi^1}{\partial x_n}(x, \alpha) \\ \vdots & & \vdots \\ \frac{\partial \Psi^n}{\partial x_1}(x, \alpha) & \cdots & \frac{\partial \Psi^n}{\partial x_n}(x, \alpha) \end{bmatrix} \tag{5}$$

evaluated at (x^0, α^0) is nonvanishing.

Let Tr(.) and Adj(.) denote trace and adjoint, respectively. Consider the system of $n^2 + n + 1$ differential equations

$$\frac{dx}{d\alpha}(\alpha) = -A(\alpha)\Psi_\alpha(x(\alpha), \alpha)/\delta(\alpha), \tag{6a}$$

$$\frac{dA}{d\alpha}(\alpha) = [A(\alpha)\mathrm{Tr}(A(\alpha)\, B(\alpha)) - A(\alpha)\, B(\alpha)\, A(\alpha)]/\delta(\alpha), \tag{6b}$$

$$\frac{d\delta}{d\alpha}(\alpha) = \mathrm{Tr}(A(\alpha)\, B(\alpha)), \tag{6c}$$

subject to the initial conditions

$$x(\alpha^0) = x^0, \tag{6d}$$

$$A(\alpha^0) = \mathrm{Adj}(J(\alpha^0)), \tag{6e}$$

$$\delta(\alpha^0) = |J(\alpha^0)|, \tag{6f}$$

where $J(\alpha) \equiv \Psi_x(x(\alpha), \alpha)$, and the ijth component $b_{ij}(\alpha)$ of the $n \times n$ matrix $B(\alpha) \equiv \mathrm{d}J(\alpha)/\mathrm{d}\alpha$ is given by

$$b_{ij}(\alpha) \equiv \sum_{m=1}^{n} \frac{\partial^2 \Psi^i}{\partial x_j \partial x_m}(x(\alpha), \alpha) \frac{\partial x_m}{\partial \alpha}(\alpha) + \frac{\partial^2 \Psi^i}{\partial x_j \partial \alpha}(x(\alpha), \alpha). \tag{6g}$$

Note, using (6a), that $B(\alpha)$ is expressible as a known function of $x(\alpha)$, $A(\alpha)$, and α.

It will now be established that system (6) has a unique solution over some open interval $N(\alpha^0)$ containing α^0. Moreover, this solution satisfies $\Psi(x(\alpha), \alpha) = 0$, $A(\alpha) = \mathrm{Adj}(J(\alpha))$, and $\delta(\alpha) = |J(\alpha)|$ for $\alpha \in N(\alpha^0)$. It should be remarked that existence and uniqueness theorems for solutions of initial value problems such as (6) have to be local in nature in view of the nonlinearity of the differential equations. In practice, however, the length of $N(\alpha^0)$ may actually be infinite. If one or both of the largest possible values ε^1 and ε^2 satisfying $(\alpha^0 - \varepsilon^1, \alpha^0 + \varepsilon^2) \subset N(\alpha^0)$ are finite, numerical integration of the initial value problem (6) can be used to locate the critical endpoints. (See the examples in Section 3, below).

LEMMA 1. Let $M(\alpha)$ be an $n \times n$ matrix function of α over an open neighborhood of a point α^0 in R such that $M_\alpha(\alpha) \equiv \mathrm{d}M(\alpha)/\mathrm{d}\alpha$ exists and is continuous over this neighborhood and $|M(\alpha^0)| \neq 0$. Then, for some open neighborhood $N(\alpha^0)$ of α^0, there exists a unique continuously differentiable $n \times n$ matrix function $A(.)$ and a unique continuously differentiable scalar function $\delta(.)$ over $N(\alpha^0)$ satisfying

$$\frac{\mathrm{d}A}{\mathrm{d}\alpha}(\alpha) = [A(\alpha)\mathrm{Tr}(A(\alpha)M_\alpha(\alpha)) - A(\alpha)M_\alpha(\alpha)A(\alpha)]/\delta(\alpha), \tag{7a}$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}\alpha}(\alpha) = \mathrm{Tr}(A(\alpha)M_\alpha(\alpha)), \quad \alpha \in N(\alpha^0), \tag{7b}$$

subject to the initial conditions

$$A(\alpha^0) = \mathrm{Adj}(M(\alpha^0)), \tag{7c}$$

$$\delta(\alpha^0) = |M(\alpha^0)|. \tag{7d}$$

Moreover, $A(.)$ and $\delta(.)$ satisfy

$$A(\alpha) = \mathrm{Adj}(M(\alpha)), \quad \alpha \in N(\alpha^0), \tag{8a}$$

$$\delta(\alpha) = |M(\alpha)|, \quad \alpha \in N(\alpha^0). \tag{8b}$$

Proof. The existence of a unique continuously differentiable solution to system (7) over some open neighborhood of α^0 follows directly from Theorem 9.1 in Hartman [8, p. 137]. To establish Lemma 1, it thus suffices to show that one solution to system (7) in an open neighborhood of α^0 is given by $A^*(\alpha) \equiv \mathrm{Adj}(M(\alpha))$ and $\delta^*(\alpha) \equiv |M(\alpha)|$.

By definition of $M(\alpha)$, the adjoint $A^*(\alpha)$ and determinant $\delta^*(\alpha)$ of $M(\alpha)$ are continuously differentiable functions of α over some open neighborhood $N'(\alpha^0)$ of α^0, and

$$A^*(\alpha)\, M(\alpha) = \delta^*(\alpha) I, \quad \alpha \in N'(\alpha^0). \tag{9}$$

By assumption, $\delta^*(\alpha^0) \neq 0$. Without loss of generality, suppose $\delta^*(\alpha) \neq 0$ for $\alpha \in N'(\alpha^0)$.

Differentiating (9) with respect to α, multiplying through by $A^*(\alpha)$, and rearranging terms, one obtains

$$\frac{dA^*}{d\alpha}(\alpha) = [A^*(\alpha)\, d\delta^*(\alpha)/d\alpha - A^*(\alpha)M_\alpha(\alpha)A^*(\alpha)]/\delta^*(\alpha), \quad \alpha \in N'(\alpha^0). \tag{10}$$

It remains to determine $d\delta^*(\alpha)/d\alpha$. Letting $m_{ij}(\alpha)$ denote the ijth element of $M(\alpha)$, and letting $C_{ij}(\alpha)$ denote the cofactor of $m_{ij}(\alpha)$, the determinant $\delta^*(\alpha)$ of $M(\alpha)$ may be expressed as

$$\delta^*(\alpha) = \sum_{i=1}^{n} m_{ij}(\alpha)\, C_{ij}(\alpha) \tag{11}$$

for any $j \in \{1, \ldots, n\}$. Thus, using the chain rule together with (11), one obtains

$$\begin{aligned}\frac{d\delta^*}{d\alpha}(\alpha) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \delta^*}{\partial m_{ij}}(\alpha)\frac{dm_{ij}}{d\alpha}(\alpha)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}(\alpha)\frac{dm_{ij}}{d\alpha}(\alpha)\\ &= \mathrm{Tr}(C^T(\alpha)\, M_\alpha(\alpha))\\ &= \mathrm{Tr}(A^*(\alpha)\, M_\alpha(\alpha)), \quad \alpha \in N'(\alpha^0),\end{aligned} \tag{12}$$

where the ijth element of the $n \times n$ matrix $C(\alpha)$ is $C_{ij}(\alpha)$, and superscript T denotes transpose. Combining (10) and (12), $A^*(.)$ and $\delta^*(.)$ are solutions to system (7) over some open neighborhood of α^0. Q.E.D.

THEOREM 1. For some open neighborhood $N(\alpha^0)$ of α^0, there exist unique continuously differentiable solution functions $\delta(.)$, $x(.)$, and $A(.)$ for the initial value problem (6) over $N(\alpha^0)$. Moreover, these solution functions satisfy

$$\Psi(x(\alpha), \alpha) = 0, \quad \alpha \in N(\alpha^0), \tag{13a}$$

$$A(\alpha) = \mathrm{Adj}(J(\alpha)), \quad \alpha \in N(\alpha^0), \tag{13b}$$

and

$$\delta(\alpha) = |J(\alpha)|, \quad \alpha \in N(\alpha^0). \tag{13c}$$

Proof. The regularity conditions imposed on $\Psi(.)$ guarantee, by the implicit function theorem, the existence of a unique continuously differentiable function $x(.)$ over some open neighborhood $N'(\alpha^0)$ of α^0 satisfying $x(\alpha^0) = x^0$, $\Psi(x(\alpha), \alpha) = 0$, $\alpha \in N'(\alpha^0)$, and

$$\frac{dx}{d\alpha}(\alpha) = -J(\alpha)^{-1}\, \Psi_\alpha(x(\alpha), \alpha), \quad \alpha \in N'(\alpha^0), \tag{14}$$

where $J(\alpha) \equiv \Psi_x(x(\alpha), \alpha)$. In particular, $J(\alpha)$ is a continuously differentiable $n \times n$ matrix function of α over $N'(\alpha^0)$ with $|J(\alpha^0)| \neq 0$. The remainder of Theorem 1 follows immediately from Lemma 1. Q.E.D.

For some comparative static studies it may be useful to have a differential system directly in terms of $x(\alpha)$ and the Jacobian inverse matrix $J(\alpha)^{-1}$. This approach was first suggested in [9]. The next theorem provides such a system.

THEOREM 2. For some open neighborhood $N(\alpha^0)$ of α^0 there exists a unique continuously differentiable $n \times 1$ vector function $x(.)$ over $N(\alpha^0)$ and a unique continuously differentiable $n \times n$ matrix function $V(.)$ over $N(\alpha^0)$ satisfying the system of $n^2 + n$ differential equations

$$\frac{dx}{d\alpha}(\alpha) = -V(\alpha)\Psi_\alpha(x(\alpha), \alpha), \quad \alpha \in N(\alpha^0), \tag{15a}$$

$$\frac{dV}{d\alpha}(\alpha) = -V(\alpha)\,B(\alpha)\,V(\alpha), \quad \alpha \in N(\alpha^0), \tag{15b}$$

subject to the initial conditions

$$x(\alpha^0) = x^0, \tag{15c}$$

$$V(\alpha^0) = J(\alpha^0)^{-1}, \tag{15d}$$

where the ijth component of $B(\alpha) \equiv dJ(\alpha)/d\alpha$ is defined as in (6g). Moreover, the solution functions $x(.)$ and $V(.)$ satisfy

$$\Psi(x(\alpha), \alpha) = 0, \quad \alpha \in N(\alpha^0), \tag{16a}$$

$$J(\alpha)\,V(\alpha) = I, \quad \alpha \in N(\alpha^0). \tag{16b}$$

Proof. As in Theorem 1, the regularity conditions imposed on $\Psi(.)$ guarantee the existence of a unique continuously differentiable function $x(.)$ over some open neighborhood $N'(\alpha^0)$ of α^0 satisfying $x(\alpha^0) = x^0$, $\Psi(x(\alpha), \alpha) = 0$, $\alpha \in N'(\alpha^0)$, and

$$\frac{dx}{d\alpha}(\alpha) = -J(\alpha)^{-1}\,\Psi_\alpha(x(\alpha), \alpha), \quad \alpha \in N'(\alpha^0), \tag{17}$$

where $J(\alpha) \equiv \Psi_x(x(\alpha), \alpha)$ is continuously differentiable and $|J(\alpha^0)| \neq 0$.

Consider the differential system (15b) and (15d), with $x(.)$ as in (17) and $B(\alpha) \equiv dJ(\alpha)/d\alpha \equiv d\Psi_x(x(\alpha), \alpha)/d\alpha$. The existence of a unique continuously differentiable solution $V(.)$ for (15b) and (15d) over some neighborhood $N''(\alpha^0)$ of α^0 follows from Theorem 9.1 in Hartman [8, p. 137]. Define $H(\alpha) \equiv J(\alpha)\,V(\alpha)$, $\alpha \in N(\alpha^0) \equiv N'(\alpha^0) \cap N''(\alpha^0)$. To complete the proof of Theorem 2, it suffices to show that $H(\alpha) = I$ for each $\alpha \in N(\alpha^0)$. Using (15b) and the definition of $B(\alpha)$,

$$\begin{aligned}\frac{dH}{d\alpha}(\alpha) &= \frac{dJ}{d\alpha}(\alpha)\,V(\alpha) + J(\alpha)\frac{dV}{d\alpha}(\alpha)\\ &= B(\alpha)\,V(\alpha) - J(\alpha)\,V(\alpha)\,B(\alpha)\,V(\alpha)\\ &= [I - H(\alpha)]B(\alpha)\,V(\alpha), \quad \alpha \in N(\alpha^0).\end{aligned} \tag{18}$$

Since $H(\alpha^0) = I$, one solution for (18) is $H(\alpha) \equiv I$. However, since (18) is a linear differential equation subject to initial conditions, it has a unique solution. Q.E.D.

3. ILLUSTRATIVE EXAMPLES

Two simple examples will now be given to illustrate the usefulness of the complete comparative static differential system (6) derived in Section 2. The first example is a profit maximization problem in which a capital and labor using industrial sector is subject to a payroll tax. The second purely numerical example illustrates the usefulness of system (6) for tracking distinct solution branches, and for locating critical parameter values α where the Jacobian matrix $J(\alpha)$ becomes singular.

Let $F: R^2_{++} \to R$ be a production function defined by

$$F(K, L) \equiv K^\gamma L^\beta, \quad 0 < \gamma, 0 < \beta, \quad \gamma + \beta < 1, \tag{19}$$

where K and L denote capital and labor services, and let P, R, and W denote output price, nominal capital rental rate, and nominal wage rate, respectively. Consider the problem of maximizing profits

$$PF(K, L) - RK - [1 + \alpha^0]WL \tag{20}$$

with respect to $K > 0$ and $L > 0$ for given $P > 0$, $R > 0$, $W > 0$, and $\alpha^0 \geqslant 0$, where α^0 denotes a payroll tax rate.

Since $F(.)$ is strictly concave over R^2_{++}, necessary and sufficient conditions for positive K^0 and L^0 to maximize profits (20) are given by

$$0 = F_K(K^0, L^0) - (R/P) \equiv F_K(K^0, L^0) - r, \tag{21a}$$

$$0 = F_L(K^0, L^0) - [1 + \alpha^0](W/P) \equiv F_L(K^0, L^0) - [1 + \alpha^0]w. \tag{21b}$$

It is easily checked that the Jacobian matrix associated with system (21) is nonsingular. Thus, using standard implicit function arguments, the existence of positive K^0 and L^0 satisfying (21) guarantees the existence of unique continuously differentiable functions $K(.)$ and $L(.)$ in a neighborhood $N(\alpha^0)$ of α^0 satisfying

$$K(\alpha^0) = K^0, \qquad L(\alpha^0) = L^0, \tag{22a}$$

$$0 = F_K(K(\alpha), L(\alpha)) - r, \quad \alpha \in N(\alpha^0), \tag{22b}$$

$$0 = F_L(K(\alpha), L(\alpha)) - w, \quad \alpha \in N(\alpha^0), \tag{22c}$$

$$\frac{\mathrm{d}K}{\mathrm{d}\alpha}(\alpha) = -\frac{F_{KL}(K(\alpha), L(\alpha))w}{|J(\alpha)|} < 0, \quad \alpha \in N(\alpha^0), \tag{22d}$$

$$\frac{\mathrm{d}L}{\mathrm{d}\alpha}(\alpha) = \frac{F_{KK}(K(\alpha), L(\alpha))w}{|J(\alpha)|} < 0, \quad \alpha \in N(\alpha^0), \tag{22e}$$

where the determinant $|J(\alpha)|$ of the Jacobian matrix $J(\alpha)$ associated with system (22b) and (22c) is given by

$$\begin{aligned} |J(\alpha)| &= F_{KK}(K(\alpha), L(\alpha))\, F_{LL}(K(\alpha), L(\alpha)) - [F_{KL}(K(\alpha), L(\alpha))]^2 \\ &= \gamma\beta K(\alpha)^{2\gamma-2}\, L(\alpha)^{2\beta-2}\, [1 - \gamma - \beta] > 0, \quad \alpha \in N(\alpha^0). \end{aligned} \tag{22f}$$

It will now be shown how Theorem 1 can be applied to this profit maximization problem to obtain explicit solution trajectories for $K(\alpha)$, $L(\alpha)$, $|J(\alpha)|$, and other relevant terms over arbitrary tax rate intervals of the form $[\alpha^0, \alpha']$, $0 \leqslant \alpha^0 < \alpha'$, given arbitrary admissible numerical values for r, w, γ, and β. Define a function $\Psi \equiv (\Psi^1, \Psi^2)^T$ taking R^3 into R^2 by

$$\Psi^1(K, L, \alpha) \equiv F_K(K, L) - r, \tag{23a}$$

$$\Psi^2(K, L, \alpha) \equiv F_L(K, L) - [1 + \alpha]w. \tag{23b}$$

Assume (21) holds for some $K^0 > 0$, $L^0 > 0$, and $\alpha^0 \geqslant 0$, hence $\Psi(K^0, L^0, \alpha^0) = 0$. Clearly $\Psi(.)$

is twice continuously differentiable in a neighborhood of the point (K^0, L^0, α^0), and, by (22f), the determinant $|J(\alpha^0)|$ of the Jacobian matrix.

$$J(\alpha) = \begin{bmatrix} \frac{\partial \Psi^1}{\partial K}(K(\alpha), L(\alpha), \alpha) & \frac{\partial \Psi^1}{\partial L}(K(\alpha), L(\alpha), \alpha) \\ \frac{\partial \Psi^2}{\partial K}(K(\alpha), L(\alpha), \alpha) & \frac{\partial \Psi^2}{\partial L}(K(\alpha), L(\alpha), \alpha) \end{bmatrix} \tag{24}$$

evaluated at α^0 is nonvanishing.

Thus, by Theorem 1, over some open neighborhood $N(\alpha^0)$ of α^0 there exist unique continuously differentiable scalar functions $K(.)$, $L(.)$, and $\delta(.)$, and a unique continuously differentiable 2×2 matrix function $A(.)$ satisfying the system of $2^2 + 2 + 1$ differential equations

$$\begin{bmatrix} \frac{dK}{d\alpha}(\alpha) \\ \frac{dL}{d\alpha}(\alpha) \end{bmatrix} = -A(\alpha)\,\Psi_\alpha(K(\alpha), L(\alpha), \alpha)/\delta(\alpha), \tag{25a}$$

$$\frac{dA}{d\alpha}(\alpha) = [A(\alpha)\,\mathrm{Tr}(A(\alpha)\,B(\alpha)) - A(\alpha)\,B(\alpha)\,A(\alpha)]/\delta(\alpha), \tag{25b}$$

$$\frac{d\delta}{d\alpha}(\alpha) = \mathrm{Tr}(A(\alpha)\,B(\alpha)), \tag{25c}$$

Subject to the initial conditions

$$K(\alpha^0) = K^0, \quad L(\alpha^0) = L^0, \tag{25d}$$

$$A(\alpha^0) = \mathrm{Adj}(J(\alpha^0)), \tag{25e}$$

$$\delta(\alpha^0) = |J(\alpha^0)|, \tag{25f}$$

where $B(\alpha) \equiv dJ(\alpha)/d\alpha$. Moreover, $K(.)$, $L(.)$, $A(.)$, and $\delta(.)$ satisfy

$$\Psi(K(\alpha), L(\alpha), \alpha) = 0, \quad \alpha \in N(\alpha^0), \tag{26a}$$

$$A(\alpha) = \mathrm{Adj}(J(\alpha)), \quad \alpha \in N(\alpha^0), \tag{26b}$$

$$\delta(\alpha) = |J(\alpha)|, \quad \alpha \in N(\alpha^0). \tag{26c}$$

The comparative static differential equations (25) were integrated on an IBM 370/Model 158 for various admissible parameter specifications (r, w, δ, β) and initial values (K^0, L^0, α^0) using a single precision Fortran program designed for arbitrary functions $\Psi: R^3 \to R^2$. Table 1 describes one such experiment. Note that the monotonic behavior of $K(.)$, $L(.)$, and $\delta(.)$ is as expected, using the analytically derivable results (22) for this simple 2×2 example. The last two columns indicate the high numerical accuracy of the computer program. The actual step size in α was 0.01, with fifty steps taken in all. The CPU execution time was 1.08 seconds. The program of course evaluates many additional interesting and useful expressions not appearing in Table 1, e.g. $A(\alpha)$, $dK(\alpha)/d\alpha$, $dL(\alpha)/d\alpha$, $d\delta(\alpha)/d\alpha$, and $dA(\alpha)/d\alpha$. A fourth order Adams–Moulton integration method with a Runge–Kutta start was employed.

Table 1. Trajectories for capital $K(\alpha)$, labor $L(\alpha)$, and the Jacobian determinant $\delta(\alpha)$ as the payroll tax α increases from 0 to 0·5, and a check of the first order conditions (26a)

α	$K(\alpha)$	$L(\alpha)$	$\delta(\alpha)$	$\Psi^1(K(\alpha), L(\alpha), \alpha)$	$\Psi^2(K(\alpha), L(\alpha), \alpha)$
0·00	1·0	1·0	0·03472	0·0	0·0
0·05	0·97115	0·92490	0·04059	-3.95×10^{-7}	$1{\cdot}04 \times 10^{-6}$
0·10	0·94442	0·85856	0·04711	-6.05×10^{-7}	$6{\cdot}66 \times 10^{-7}$
0·15	0·91956	0·79962	0·05431	$6{\cdot}63 \times 10^{-7}$	$3{\cdot}66 \times 10^{-7}$
0·20	0·89638	0·74698	0·06223	-6.61×10^{-7}	$7{\cdot}43 \times 10^{-7}$
0·25	0·87469	0·69975	0·07091	$-2{\cdot}92 \times 10^{-7}$	6.19×10^{-7}
0·30	0·85435	0·65719	0·08039	$-8{\cdot}80 \times 10^{-7}$	$2{\cdot}84 \times 10^{-7}$
0·35	0·83522	0·61868	0·09071	$-5{\cdot}66 \times 10^{-7}$	2.35×10^{-7}
0·40	0·81719	0·58371	0·10191	$4{\cdot}51 \times 10^{-7}$	$-1{\cdot}24 \times 10^{-6}$
0·45	0·80016	0·55184	0·11402	$1{\cdot}59 \times 10^{-6}$	$-1{\cdot}90 \times 10^{-6}$
0·50	0·78405	0·52270	0·12709	$4{\cdot}75 \times 10^{-7}$	$5{\cdot}35 \times 10^{-7}$

Parameter values: $\gamma = r = \frac{1}{3}$, $\beta = w = \frac{1}{4}$.
Initial values: $K^0 = L^0 = 1$, $\alpha^0 = 0$.

The comparative static differential equations (25) also facilitate multiparameter sensitivity studies. For example, as indicated in Fig. 1, a simultaneous increase in the Table I parameters β and w from $\frac{1}{4}$ to $\frac{1}{3}$ uniformly lowers capital usage from $K(\alpha)$ to $K^*(\alpha)$ and uniformly increases the value of the Jacobian determinant from $\delta(\alpha)$ to $\delta^*(\alpha)$. Similarly, labor usage $L(\alpha)$ uniformly decreases. Such monotonic shifts in $K(\alpha)$, $L(\alpha)$, and $\delta(\alpha)$ in response to multiple changes in r, w, γ, and β do not appear to be easily detectable from the usual comparative static relations (22).

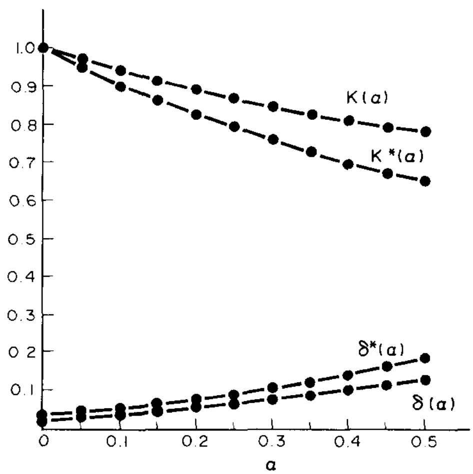

Fig. 1. Response of $K(\alpha)$ and $\delta(\alpha)$ to a simultaneous increase in the production function coefficient β and the real wage rate w.

Now consider the following simple system of equations

$$0 = xy - \alpha \equiv \Psi^1(x, y, \alpha), \tag{27a}$$

$$0 = x + 3y - 1 \equiv \Psi^2(x, y, \alpha). \tag{27b}$$

Given any real α, system (27) has the two solutions

$$x = 1 - 3y, \qquad y = [1 + \sqrt{1 - 12\alpha}]/6, \tag{28a}$$

$$x = 1 - 3y, \qquad y = [1 - \sqrt{1 - 12\alpha}]/6. \tag{28b}$$

The Jacobian matrix associated with system (27) is singular if and only if $3y = x$, i.e. using (28), if and only if $\alpha = \frac{1}{12}$. Note that the solutions x and y in (28) are real if and only if α lies in the interval $(-\infty, \frac{1}{12})$. Given any x^0, y^0, and $\alpha^0 \in (-\infty, \frac{1}{12})$ satisfying (27), implicit function arguments guarantee the existence of unique real-valued continuously differential functions $x(.)$ and $y(.)$ over an open neighborhood $N(\alpha^0)$ of α^0 satisfying

$$x(\alpha^0) = x^0, \qquad y(\alpha^0) = y^0, \tag{29a}$$

$$x(\alpha)\, y(\alpha) - \alpha = 0, \quad \alpha \in N(\alpha^0), \tag{29b}$$

$$x(\alpha) + 3y(\alpha) - 1 = 0, \quad \alpha \in N(\alpha^0), \tag{29c}$$

$$\frac{dx}{d\alpha}(\alpha) = \frac{3}{|J(\alpha)|}, \quad \alpha \in N(\alpha^0), \tag{29d}$$

$$\frac{dy}{d\alpha}(\alpha) = \frac{-1}{|J(\alpha)|}, \quad \alpha \in N(\alpha^0), \tag{29e}$$

where $|J(\alpha)|$ denotes the determinant of the Jacobian matrix

$$J(\alpha) \equiv \begin{bmatrix} y(\alpha) & x(\alpha) \\ 1 & 3 \end{bmatrix}. \tag{29f}$$

It is immediate that Theorem 1 is applicable to system (27), given any initial solution values x^0, y^0, and $\alpha^0 \in (-\infty, \frac{1}{12})$. Specifically, over some open neighborhood $N(\alpha^0)$ of α^0 there exist unique continuously differentiable scalar functions $x(.)$, $y(.)$, and $\delta(.)$ and a unique continuously differentiable 2×2 matrix function $A(.)$ satisfying

$$\begin{bmatrix} \dfrac{dx}{d\alpha}(\alpha) \\ \\ \dfrac{dy}{d\alpha}(\alpha) \end{bmatrix} = -A(\alpha)\, \Psi_\alpha(x(\alpha), y(\alpha), \alpha)/\delta(\alpha), \tag{30a}$$

$$\frac{dA}{d\alpha}(\alpha) = [A(\alpha)\mathrm{Tr}(A(\alpha)\, B(\alpha)) - A(\alpha)\, B(\alpha)\, A(\alpha)]/\delta(\alpha), \tag{30b}$$

$$\frac{d\delta}{d\alpha}(\alpha) = \mathrm{Tr}(A(\alpha)\, B(\alpha)), \tag{30c}$$

subject to the initial conditions

$$x(\alpha^0) = x^0, \qquad y(\alpha^0) = y^0, \tag{30d}$$

$$A(\alpha^0) = \mathrm{Adj}(J(\alpha^0)), \tag{30e}$$

$$\delta(\alpha^0) = |J(\alpha^0)|, \tag{30f}$$

where $B(\alpha) \equiv \mathrm{d}J(\alpha)/\mathrm{d}\alpha$. Moreover, $x(.)$, $y(.)$, $A(.)$, and $\delta(.)$ satisfy

$$x(\alpha)\, y(\alpha) - \alpha = 0, \quad \alpha \in N(\alpha^0), \tag{31a}$$

$$x(\alpha) + 3y(\alpha) - 1 = 0, \quad \alpha \in N(\alpha^0), \tag{31b}$$

$$A(\alpha) = \mathrm{Adj}(J(\alpha)), \quad \alpha \in N(\alpha^0), \tag{31c}$$

$$\delta(\alpha) = |J(\alpha)|, \quad \alpha \in N(\alpha^0). \tag{31d}$$

The comparative differential equations (30) were integrated over the α interval $[0, \frac{1}{12}]$ for each of the two distinct initial solution values x^0 and y^0 corresponding to $\alpha^0 = 0$, namely,

$$x^0 = 1 \quad \text{and} \quad y^0 = 0 \tag{32}$$

and

$$x^0 = 0 \quad \text{and} \quad y^0 = \tfrac{1}{3}. \tag{33}$$

The step size in α was $\frac{1}{120} \cong 0{\cdot}00833$, and the CPU execution time was on the order of 0·36 seconds. As expected, in each case the singularity of $J(\alpha)$ at $\alpha = \frac{1}{12}$ was strongly indicated by a blow-up of the determinant derivative $\mathrm{d}\delta(\alpha)/\mathrm{d}\alpha$ as α neared the critical point $\frac{1}{12} \cong 0{\cdot}08333$. Table 2 describes the trajectories for $x^1(\alpha)$, $y^1(\alpha)$, $\delta^1(\alpha)$, and $\mathrm{d}\delta^1(\alpha)/\mathrm{d}\alpha$ corresponding to the first root (32). The final two columns of Table 2 indicate the high numerical accuracy of the computer program, even near the critical point $\alpha = \frac{1}{12}$.

Similar results were obtained for the trajectories $x^2(\alpha)$, $y^2(\alpha)$, $\delta^2(\alpha)$, and $\mathrm{d}\delta^2(\alpha)/\mathrm{d}\alpha$ corresponding to the second root (33). Figure 2 compares the $x(\alpha)$ and $y(\alpha)$ trajectories corresponding to the two roots (32) and (33).

Table 2. Trajectories for $x^1(\alpha)$, $y^1(\alpha)$, $\delta^1(\alpha)$, and $\mathrm{d}\delta^1(\alpha)/\mathrm{d}\alpha$ as the parameter α varies from 0 to $\frac{1}{12}$, and a check of the first-order conditions (31a) and (31b).

α	$x^1(\alpha)$	$y^1(\alpha)$	$\delta^1(\alpha)$	$\mathrm{d}\delta^1(\alpha)/\mathrm{d}\alpha$	$x^1(\alpha)\,y^1(\alpha) - \alpha$	$x^1(\alpha) + 3y^1(\alpha) - 1$
0	1·0	0·0	−1·0	—	0·0	0·0
0·00833	0·97434	0·00855	−0·94869	6·3246	$6{\cdot}07 \times 10^{-7}$	$-1{\cdot}00 \times 10^{-5}$
0·01667	0·94721	0·01760	−0·89443	6·7082	$8{\cdot}96 \times 10^{-7}$	$1{\cdot}00 \times 10^{-5}$
0·02500	0·91833	0·02722	−0·83666	7·1714	$-3{\cdot}06 \times 10^{-6}$	$-1{\cdot}00 \times 10^{-5}$
0·03333	0·88730	0·03757	−0·77459	7·7460	$1{\cdot}00 \times 10^{-5}$	$1{\cdot}00 \times 10^{-5}$
0·04167	0·85355	0·04882	−0·70710	8·4854	$3{\cdot}11 \times 10^{-7}$	$1{\cdot}00 \times 10^{-5}$
0·05000	0·81622	0·06126	−0·63244	9·4871	$1{\cdot}64 \times 10^{-6}$	0·0
0·05833	0·77384	0·07539	−0·54767	$1{\cdot}10 \times 10^1$	$1{\cdot}00 \times 10^{-5}$	$1{\cdot}00 \times 10^{-5}$
0·06667	0·72353	0·09216	−0·44707	$1{\cdot}34 \times 10^1$	$1{\cdot}00 \times 10^{-5}$	$1{\cdot}00 \times 10^{-5}$
0·07500	0·65782	0·11406	−0·31563	$1{\cdot}90 \times 10^1$	$3{\cdot}00 \times 10^{-5}$	0·0
0·08333	0·52435	0·15855	−0·04870	$1{\cdot}23 \times 10^2$	$1{\cdot}94 \times 10^{-4}$	0·0
0·09167	—	—	0·72655	—	—	—

Initial values: $x^0 = 1$, $y^0 = 0$, $\alpha^0 = 0$.

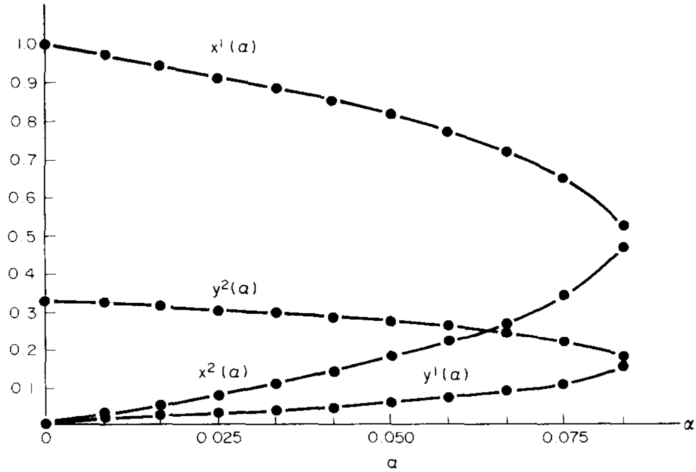

Fig. 2. Solution trajectories for $x(\alpha)$ and $y(\alpha)$ corresponding to the initial solution values (32) and (33).

4. DISCUSSION

Given an economic model characterized by a parameterized system of equations $\Psi(x, \alpha) = 0$, the usual comparative static differential equations $dx/d\alpha = -J(\alpha)^{-1}\Psi_\alpha$ are often analytically incomplete in the sense that their integration requires the algebraic determination of the Jacobian inverse $J(\alpha)^{-1}$ at each step in the integration process. The present paper supplements these equations by differential equations for the adjoint $A(\alpha)$ and determinant $\delta(\alpha)$ of $J(\alpha)$. The resulting initial value problem can be solved with great speed and accuracy on present-day computers to provide explicit trajectories for $x(\alpha)$, $A(\alpha)$, and $\delta(\alpha)$ over parameter intervals of interest. Alternatively, the complete differential system can be used in a qualitative manner to investigate sign retention over parameter intervals.

In optimization and equilibrium investigations the definiteness and stability properties of $J(\alpha)$ are of major concern. In future papers it will be shown how the differential equations for $x(\alpha)$, $A(\alpha)$, and $\delta(\alpha)$ can be supplemented by differential equations for the eigenvalues and right and left eigenvectors of $J(\alpha)$. It may also be possible to bypass the eigenvectors and directly treat the coefficients of the characteristic polynomial and the number of eigenvalues in any given region of the complex plane. (See [13, 14, 15].)

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