Leverage Causes Fat Tails and Clustered Volatility

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We study a very simple model of leveraged asset purchases with margin calls. Investment funds use what is perhaps the most basic financial strategy, called "value investing", i.e. systematically attempting to buy underpriced assets. When funds do not borrow, the price fluctuations of the asset are normally distributed and uncorrelated across time. All this changes when the funds are allowed to leverage, i.e. borrow from a bank, to purchase more assets than their wealth would otherwise permit. When funds use leverage, price fluctuations become heavy tailed and display clustered volatility, similar to what is observed in real markets. Previous explanations of fat tails and clustered volatility depended on "irrational behavior", such as trend following. We show that the immediate cause of the increase in extreme risks in our model is the risk control policy of the banks: A prudent bank makes itself locally safer by putting a limit to leverage, so when a fund exceeds its leverage limit, it must partially repay its loan by selling the asset. Unfortunately this sometimes happens to all the funds simultaneously when the price is already falling. The resulting nonlinear feedback amplifies downward price movements. At the extreme this causes crashes, but the effect is seen at every time scale, producing a power law of price disturbances. A standard (supposedly more sophisticated) risk control policy by individual banks makes these extreme fluctuations even worse. Thus it is the very effort to control risk at the local level that creates excessive risk at the aggregate level, which shows up as fat tails and clustered volatility.
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Recent events in financial markets have underscored the dangerous consequences of the use of excessive credit. At the most basic level the problem is obvious: If a firm buys assets with borrowed money, then under extreme market conditions it may owe more money than it has and default. If this happens on a sufficiently wide scale then it can severely stress creditors and cause them to fail as well.

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We show here that a special but extremely widespread kind of credit called collateralized loans with margin calls has a more pervasive effect: when used excessively it can cause default and crashes, but it also leaves a signature even when there is no default or crash. These kinds of loans have already been identified as a major culprit in the recent crisis, and in previous near crises as well[1]. But we show here that they create a dynamic in asset price fluctuations that manifests itself at all time scales and to all degrees. The extraordinary crisis of the last couple years is just one extreme (but not extremal) point on a continuum.

By taking out a collateralized loan a buyer of stocks or mortgage backed securities can put together a portfolio that is worth a multiple of the cash he has available for their purchase. In 2006 this multiple or "leverage" reached 60 to 1 for AAA rated mortgage securities, and 16 to 1 for what are now called the toxic mortgage securities. The outstanding volume of these leveraged asset purchases reached many trillions of dollars. Leverage has fluctuated up and down in long cycles over the last 30 years.

Conventional credit is for a fixed amount and a fixed maturity, extending over the period the borrower needs the money. In a collateralized loan with margin calls, the debt is guaranteed not by the reputation (or punishment) of the borrower, but by an asset which is confiscated if the loan is not repaid. Typically the loan maturity is very short, say a day, much shorter than the length of time the borrower anticipates needing the money. The contract usually specifies that after the daily interest is paid, as long as the loan to value ratio remains below a specified threshold, the debt is rolled over another day (up to some final maturity, when the threshold ratio might be changed). If, however, the collateral asset value falls, the lender makes a margin call and the borrower is expected to repay part of the debt and so roll over a smaller loan to maintain the old loan to value threshold. Quite often the borrower will obtain the cash for this extra downpayment by selling some of the collateral. The nature of the collateralized loan contract thus sometimes turns buyers of the collateral into sellers, even when they might think it is the best time to buy.

Needless to say, the higher the loan to value, or equivalently, the higher the *leverage* ratio of asset value to cash downpayment, the more severe will be the feedback mechanism. A buyer who is at his threshold of λ times leveraged loses $\lambda\%$ of his investment for every 1% drop in the asset price, and on top of that will have to come up with $\$(\lambda - 1)/\lambda$ of new cash for every \$1 drop in the price of the asset. When there is no leverage, and $\lambda = 1$, there is no feedback, but as the leverage increases, so does the feedback.

The feedback from falling asset prices to margin calls to the transformation of buyers into sellers back to falling asset prices creates a nonlinear dynamic to the system. The nonlinearity rises as the leverage rises. This nonlinear feedback would be present in the most sophisticated rational expectations models or in the most simple minded behavioral models: it is a mechanical effect that stems directly from the nonlinear dynamics caused by the use of leverage and margin calls. We therefore build the simplest model possible and then simulate it over tens of thousands of periods, measuring and quantifying the effect of leverage on asset price fluctuations.

Our model provides a new explanation for the fat tails and clustered volatility that are commonly observed in price fluctuations (Mandelbrot 1963, Engle 1982). Clustered volatility and fat tails emerge in the model on a broad range of time scales, including very rapid ones and very slow ones. Mandelbrot and Engle found that actual price fluctuations did not display the independent and normally distributed properties assumed by the pioneers of classical finance (Bachelier and Black-Scholes). Though their work has been properly celebrated, no consensus has formed on the mechanism which creates fat tails

[1]For previous equilibrium-based analyses of leverage, which show that prices crash before default actually occurs, see Geanakoplos (Geanakoplos 1997, Geanakoplos 2003, Fostel 2008, Geanakoplos 2009).

and clustered volatility. Common sense suggests there must be some endogenous dynamic at work, since it is unlikely that information itself (which moves markets) is heavy tailed and clustered.

Previous endogenous explanations assume that market traders are of at least two types: value investors, who make investments based on fundamentals, and trend followers, who make investments in the direction of recent price movements[2]. Trend followers are inherently destabilizing, and many would dispute whether such behavior is rational. Value investors, in contrast, are essential to maintain a reasonably efficient market: They gather information about valuations, and incorporate it into prices. Thus in this sense value investing is rational. In typical models of this type, investors move their money back and forth between trend strategies and value strategies, depending on who has recently been more successful, and fat tails and clustered volatility are generated by temporary increases in destabilizing trend strategies. The mechanism that we propose here for fat tails and clustered volatility only involves value investors, who leverage their investments by borrowing from a bank. Clustered volatility and fat tails emerge on a broad range of time scales, including very rapid ones; our explanation has the advantage that it can operate on such time scales (whereas it is not obviously plausible that real agents switch from value investing to trend following at rapid speed).

An important aspect of our model is that even though the risk control policies used by the individual bank lenders are reasonable from a narrow, bank-centric point of view, when a group of banks inadvertently acts together, they can dramatically affect prices, inducing nonlinear behavior at a systemic level that gives rise to excessive volatility and even crashes. Attempts to regulate risk without taking into account systemic effects can backfire, accentuating risks or even creating new ones.[3]

In our model traders have a choice between owning a single asset, such as a stock or a commodity, or owning cash. There are two types of traders, noise traders and funds. The noise traders buy and sell nearly at random, with a slight bias that makes the price weakly mean-revert around a perceived fundamental value V. The funds use a strategy that exploits mispricings by taking a long position (holding a net positive quantity of the asset) when the price is below V, and otherwise staying out of the market. The funds can augment the size of their long position by borrowing from a bank at an interest rate that for simplicity we fix at zero, using the asset as collateral. This borrowing is called leverage. The bank will of course be careful to limit its lending so that the value of what is owed is less than the current price of the assets held as collateral. Default occurs if the asset price falls sufficiently far before the loan comes due in the next period.

In addition to the two types of traders there is a representative investor who either invests in a fund or holds cash. The amount she invests in a given fund depends on its recent historical performance relative to a benchmark return $r^{\mathbf{b}}$. Thus successful funds attract additional capital above and beyond what they gain in the market and similarly unsuccessful funds lose additional capital.

The funds in our model are *value investors* who base their demand on a mispricing signal $m(t) = V - p(t)$, where $p(t)$ is the price of the asset at time t. The perceived fundamental value V is held constant and is the same for the noise traders and for all funds. As shown in Figure 1, each fund h computes its demand $D_h(t)$ based on the mispricing. As the mispricing increases, the dollar value of the asset the fund wishes to hold increases linearly, but the position size is capped when the fund reaches the

[2]See (Palmer 1994, Arthur 1997, Brock 1997, Brock 1998, Lux 1999, Lux 1999, Caldarelli 1997, Giardina 2003). See also (Friedman 2007), who induce bubbles and crashes via myopic learning dynamics.

[3]Another good example from the recent meltdown illustrating how individual risk regulation can create systemic risk is the use of derivatives.

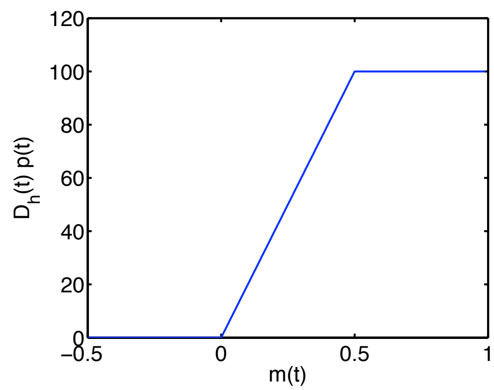

FIGURE 1: DEMAND FUNCTION $D_h(t)p(t)$ OF A FUND (MEASURED IN DOLLARS) VS. THE MISPRICING SIGNAL $m(t) = V - p(t)$.

maximum leverage. This can be written:

$$m < 0 \quad : \quad D_h = 0 \tag{1}$$

$$0 < m < m_{\text{crit}} \quad : \quad D_h p = \beta_h m W_h \tag{2}$$

$$m \geq m_{\text{crit}} \quad : \quad D_h p = \lambda_h^{\text{MAX}} W_h. \tag{3}$$

In (1) the asset is over-priced and the fund holds nothing. In (2) the asset is underpriced but the mispricing is not too large. The fund takes a position whose monetary value is proportional to the mispricing m, the fund's wealth W, and the aggression parameter β_h, which can vary from fund to fund. In (3) the asset is even more underpriced so that the fund has reached its maximum leverage $\lambda_h(t) = \lambda_h^{\text{MAX}}$. This occurs when $m \geq m_{crit} = \lambda_h^{\text{MAX}}/\beta_h$. The *leverage* λ_h is the ratio of the dollar value of the fund's asset holdings to its wealth, i.e. $\lambda_h(t) = D_h p/W_h = D_h p/(D_h p + C_t)$, where C_t is cash. When the fund is borrowing money, C_t is negative and represents the loan amount. The percent change in wealth from a loss or gain in the asset price is λ_h times the percent change in the value of the asset, hence the name "leverage". If $\lambda_h = \lambda_h^{\text{MAX}} > 1$ and then in the next period the price decreases, staying under the maximum leverage will then require the fund to sell the asset. This is called meeting a *margin call*.

At the beginning of the simulation the funds are all given the same wealth $W(0) = 2$. Their wealth automatically grows or shrinks according to the success or failure of their trading. In addition it changes due to additions or withdrawals of funds by investors. The size of the addition or withdrawal is determined by the difference between a trailing exponential average of the fund's recent performance and the benchmark return $r^{\mathbf{b}}$. If a fund's wealth goes below a critical threshold, here set to $W(0)/10$, the fund goes out of business[4], and after a period of time has passed it is replaced by a new fund with wealth $W(0)$ and the same parameters β_h and λ^{MAX}. If $W(t) < 0$ then the fund defaults. Prices are set by equating the demand of the funds plus the noise traders to the fixed supply of the asset. The details of the simulation and the fixed parameter values are given in the

[4]Using a positive survival threshold for removing funds avoids the creation of "zombie funds" that persist for long periods of time with almost no wealth.

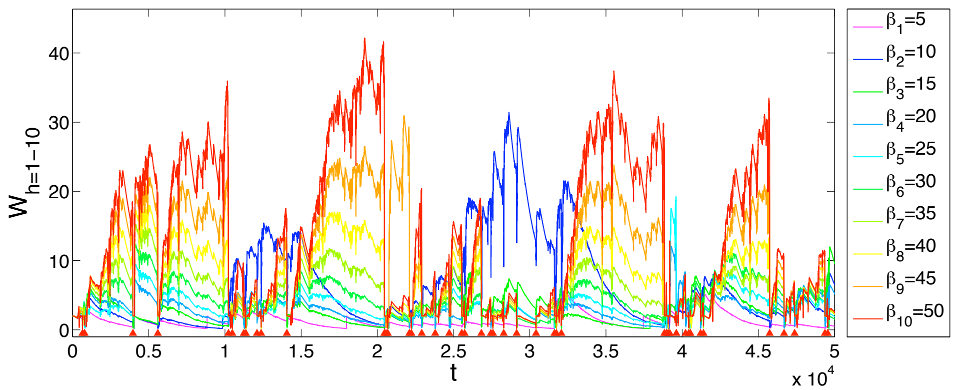

FIGURE 2: WEALTH TIMESERIES $W_h(t)$ FOR 10 FUNDS WITH $\beta_h = 5, 10, \ldots, 50$ AND $\lambda_h^{\max} = 20$ FOR ALL FUNDS. TIMES AT WHICH (AT LEAST) ONE FUND COLLAPSES ARE MARKED BY TRIANGLES.

Supplementary Material[5].

The benchmark return $r^{\mathbf{b}}$ plays an important role. If the benchmark return is set very low then funds will become very wealthy and will buy a large quantity of the asset under even small mispricings, preventing the mispricing from ever growing large. This effectively induces a hard floor on prices. If the benchmark return is set very high, funds accumulate little wealth and play a negligible role in price formation. The interesting behavior is observed at intermediate values of $r^{\mathbf{b}}$ where the funds' demand is comparable to that of the noise traders.

In Fig. 2 we illustrate the wealth dynamics for a simulation with 10 funds whose aggression parameters are $\beta_h = 5, 10, \ldots, 50$. They all begin with the same low wealth $W(0) = 2$; at the outset they make good returns and their wealth grows quickly. This is particularly true for the most aggressive funds; with higher leverage they make higher returns so long as the asset price is increasing. As their wealth grows the funds have more impact, i.e. they themselves affect prices, driving them up when they are buying and down when they are selling. This limits their profit-making opportunities and imposes a ceiling of wealth at about $W = 40$. There is a series of crashes which cause defaults, particularly for the most highly leveraged funds. Twice during the simulation, at around $t = 10,000$ and $25,000$, crashes wipe out all but the two least aggressive funds with $\beta_h = 5, 10$. While funds $\beta_3 - \beta_{10}$ wait to get reintroduced, fund β_2 manages to become dominant for extended periods of time.

The presence of the funds dramatically alters the statistical properties of price returns. This is illustrated in Fig. 3, where we compare the distribution of logarithmic price returns $r(t) = \log p(t) - \log p(t-1)$, for three cases: (1) Noise traders only. (2) Hedge funds with no leverage ($\lambda^{\mathrm{MAX}} = 1$). (3) Substantial leverage, i.e. $\lambda^{\mathrm{MAX}} = 10$. With only noise traders the log returns are (by construction) nearly normally distributed. When funds are added without leverage the volatility of prices drops slightly, but the log returns remain approximately normally distributed. When we increase leverage to $\lambda^{\mathrm{MAX}} = 10$, however, the distribution becomes much more concentrated in the center and the negative returns develop fat tails. (Recall that since the funds are long-only, they are only active when the asset is undervalued, i.e. when the mispricing $m > 0$. This

[5]The Supplementary Material can be found at http://www.santafe.edu/ jdf/SFI%20Template/PubsEconomics.html.

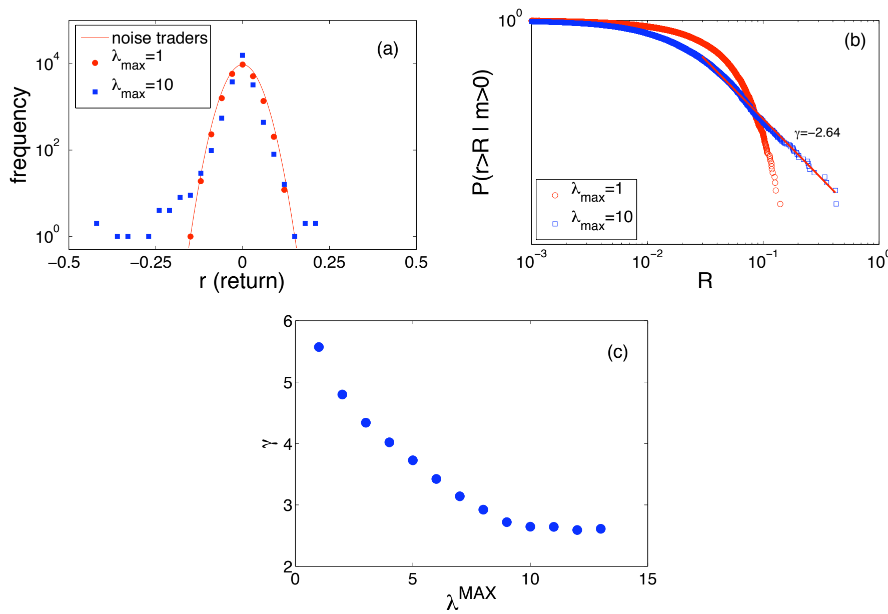

Figure 3: The distribution of log returns r. (a) plots the density of log returns $p(r|m > 0)$ on semi-log scale. The results are conditioned on positive mispricing, i.e. only when the funds are active. The unleveraged case (red circles) closely matches the noise trader only case (red curve). When the maximum leverage is raised to ten (blue squares) the body of the distribution becomes thinner but the tails become heavy on the negative side. This is seen from a different point of view in (b), which plots the cumulative distribution for negative returns, $P(r > R|m > 0)$, in log-log scale. For $\lambda^{\mathrm{MAX}} = 10$ we fit a power law to the data across the indicated region and show a line for comparison. In (c) we vary λ_{MAX} and plot fitted values of γ, illustrating how the tails become heavier as the leverage increases. We use the parameter settings described in the Supplementary material with the same β values as in Figure 1.

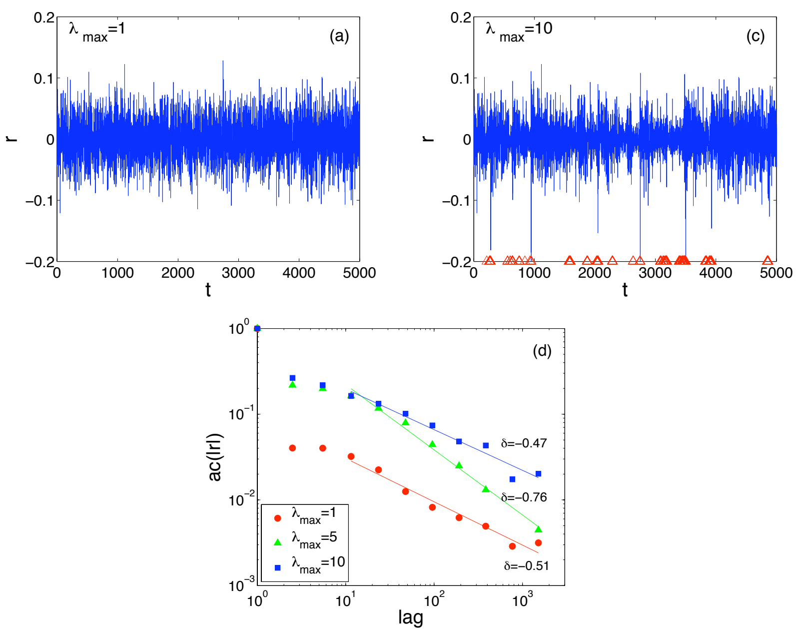

FIGURE 4: LOG-RETURN TIMESERIES (A) $\lambda^{\mathrm{MAX}} = 1$; (B) $\lambda^{\mathrm{MAX}} = 10$. TRIANGLES MARK MARGIN CALLS IN THE SIMULATION, INDICATING A DIRECT CONNECTION BETWEEN LARGE PRICE MOVES AND MARGIN CALLS. (C) AUTOCORRELATION FUNCTION OF THE ABSOLUTE VALUES OF LOG-RETURNS FOR (A-B) OBTAINED FROM A SINGLE RUN WITH $100,000$ TIMESTEPS. THIS IS PLOTTED ON LOG-LOG SCALE IN ORDER TO ILLUSTRATE THE POWER LAW TAILS. (THE AUTOCORRELATION FUNCTION IS COMPUTED ONLY WHEN THE MISPRICING IS POSITIVE.) SAME β VALUES AS IN FIGURE 1.

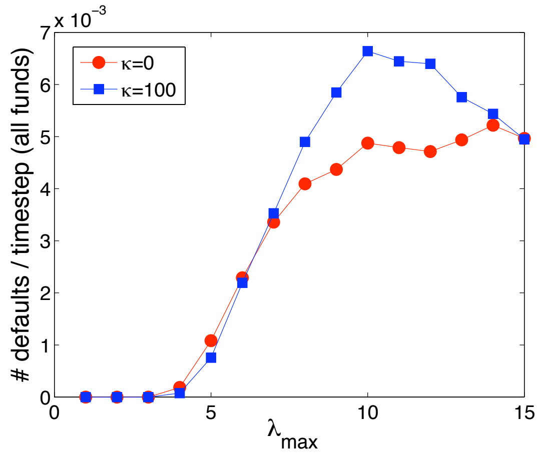

Figure 5: Comparison of the default rate with constant maximum leverage (red circles) vs. adjustable leverage based on recent historical volatility (blue squares). For details of how leverage is adjusted see Supplementary Material. 10 funds were considered with the same β values as in Figure 1.

creates an asymmetry between positive and negative returns.) As shown in Fig. 3(b), for $\lambda^{\rm MAX} = 10$ the cumulative distribution for the largest negative returns roughly follows a straight line in a double logarithmic scale, suggesting that it is reasonable to approximate the tails of the distribution as a power law, of the form $P(r > R|m > 0) \sim R^{-\gamma}$.

The exponent γ may be regarded as a measure of the concentration of extreme risks. The transition from normality to fat tails occurs more or less continuously as $\lambda^{\rm MAX}$ varies. This is in contrast to the conjecture of Plerou et al.(Plerou 1999, Gabaix 2003, Gabaix 2006, Plerou 2008) that γ has a universal value $\gamma \approx 3$. In Figure 3(c) we measure γ as a function of $\lambda^{\rm MAX}$. As $\lambda^{\rm MAX}$ increases γ decreases, corresponding to heavier tails[6]. This trend continues until $\lambda^{\rm MAX} \approx 10$, where γ reaches a floor at $\gamma \approx 2.5$. A typical value measured for financial time series, such as American stocks (Plerou 1999, Cont 2000), is $\gamma \approx 3$. In our model this corresponds to a maximum leverage $\lambda^{\rm MAX} \approx 7.5$. It is perhaps a coincidence that 7.5 is the maximum leverage allowed for equity trading in the United States, but in any case this demonstrates that the numbers produced by this model are reasonable.

In Fig. 4 we show the log-returns $r(t)$ as a function of time. The case $\lambda^{\rm MAX} = 1$ is essentially indistinguishable from the pure noise trader case; there are no large fluctuations and little temporal structure. The case $\lambda^{\rm MAX} = 10$, in contrast, shows large, temporally correlated fluctuations. The autocorrelation function shown in panel (c) is similar to that observed in real price series. This suggests that this model may also explain clustered volatility (Engle 1982).

The fat tails of price movements in our model are explained by the nonlinear positive feedback that occurs when leverage hits its maximum value. When leverage is below its maximum, funds damp volatility. Intuitively this is because they buy when the price falls, opposing and therefore damping price movements. In comparison to the volatility

[6]The value of γ when $\lambda = 1$ should be infinite, in contrast to the measured value. Large values of γ are difficult to measure correctly, whereas small values are measured much more accurately.

for noise traders alone, the expected volatility $E[r_t^2]$ is damped by a factor approximately $1/(1 + \frac{\beta}{N}(C_h + D_h V)) < 1$, where N is the total number of shares of the asset. This is shown in the Supplementary Material.

When funds reach their maximum leverage this reverses and funds instead amplify volatility. To remain below λ^{MAX} the fund is forced to sell when the price falls. The volatility in this case is amplified by a factor approximately $1/(1 - \frac{\lambda^{\mathrm{MAX}}}{N} V) > 1$. This creates a positive feedback loop: Dropping prices cause funds to sell, which causes a further drop in prices, which causes funds to sell. This is clearly seen in Fig. 4(b), where we have placed red triangles whenever at least one of the funds is at its maximum leverage. All the largest negative price changes occur when leverage is at a maximum. Thus we see that the final cause of the extreme price movements is the margin call, which funds can meet only by selling and driving prices further down. Of course we are not saying banks should not maintain leverage at a reasonable level; we are only saying that if they all maintain leverage at a similar level, many funds may make margin calls at nearly the same time, inducing an instability in prices.

In an attempt to achieve better risk control, banks often vary the maximum leverage based on the recent historical volatility of the market, lowering maximum leverage when volatility is high and raising it when it is low. This is prudent practice when lending to a single fund. But as shown in Fig. 5 this can be counterproductive when all the funds might be deleveraged at the same time. The reason for this is simple: Lowering the maximum leverage across all funds can cause massive selling at just the wrong time, creating more defaults rather than less. Once again, an attempt to improve risk control that is sensible if one bank does it for one fund can backfire and create more risk if every bank does it with every fund.

The use of leverage in the economy is not just an esoteric matter relating to funds: It is unavoidable. It is the mechanism through which most people are able to own homes and corporations do business. Credit (and thus leverage) is built into the fabric of society. The current financial crisis perfectly illustrate the dangers of too much leverage followed by too little leverage. Like Goldilocks, we are seeking a level that is "just right" (Peters 2009).

We are not the first to recognize the downward spiral of margin calls. After the Great Depression the Federal Reserve was empowered to regulate margins and leverage. The model we have developed here provides a *quantifiable* framework to explore the consequences of leverage and its regulation. Recent empirical work has found a correlation betweeen leverage and volatility (Adrian 2008), but our work suggests a more subtle relationship. We make the falsifiable prediction that high leverage limits, such as we had in reality until very recently, cause increased clustering of volatility and fat tails, and that these effects should go up and down as leverage goes up and down. We have shown that when individual lenders seek to control risk through adjusting leverage, they may collectively amplify risk. Our model can be used to search for a better collective solution, perhaps coordinated through government regulation.

At a broader level, this work shows how attempts to regulate risk at a local level can actually generate risks at a systemic level. The key element that creates the risk is the nonlinear feedack on prices that is created due to repaying loans at a bad time. This mechanism is actually quite general, and also comes into play with other risk control mechanisms, such as stop-loss orders and many types of derivatives, whenever they generate buying or selling in the same direction as price movement. We suspect that this is a quite general phenomenon, that occurs in many types of systems whenever optimization for risk reduction is done locally without fully taking collective phenomena into account.

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