Introductory Notes on the Structural and Dynamical Analysis of Networks

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AGNLD, Institute für Physik
Universität Potsdam

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What is a Network?

• A network is a (finite) collection of entities together with a specified pattern of relationships among these entities.

• Three main tools have been used for the quantitative study of networks:
  – graph theory;
  – statistical and probability theory;
  – algebraic models.
1. INTRODUCTION

Technological Networks

World-Wide Web

Internet

Power Grid
1. INTRODUCTION

Social Networks

Friendship Net

Movie Actors

Citation Networks

Sexual Contacts

Collaboration Networks
1. INTRODUCTION

Transportation Networks

Airport Networks

Local Transportation

Road Maps
1. INTRODUCTION

**Biological Networks**

- Ecological Webs
- Neural Networks
- Genetic Networks
- Metabolic Networks

- Protein interaction
**GOAL:** A unified approach enabling analysis of the connection topology underlying a wide variety of Complex Systems

Example: Food Web
Graphical Approach: Vertices and Edges

Example: Simple graph $G$ with Vertex Set $V(G)=\{V_1,\ldots,V_8\}$

$A_{ij} = 1$ iff $(i,j)$ is in the Edge Set $E(G)$

Symmetrical Adjacency Matrix $A$ for the Simple Graph $G$
Graphical Approach...

Simple Graph

Directed Graph G (DiGraph)

A_{ij} = 1 \text{ iff } (i,j) \text{ is in the edge set } E(G)

Non-Symmetrical Adjacency Matrix A for the DiGraph

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2. NETWORKS...

Graphical Approach...

Simple Graph

DiGraph

Weighted DiGraph
Structural Characterization

Vertex Degree: \( k(v) \)

Simple Graph

e.g. Trade Network

\( k(\bullet) = 3 \)
2. NETWORKS...

Structural Characterization…

Clustering Coefficient: $C(v)$

Simple Graph

e.g. Trade Network
Structural Characterization...

Clustering Coefficient: $C(v)$

- Degree of vertex $v$ (number of directly connected vertices): $k(v) = 3$
2. NETWORKS...

Structural Characterization...

Clustering Coefficient: $C(v)$

- Degree of vertex $v$: $k(v) = 3$
- Total number of possible connections among these 3 neighbors:
  \[
  \frac{1}{2} \cdot k(v) \cdot [k(v) - 1] = \frac{1}{2} \cdot [3 \cdot 2] = 3
  \]
Structural Characterization...

Clustering Coefficient: $C(v)$

- Number of actual connections among the three neighbors $= 1$
- Total number of possible connections: $\frac{1}{2} \cdot k(v) \cdot [k(v) - 1] = \frac{1}{2} \cdot [3 \cdot 2] = 3$
- $C(v) = \frac{1}{3} = 0.33333$
- Measures how well my neighbors are connected to each other!
2. NETWORKS...

Structural Characterization…

Simple *Connected* Graph

e.g. Trade Network

“Distance” vi to vj?
2. NETWORKS...

**Structural Characterization** …

Simple Connected Graph

Length of this path $v_i$ to $v_j = 4$

e.g. Trade Network
Simple Connected Graph

Length of this path $v_i$ to $v_j = 3$

e.g. Trade Network
2. NETWORKS...

Structural Characterization...

Distance \( v_i \) to \( v_j \) = \text{Shortest} path length \( v_i \) to \( v_j \), here equal to 3

Simple Connected Graph

e.g. Trade Network
2. NETWORKS...

**Structural Characterization...**

Simple Connected Graph

Distance from vertex $v_i$ to each other vertex $v$?

- e.g. Trade Network

- $v_i$
2. NETWORKS...

**Structural Characterization...**

Simple Connected Graph

Distance-1 Vertices from Vertex vi

e.g. Trade Network
2. NETWORKS...

Structural Characterization...

Simple Connected Graph

Distance-2 Vertices from Vertex $v_i$

e.g. Trade Network
2. NETWORKS...

Characterization

Distance 3-Vertices from Vertex vi

Simple Connected Graph

Distance $L_{ij}$: Length of the shortest path(s) from vi to vj
L(G) = Characteristic Path Length of Graph G

- All-to-all distance matrix:

\[ L_{ij} \] Length of the shortest path(s)

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\[ L(G) \] = Average of \( L_{ij} \) over all vertices \( v_i \) and \( v_j \) \( (i \neq j) \) in \( V(G) = 1.94 \)
E-R Random Graph Model

Paul Erdös & Alfréd Rényi *(Hungarian Academy of Sciences, 1960)*:

Start with a collection of \( N \) unconnected vertices.

Then, for each distinct pair of vertices, connect them by an edge with probability \( p \).

Denote the resulting graph as \( G = G(N,p) \).
2. NETWORKS...

E-R Random Graph Model…Continued

- Degree distribution: $P_G(k)$

$P_G(k) = \text{Probability that a randomly selected vertex in G will have degree } k$

$P_G(k) \sim \left[ \frac{e^{-z} z^k}{k!} \right]$ for $G=G(N,p)$

where $z = \text{mean } k$ (depends on $N,p$)
2. NETWORKS...

Graph G for a Regular Ring Lattice

- Regular = Every vertex has the same degree
- \(|V(G)| = \text{No. of Vertices} = 16\)
- Degree \(k = 4\)
- Clustering: \(C(G) = \frac{1}{2}\)
- Characteristic Path Length:
  \[ L(G) = \frac{36}{15} = \frac{12}{5} \]

**Small-World Network (SWN) Models**

Construction of SWN $G(p)$, $0 \leq p \leq 1$

Choose a vertex $v$ and edge $e^*$ that connects $v$ to its nearest neighbor $v^*$ in clockwise direction.

With probability $p$, reconnect edge to a vertex $v^{**}$ chosen uniformly at random over the ring but with duplicate edges forbidden.

Continue process clockwise around ring until 1 lap is complete.
SWN Models...Continued

Watts-Strogatz 1998: Construction of Small-World Network $G(p)$

Next consider edges $e'$ at distance 2 from each $v$ in clockwise direction, and randomly rewire with probability $p$.

Moving clockwise, complete a full lap of distance-2 rewiring.

In general, for a ring of any even degree $k$, successively rewire ALL edges with probability $p$ by completing $k/2$ laps around the ring.

Rewired edges are called "SHORT-CUTS"
2. NETWORKS...

SWN Models…Continued

Watts-Strogatz 1998: Construction of Small-World Network $G(p)$

- For a range of $p$’s with $0 < p < 1$, the SWN $G(p)$ is characterized by
  - High clustering $C(p)/C(0)$
  - Short path length $L(p)/L(0)$

2. NETWORKS...

SWN Models...Continued

Albert-Lázló Barabási (A-B) **Scale-Free Network** *(Science, 1999)*:

- At each step add new vertex \( v \) to graph and connect it to 2 randomly selected existing vertices \( v_i \) using “preferential attachment” prob’s

\[
p_i = \frac{k_i}{\sum_j k_j} = \text{Prob}(v_i)
\]

- Results:
  - “Richer-Get-Richer”
  - \( P_G(k) \sim k^{-3} \) (Power Law = Scale Free)
## Properties of the Network Models

<table>
<thead>
<tr>
<th></th>
<th>Regular</th>
<th>WS Small-World</th>
<th>AB Small-World</th>
<th>E-R Random</th>
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<tbody>
<tr>
<td>Path length</td>
<td>Long</td>
<td>Short</td>
<td>Short</td>
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<td>Clustering</td>
<td>Large</td>
<td>Large</td>
<td>Large</td>
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Small-world networks fall “between” regular and E-R random networks!
2. NETWORKS...

Properties of the Network Models...

Regular Lattice  A-B Scale-Free SWN  E-R Random Graph

\[ \mathbb{P}_G(k) = \delta(k - k_{\text{true}}) : \] Delta Function equals 1 at true degree \( k \) and 0 elsewhere

\[ \mathbb{P}_G(k) \sim k^{-3} \] power law

\[ \mathbb{P}_G(k) \sim \left[ e(-z)z^k \right]/k! \] thick tailed

\( z = \text{mean } k \)
Small-World Nets: Robustness to Shocks

• **Network Resilience:**
  
  – Highly robust against RANDOM failures of vertices \( v \)
Small-World Nets: Significant Impacts

- **Network Resilience:**
  - Highly robust against RANDOM failures of vertices v
2. NETWORKS...

Small-World Nets: Significant Impacts

- **Network Resilience:**
  - Highly robust against RANDOM failures of vertices
  - **BUT** highly vulnerable to deliberate attack on HUBS (v’s having a relatively high degree k)
2. NETWORKS...

Small-World Nets: Significant Impacts

• **Network Resilience:**
  
  – Highly robust against RANDOM failures of vertices
  
  – **BUT** highly vulnerable to deliberate attack on HUBS
So how well do YOU know Kevin Bacon?

- **Small-World Effect** = Hypothesis that every two people in the world are connected by a surprisingly short chain of social acquaintances.

- **Example:** The trivia game *Six Degrees of Kevin Bacon*
Six Degrees of Kevin Bacon…

- Name taken from 1990 stage play by American playwright John Guare: *Six Degrees of Separation*

- Play loosely based on 1967 small-world experiment by Stanley Milgrom suggesting random pairs of U.S. citizens were connected on average by a chain of six social acquaintances (people on a first-name basis).

- Pick any **film actor A**, then try to link this actor to Bacon via a chain of films.

- Actor set for first film in chain must include A, each successive film must include an actor from previous film, and final film must include Bacon among its actors.
Six Degrees of Kevin Bacon....

Example: (from Wikipedia, accessed 4/8/07)

- **Elvis Presley** was in *Change of Habit* (1969) with Edward Asner
- Edward Asner was in *JFK* (1991) with Kevin Bacon
- Therefore Elvis Presley has a Bacon Number = 2.
2. NETWORKS...

What’s the average distance between Kevin Bacon and all other actors? (from Albert-Lázló Barabási, www.nd.edu/~networks)

<table>
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<th>Rank</th>
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<th># of movies</th>
<th># of links</th>
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... 876 Kevin Bacon 2.786981 46 1811 ...

***Is Kevin Bacon the most connected actor?***

**NO!**