A New Differential Equation Method for Finding the Perron Root of a Positive Matrix*

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ABSTRACT

A basic problem in linear algebra is the determination of the largest eigenvalue (Perron root) of a positive matrix. In the present paper a new differential equation method for finding the Perron root is given. The method utilizes the initial value differential system developed in a companion paper for individually tracking the eigenvalue and corresponding right eigenvector of a parametrized matrix.

1. INTRODUCTION

A basic problem in linear algebra [1-4] is the determination of the largest eigenvalue of an \( n \times n \) matrix \( Q \). When the entries of \( Q \) are nonnegative, the largest eigenvalue is referred to as the Perron or Perron-Frobenius root.

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In two companion papers [5, 6] it is shown how the eigenvalues and the right and left eigenvectors of a parametrized matrix $M(\alpha)$ can be tracked as functions of a scalar parameter $\alpha$ by integrating a system of ordinary differential equations from initial conditions.$^1$ In the present paper it is shown how the initial value differential system developed in Ref. [6] can be modified to obtain an initial value differential system for tracking the Perron root and a corresponding unit normalized right eigenvector for a positive matrix $Q$, i.e., a matrix $Q$ with all positive elements.

For completeness, the initial value system developed in Ref. [6] is outlined in Sec. 2. The modified initial value system for tracking the Perron root is developed in Sec. 3. A numerical example is given in the final Sec. 4.

2. INDIVIDUAL TRACKING OF AN EIGENVALUE AND EIGENVECTOR

Let $M(\alpha)$ be an $n \times n$ complex matrix-valued continuously differentiable function of a parameter $\alpha$ varying over a simply connected region of the complex plane. In a companion paper [6] it is shown that an initial value system can be developed for tracking a single eigenvalue $\lambda(\alpha)$ and corresponding unit normalized right eigenvector $x(\alpha)$ of $M(\alpha)$ if, at the initial point $\alpha^0$, a certain Jacobian matrix $J(\alpha^0)$ is nonsingular.

Specifically, the differential system takes the form

$$\begin{bmatrix} \dot{x}(\alpha) \\ \dot{\lambda}(\alpha) \end{bmatrix} = \frac{A(\alpha) + \delta(\alpha)}{\delta(\alpha)} \begin{bmatrix} M(\alpha)x(\alpha) \\ 0 \end{bmatrix},$$

with initial conditions

$$x(\alpha^0) = x^0,$$

$$\lambda(\alpha^0) = \lambda^0,$$

$$A(\alpha^0) = \text{Adj}(J(\alpha^0)),$$

$$\delta(\alpha^0) = \text{Det}(J(\alpha^0)).$$

$^1$The right and left eigenvectors of a given $n \times n$ matrix $M$ corresponding to an eigenvalue $\lambda$ are defined to be the nontrivial solutions $x$ and $w^T$ to $Mx = \lambda x$ and $w^TM = \lambda w^T$, respectively, where superscript $T$ denotes transpose.
where

\[
J(\alpha) \equiv \begin{bmatrix}
\lambda(\alpha)I - M(\alpha) & x(\alpha) \\
x(\alpha)^T & 0
\end{bmatrix},
\]

and a dot denotes differentiation with respect to \( \alpha \).

The initial conditions (1d)–(1g) are obtained by solving

\[
M(\alpha^0)x = \lambda x, \\
x^T x = 1
\]

for \( x^0 \) and \( \lambda^0 \), and then determining the adjoint \( A(\alpha^0) \) and determinant \( \delta(\alpha^0) \) of the Jacobian matrix

\[
J(\alpha) \equiv \begin{bmatrix}
\lambda^0I - M(\alpha^0) & x^0 \\
x^{0T} & 0
\end{bmatrix}
\]

for the system (2).

3. VARIATIONAL EQUATIONS FOR THE PERRON ROOT

The presently proposed procedure for determining the Perron root \( \lambda_p \) of an arbitrary \( n \times n \) positive matrix \( Q \) is as follows. Define

\[
M(\alpha) \equiv [1 - \alpha]C + \alpha Q, \quad 0 < \alpha < 1,
\]

where

\[
C = \begin{bmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & 1
\end{bmatrix}_{n \times n}
\]

The Perron root \( \lambda(0) \) of \( M(0) = C \) is easily determined to be \( \lambda(0) = n \), and the unit length normalized right eigenvector corresponding to \( \lambda(0) \) may be taken to be \( x(0)^T = (1/\sqrt{n}, \ldots, 1/\sqrt{n}) \). Clearly the Perron root \( \lambda(1) \) of \( M(1) \) is the
desired Perron root \( \lambda_p \) of \( Q \). Thus, in principle, the Perron root \( \lambda_p \) of \( Q \) may be found by integrating system (1) from \( \alpha = 0.0 \) to \( \alpha = 1.0 \) using the initial conditions

\[
\begin{align*}
x(0)^T &= \left( 1/\sqrt{n}, \ldots, 1/\sqrt{n} \right), \\
\lambda(0) &= \lambda, \\
A(0) &= \text{Adj}(J(0)), \\
\delta(t) &= \text{Det}(J(0)),
\end{align*}
\]  

where

\[
J(0) = \begin{bmatrix}
\lambda(0)I - M(0) & x(0) \\
x(0)^T & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
nI - C & x(0) \\
x(0)^T & 0
\end{bmatrix}.
\]  

Due to the nonlinearity of the differential equation system (1), the integration may be stopped prior to reaching \( \alpha = 1.0 \). However, as the following theorem demonstrates, the Jacobian matrix

\[
J(\alpha) = \begin{bmatrix}
\lambda(\alpha)I - M(\alpha) & x(\alpha) \\
x(\alpha)^T & 0
\end{bmatrix}
\]

is nonsingular for \( 0 < \alpha < 1 \), where \( \lambda(\alpha) \) is the Perron root of \( M(\alpha) \) and \( x(\alpha) \) is a corresponding unit normalized right eigenvector with all components taken to be positive. Thus, \( \delta(\alpha) = \text{Det}(J(\alpha)) \) is uniformly bounded away from zero over the compact interval \([0, 1]\).

**Theorem.** Let \( \lambda_p \) be the Perron root of a positive \( n \times n \) matrix \( M \), and let \( x_p \) be a corresponding right eigenvector with all elements taken to be positive. Then

\[
\text{Det} \begin{pmatrix}
\lambda_p I - M & x_p \\
x_p^T & 0
\end{pmatrix} \neq 0.
\]
Proof. It is well known that the dominant (Perron) root $\lambda_p$ of a positive $n \times n$ matrix $M$ is positive and simple, and that the elements of the right and left eigenvectors $x_p$ and $u_p$ corresponding to $\lambda_p$ can be taken to be positive.

Define $N \equiv \lambda_p I - M$. Suppose (12) is false, i.e., suppose there exists a non-zero vector $(y, s)^T$ satisfying

$$
\begin{pmatrix}
N & x_p \\
x_p^T & 0
\end{pmatrix}
\begin{pmatrix}
y \\
s
\end{pmatrix}
= 0.
$$

Then

$$
Ny + x_p s = 0,
$$

(14a)

$$
x_p^T y = 0.
$$

(14b)

Multiplying through (14a) by $u_p^T$, one obtains $u_p^T Ny + u_p^T x_p s = 0$, which in turn implies $s = 0$, since $u_p^T N = 0$ and $u_p^T x_p > 0$. It follows from (14) that $Ny = 0$ and $x_p^T y = 0$. However, since $\lambda_p$ is simple, the right eigenvector $x_p$ is unique up to positive linear transformation. Thus $y = 0$, a contradiction.

It follows from this proof by contradiction that the condition (12) must be true. \qed

4. AN ILLUSTRATIVE NUMERICAL EXAMPLE

Consider the $2 \times 2$ positive matrix $Q$ defined by

$$
Q \equiv \begin{bmatrix}
1 & 2 \\
4 & 5
\end{bmatrix}.
$$

(15)

It is easily verified that, to six decimal places, the Perron root of $Q$ is

$$
\lambda_p = 6.464102,
$$

(18)

with corresponding positive unit normalized right eigenvector given by

$$
x_p = \begin{bmatrix}
u_p \\
v_p
\end{bmatrix} = \begin{bmatrix}
0.343724 \\
0.939071
\end{bmatrix}.
$$

(17)
To find the Perron root of $Q$ by means of the differential equation method developed in Sec. 3, we first define

$$M(\alpha) \equiv (1-\alpha)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \alpha Q, \quad 0 < \alpha < 1. \quad (18)$$

The appropriate initial value system (1) for the problem at hand then reduces to

$$\begin{bmatrix} \dot{u}(\alpha) \\ \dot{v}(\alpha) \\ \dot{\lambda}(\alpha) \end{bmatrix} = \frac{A(\alpha)}{\delta(\alpha)} \begin{bmatrix} v(\alpha) \\ 3u(\alpha) + 4v(\alpha) \\ 0 \end{bmatrix}, \quad (19a)$$

$$\dot{\delta}(\alpha) = \frac{A(\alpha)\text{Trace}(A(\alpha)B(\alpha)) - A(\alpha)B(\alpha)A(\alpha)}{\delta(\alpha)}, \quad (19b)$$

$$\delta(0) = \text{Trace}(A(\alpha)B(\alpha)), \quad (19c)$$

with initial conditions

$$u(0) = \frac{1}{\sqrt{2}}, \quad (19d)$$

$$v(0) = \frac{1}{\sqrt{2}}, \quad (19e)$$

$$\lambda(0) = 2, \quad (19f)$$

$$A(0) = \text{Adj}(J(0)), \quad (19g)$$

$$\delta(0) = \text{Det}(J(0)), \quad (19h)$$

where

$$J(0) = \begin{bmatrix} 1 & -1 & 1/\sqrt{2} \\ -1 & 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \quad (19i)$$

and

$$B(\omega) = \frac{\dot{J}(\alpha)}{J(\alpha)} = \begin{bmatrix} \dot{\lambda}(\alpha) & -1 & \dot{u}(\alpha) \\ -3 & \dot{\lambda}(\alpha) - 4 & \dot{v}(\alpha) \\ \dot{u}(\alpha) & \dot{v}(\alpha) & 0 \end{bmatrix}. \quad (19j)$$
TABLE 1

**PERRON EIGENVALUE \( \lambda(\alpha) \) AND CORRESPONDING RIGHT EIGENVECTOR \( x(\alpha)^T = (u(\alpha)v(\alpha)) \) OF \( M(\alpha) \), AND DETERMINANT \( \delta(\alpha) \) OF \( J(\alpha) \)**

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( u(\alpha) )</th>
<th>( v(\alpha) )</th>
<th>( \lambda(\alpha) )</th>
<th>( \delta(\alpha) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.707107</td>
<td>0.707107</td>
<td>2.0</td>
<td>-2.0</td>
</tr>
<tr>
<td>0.1</td>
<td>0.614441</td>
<td>0.788963</td>
<td>2.41244</td>
<td>-2.42487</td>
</tr>
<tr>
<td>0.2</td>
<td>0.548506</td>
<td>0.837912</td>
<td>2.84222</td>
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</tr>
<tr>
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</tr>
<tr>
<td>0.4</td>
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<td>0.889731</td>
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<tr>
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</tr>
<tr>
<td>0.6</td>
<td>0.403041</td>
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</tr>
<tr>
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<td>0.383904</td>
<td>0.923373</td>
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</tr>
<tr>
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<td>5.54618</td>
<td>-5.89237</td>
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</tr>
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<td>1.0</td>
<td>0.343724</td>
<td>0.939071</td>
<td>6.46140</td>
<td>-6.92830</td>
</tr>
</tbody>
</table>

*The Perron \( \lambda \) of \( Q \) is given by \( \lambda(\alpha) \) at \( \alpha = 1 \).

A fourth-order Runge-Kutta method was used to integrate the system (19) from \( \alpha = 0 \) to \( \alpha = 1 \), with grid intervals equal to 0.01. As indicated in Table 1, the Perron root \( \lambda(1) \) obtained for \( M(1) = Q \) agrees to at least six digits with the analytically derived Perron root \( \lambda_p \) of \( Q \) given by (16).

REFERENCES