Solution Analysis Techniques for General Parameterized Nonlinear Systems

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Solution Analysis for Parameterized Systems of Equations

- Consider a system of equations $S(\lambda)$ characterized by a parameter vector $\lambda$ in a parameter space $\Lambda$.
- Suppose $S(\lambda)$ has a unique solution $z(\lambda)$ for each $\lambda$ in $\Lambda$.
- Let $F: \Lambda \rightarrow Z$ denote the mapping $\lambda \rightarrow z(\lambda)$, where $Z = \{ z(\lambda) : \lambda \in \Lambda \}$.
- Let $\text{Par}(Z) = \{Z', \ldots\}$ denote any given partition of $Z$.
- $\text{Par}(Z)$ induces a partition $\text{Par}(\Lambda) = \{\Lambda', \ldots\}$ of $\Lambda$, where $\Lambda' = \{\lambda \in \Lambda : z(\lambda) \in Z'\}$ for each $Z'$ in $\text{Par}(Z)$.
- Define a function $G: \text{Par}(Z) \rightarrow \text{Par}(\Lambda)$ by $G(Z') = \Lambda'$.
Applications

❑ Multiparametric Programming (MPP)

For each $\lambda$ in $\Lambda$, the solution $z(\lambda)$ for a system of equations $S(\lambda)$ is an optimal solution for a programming problem parameterized by $\lambda$. Any partition of the solution space $Z$ based on solution attribute differences induces a corresponding partition of the parameter space $\Lambda$.

**Example: Solution state partitioning based on binding inequality constraints**


**Illustration:** The DC Optimal Power Flow (OPF) solutions for a two-bus grid are parameterized by the fixed load vector $(L_1, L_2)$ for buses 1 and 2. The partitioning of the solution space based on differences in the set of binding inequality constraints for transmission line and generation capacities induces a partition of the parameter space into nine *System Pattern Regions (SPRs)*. **Source:** Fig. 3 in Ch. 2 of DY Heo, PhD Thesis, Iowa State U, 2015, illustrating the basic SPR approach developed by Zhou et al. (2011).
Applications ... Continued

Multiparametric Equilibrium Analysis

For each $\lambda$ in $\Lambda$, the solution $z(\lambda)$ for a system of equations $S(\lambda)$ is an equilibrium solution for a dynamic system parameterized by $\lambda$. Any partition of the solution space $Z$ based on solution attribute differences induces a corresponding partition of the parameter space $\Lambda$.

Example: Solution space partitioning based on decision-maker behaviors


Illustration: The equilibrium solutions for a dynamic overlapping generations model with 3-period lived agents are parameterized by the initial good endowment $\varpi$ of young agents and a government tax rate $T$, among other parameters. The partition of these equilibria by interest rate $i$ outcome (zero or positive) and the money-holding behavior (zero or positive) of young and middle-aged agents induces a partition of the $(T, \varpi)$ parameter space, all else equal. **Source:** Fig. 6 in L. Tesfatsion, “Macro Implications...,” *Journal of Public Economics* 19(2), Nov. 1982, 139-169.
Applications ...Continued

❑ Chaos and Fractal Studies

For each $\lambda$ in $\Lambda$, the corresponding $z(\lambda)$ is the solution for a nonlinear dynamic system $S(\lambda)$ parameterized by $\lambda$. Any partition of the solution space $Z$ based on differences in asymptotic solution properties induces a corresponding partition of the parameter space $\Lambda$.

Example: The Mandelbrot Set $M$

Let $C$ denote the complex plane. For each $\lambda$ in $C$, define a nonlinear dynamic system $S(\lambda)$ as follows:

$$x_{n+1} = [x_n]^2 + \lambda, \quad \text{for } n = 0, 1, ...$$

$$x_0 = \lambda$$

Let $z(\lambda) = \{x_0(\lambda), x_1(\lambda), x_2(\lambda), ...\}$ denote the solution for $S(\lambda)$. Partition $C$ into two subsets:

$C_1 = \{\text{all } \lambda \text{ in } C \text{ for which } x_n(\lambda) \text{ does NOT diverge to infinity as } n \text{ approaches infinity}\}$.

$C_2 = \{\text{all } \lambda \text{ in } C \text{ that are not an element of } C_1\}$

The $\textit{Mandelbrot Set } M = C_1$
Mandelbrot Set Depiction

Beautiful computer-generated plots of M can be created by coloring \textit{nonmember} points \( \lambda \) in \( \mathbb{C} \) in graded fashion, depending on how quickly the magnitude of the elements in their corresponding divergent solution sequences \( z(\lambda) \) reach a user-specified number \( R \geq 2 \). As one “zooms into” M at ever greater resolution, incredibly detailed structure appears; see https://en.wikipedia.org/wiki/Mandelbrot_set

The following figure shows the “boundary points” of the Mandelbrot set colored in blue, where these boundary points are \( \lambda \) points for which the dynamical behavior of \( z(\lambda) \) substantially changes. This figure was simulated on a computer by Wolfram Inc. [1] using the following steps: (1) Choose a (large) \textbf{max} number \( N \) of iterations \( n \); (2): Set \( R = 2 \); (3): Color a \( \lambda \) point blue if \( |x_n(\lambda)| \leq 2 \) for all \( n \leq N \).

Solution Tracking for General Parameterized Nonlinear Systems


**Abstract:** This article presents and illustrates the NASA program for the Nonlocal Automated Sensitivity Analysis of nonlinear systems $H(x, \lambda) = 0$ parametrized by a vector $\lambda$. The NASA program incorporates automated procedures for initialization, derivative evaluation, and tracking of solutions $x(\lambda)$ along any fixed or adaptively-generated path for $\lambda$ for which ill-conditioning of the solution does not arise. This tracking can also be used to identify regions of the parameter space where ill-conditioning arises, such as neighborhoods of singularities or bifurcation points.
Tracking of Eigenvalues and Eigenvectors for General Parameterized Matrices

*Example:* [http://www2.econ.iastate.edu/tesfatsi/VariationalEquationsEigen.LT.pdf](http://www2.econ.iastate.edu/tesfatsi/VariationalEquationsEigen.LT.pdf)


**Abstract:** This article develops a complete system of ordinary differential equations for tracking the eigenvalues and right and left eigenvectors of *nonsymmetric* parameterized matrices $M(\lambda)$ along any fixed or adaptively-generated path for the parameter vector $\lambda$ for which matrix ill-conditioning does not arise. A simpler reduced form of the ODE system is then derived for tracking the eigenvalues and eigenvectors of *symmetric* parameterized matrices. This tracking can also be used to identify regions in parameter space where matrix ill-conditioning arises, such as neighborhoods of singularities or bifurcation points. The feasibility and accuracy of the tracking method are illustrated by numerical examples.