Solution Analysis Techniques for General Parameterized Nonlinear Systems

Leigh Tesfatsion
Research Professor
Iowa State University
Ames, IA 50011
6 September 2020

Solution Analysis for Parameterized Systems of Equations

- Consider a system of equations S(λ) characterized by a parameter vector λ in a parameter space Λ.
- Suppose S(λ) has a unique solution z(λ) for each λ in Λ.
- Let F: Λ → Z denote the mapping λ → z(λ), where Z = { z(λ) : λ in Λ }.
- Let Par(Z) = {Z’, …} denote any given partition of Z.
- Par(Z) induces a partition Par(Λ) = {Λ’, … } of Λ, where Λ’= {λ in Λ : z(λ) in Z’} for each Z’ in Par(Z).
- Define a function G: Par(Z) → Par(Λ) by G(Z’) = Λ’ .

$$
\begin{tikzcd}
\mathrm{Par}(Z) \arrow[r, "G"] & \mathrm{Par}(\Lambda) \arrow[d, "U\ \text{(union of subsets)}"] \\
Z \arrow[u, "\text{given partition}"] & \Lambda \arrow[l, "F"]
\end{tikzcd}
$$

Applications

❑ Multiparametric Programming (MPP)

For each λ in Λ, the solution z(λ) for a system of equations S(λ) is an optimal solution for a programming problem parameterized by λ. Any partition of the solution space Z based on solution attribute differences induces a corresponding partition of the parameter space Λ.

Example: Solution state partitioning based on binding inequality constraints

- X. Geng, L. Xie (2017), Learning the LMP-Load Coupling from Data: A Support Vector Machine Approach, *IEEE Transactions on Power Systems* 32(2), March, 1127-1138. https://doi.org/10.1109/TPWRS.2016.2572138
- Q. Zhou, L. Tesfatsion, C-C Liu (2011), Short-Term Congestion Forecasting in Wholesale Power Markets, *IEEE Transactions on Power Systems* 26(4), November, 2185-2196. https://doi.org/10.1109/TPWRS.2011.2123118

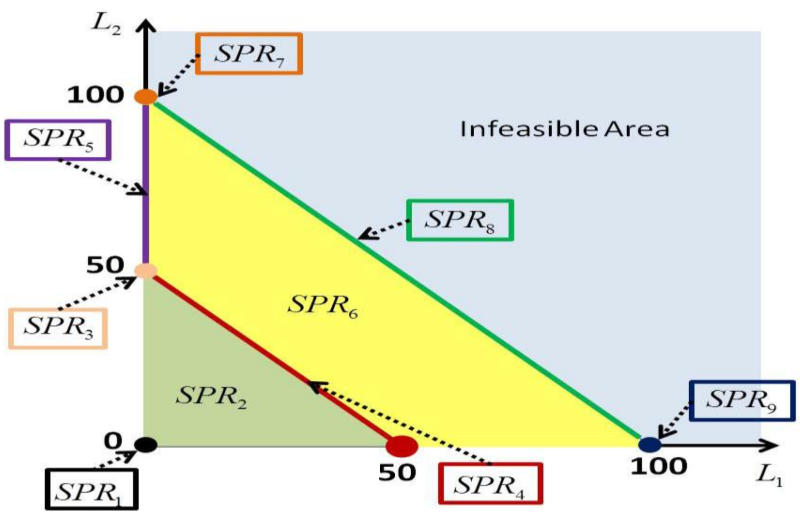


Illustration: The DC Optimal Power Flow (OPF) solutions for a two-bus grid are parameterized by the fixed load vector (L_1,L_2) for buses 1 and 2. The partitioning of the solution space based on differences in the set of binding inequality constraints for transmission line and generation capacities induces a partition of the parameter space into nine *System Pattern Regions (SPRs)*. ***Source:*** Fig. 3 in Ch. 2 of DY Heo, PhD Thesis, Iowa State U, 2015, illustrating the basic SPR approach developed by Zhou et al. (2011).

Applications … Continued

❑ Multiparametric Equilibrium Analysis

For each λ in Λ, the solution z(λ) for a system of equations S(λ) is an equilibrium solution for a dynamic system parameterized by λ. Any partition of the solution space Z based on solution attribute differences induces a corresponding partition of the parameter space Λ.

Example: Solution space partitioning based on decision-maker behaviors

— Leigh Tesfatsion (1982), "Macro Implications of Government Redistributive Tax-Transfer Policies," *Journal of Public Economics*, Vol. 19, Issue 2, November, 139-169

http://www2.econ.iastate.edu/tesfatsi/MacroImplicationsRedistributivePoliciesOG.JPubE1982.LT.pdf

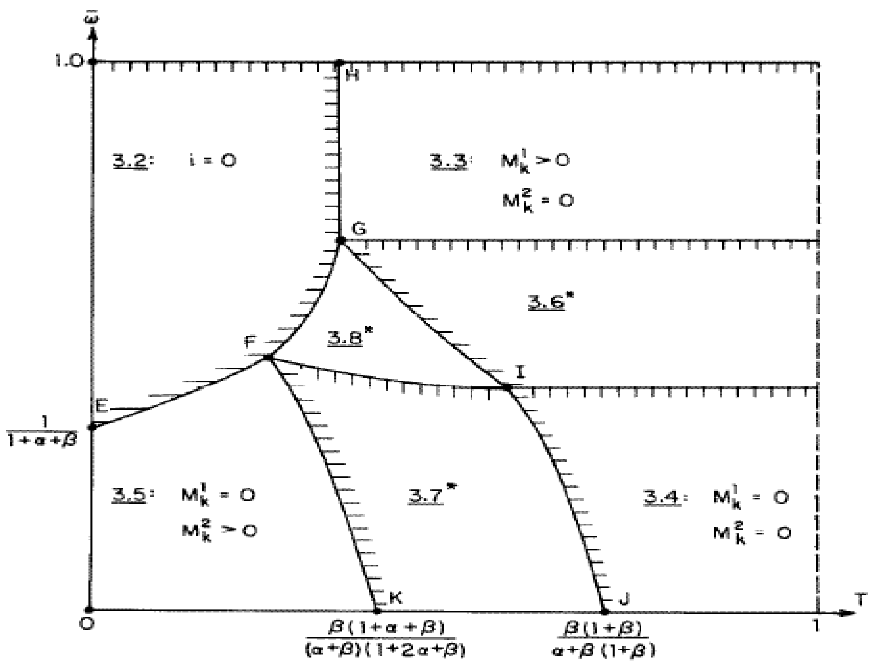


Fig. 6. Existence of equilibria with $S_1>0$ and $S_2=\bar{B}=0$ for the case $N_1=N_2$, $\omega_1^1=\omega_2^1\equiv\bar{\omega}$, and $\omega_1^2=\omega_2^2=1-\bar{\omega}$.

Illustration: The equilibrium solutions for a dynamic overlapping generations model with 3-period lived agents are parameterized by the initial good endowment ϖ of young agents and a government tax rate T, among other parameters. The partition of these equilibria by interest rate i outcome (zero or positive) and the money-holding behavior (zero or positive) of young and middle-aged agents induces a partition of the (T, ϖ) parameter space, all else equal. ***Source:*** Fig. 6 in L. Tesfatsion, "Macro Implications…," *Journal of Public Economics* 19(2), Nov. 1982, 139-169.

Applications …Continued

❑ Chaos and Fractal Studies

For each λ in Λ, the corresponding z(λ) is the solution for a nonlinear dynamic system S(λ) parameterized by λ. Any partition of the solution space Z based on differences in asymptotic solution properties induces a corresponding partition of the parameter space Λ.

Example: The Mandelbrot Set M

Let C denote the complex plane. For each λ in C, define a nonlinear dynamic system S(λ) as follows:

$$x_{n+1} = [\, x_n \,]^2 + \lambda \,, \quad \text{for } n = 0, 1, \ldots$$

$$x_0 = \lambda$$

Let z(λ) = {$x_0(\lambda)$, $x_1(\lambda)$, $x_2(\lambda)$, … } denote the solution for S(λ). Partition C into two subsets:

C1 = { all λ in C for which $x_n(\lambda)$ does NOT diverge to infinity as n approaches infinity} .

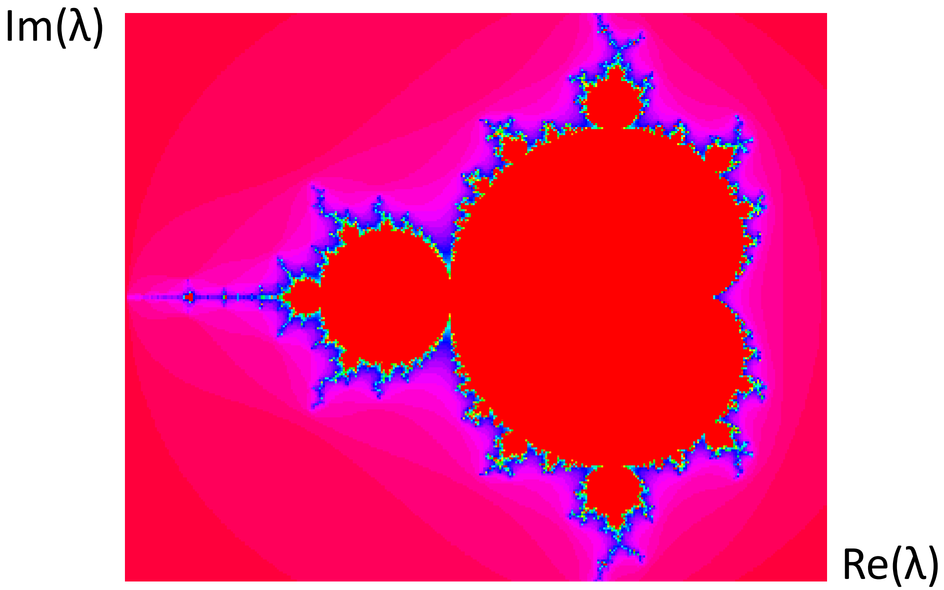
C2 = { all λ in C that are not an element of C1}

The *Mandelbrot Set M* = C1

Mandelbrot Set Depiction

Beautiful computer-generated plots of M can be created by coloring *nonmember* points λ in C2 in graded fashion, depending on how quickly the magnitude of the elements in their corresponding divergent solution sequences z(λ) reach a user-specified number R ≥ 2. As one "zooms into" M at ever greater resolution, incredibly detailed structure appears; see https://en.wikipedia.org/wiki/Mandelbrot_set

The following figure shows the "boundary points" of the Mandelbrot set colored in blue, where these boundary points are λ points for which the dynamical behavior of z(λ) substantially changes. This figure was simulated on a computer by Wolfram Inc. [1] using the following steps: (1) Choose a (large) max number N of iterations n; (2): Set R = 2; (3): Color a λ point blue if $|x_n(\lambda)| \leq 2$ for all $n \leq N$.



[1] https://mathworld.wolfram.com/MandelbrotSet.html

Solution Tracking for Parameterized Systems

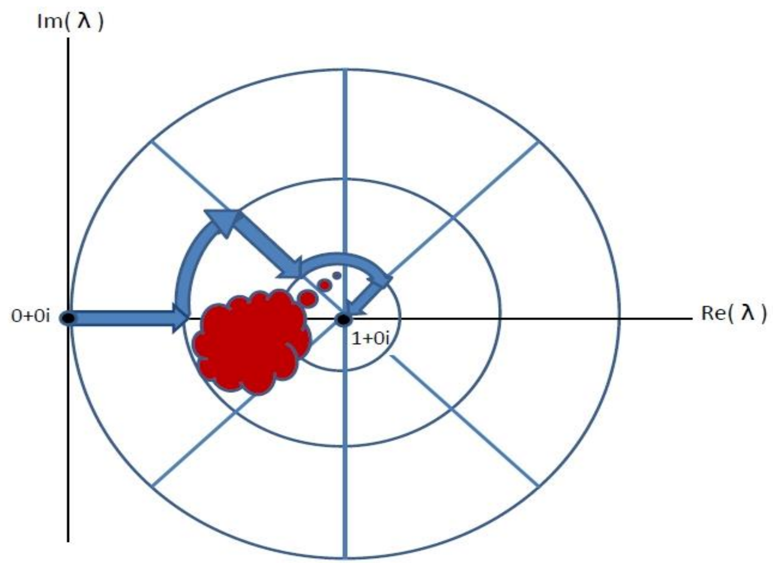
http://www2.econ.iastate.edu/tesfatsi/nasahome.htm

- **Solution Tracking for General Parameterized Nonlinear systems**

Example: http://www2.econ.iastate.edu/tesfatsi/NASA.CMWA1990.pdf

Robert E. Kalaba and Leigh Tesfatsion (1990), "Nonlocal Automated Sensitivity Analysis," *Computers and Mathematics with Applications*, Vol. 20, Issue 2, 53-65.

Abstract: This article presents and illustrates the ***NASA*** program for the ***N****onlocal* ***A****utomated* ***S****ensitivity* ***A****nalysis* of nonlinear systems H(x,λ)=0 parametrized by a vector λ. The NASA program incorporates automated procedures for initialization, derivative evaluation, and tracking of solutions x(λ) along any fixed or adaptively-generated path for λ for which ill-conditioning of the solution does not arise. This tracking can also be used to identify regions of the parameter space where ill-conditioning arises, such as neighborhoods of singularities or bifurcation points.



Adaptive movement of the homotopy continuation parameter λ from 0+0i to 1+0i in the complex plane, avoiding regions where computation becomes ill-conditioned

Solution Tracking for Parameterized Systems … Continued

http://www2.econ.iastate.edu/tesfatsi/nasahome.htm

- **Tracking of Eigenvalues and Eigenvectors for General Parameterized Matrices**

Example: http://www2.econ.iastate.edu/tesfatsi/VariationalEquationsEigen.LT.pdf

Robert E. Kalaba, Karl Spingarn, and Leigh Tesfatsion (1981), "Variational Equations for the Eigenvalues and Eigenvectors of Nonsymmetric Matrices," *Journal of Optimization Theory and Applications*, Vol. 33, 1-8.

Abstract: This article develops a complete system of ordinary differential equations for tracking the eigenvalues and right and left eigenvectors of *nonsymmetric* parameterized matrices M(λ) along any fixed or adaptively-generated path for the parameter vector λ for which matrix ill-conditioning does not arise. A simpler reduced form of the ODE system is then derived for tracking the eigenvalues and eigenvectors of *symmetric* parameterized matrices. This tracking can also be used to identify regions in parameter space where matrix ill-conditioning arises, such as neighborhoods of singularities or bifurcation points. The feasibility and accuracy of the tracking method are illustrated by numerical examples.