Illustration of Agent-Oriented Programming: Per Bak’s Sand Pile Model
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26 June 2004

NOTE: See the 1997 paper by Nathan Winslow (Geological Sciences, University of Michigan) linked to the RePast Study Group syllabus, titled “Introduction to Self-Organized Criticality and Earthquakes.” The following notes summarize the discussion in Winslow.

• When you first start building a sand pile on a tabletop, the system is weakly interactive. Sand grains drizzled from above onto the center of the sand pile have little effect on sand grains at the edges.

• As you keep adding sand grains to the center, a small number at a time, eventually the slope of the sand pile “self organizes” to a critical state where the sand pile cannot grow any larger and breakdowns of all different sizes are possible in response to further drizzlings of sand grains.

• Bak refers to this critical state as a state of self-organized criticality (SOC), since the sand grains on the surface of the sand pile have self-organized to a point where they are just barely stable.
What does it mean to say that “breakdowns of all different sizes” can happen at the SOC state?

- Starting in this state, the addition of one more grain can result in an “avalanche” or “sand slide,” i.e., a cascade of sand down the edges of the sand pile and (possibly) off the edge of the table.

- The size of this avalanche can range from one grain to catastrophic collapses involving large portions of the sand pile.

- The size distribution of these avalanches follows a “Power Law” over any specified period of time T. That is, the average frequency of a given size of avalanche is inversely proportional to some power of its size, so that big avalanches are rare and small avalanches are frequent.
So what’s the formal definition of a “Power Law”?

Two variables $N$ and $C$ are said to satisfy a \textit{POWER LAW} relationship if there exist constants $K$ and $s$ such that

\[ N = KC^{-s} \quad . \] (1)

Letting $n = \log(N)$, $k = \log(K)$ and $c = \log(C)$, it can be shown that equation (1) implies the linear relationship

\[ n = k - sc \quad . \] (2)
EXAMPLE:

Over 24 hours you might observe one avalanche involving 1000 sand grains, 10 avalanches involving 100 sand grains, and 100 avalanches involving 10 sand grains.

This is consistent with a power law of the form \( \log(N) = \log(K) - \text{slog}(C) \) with \( N = \) number of avalanches, \( K = 1000 \), \( C = \) number of sand grains involved in the avalanche, and \( s = 1 \).
Winslow (1997) gives an algorithmic description of Bak’s sand pile model, summarized below, but no actual code. The following pages translate Winslow’s description into pseudo-code that could be fleshed out into an actual working program.

- A sand pile on a tabletop can be modelled as a two-dimensional “cellular automaton” (checkerboard grid) in which each cell (checkerboard square) keeps numerical track of the “average gradient” $G$ of the sand pile in that cell as successive sand grains are added to the sand pile.

- Each cell is assigned a common user-specified critical value $CV$, which can be any number greater than or equal to 3.

- Starting from some initial distribution of $G$ values across the entire automaton (e.g., all $G$ values set to 0), a cell is initially chosen at random and its $G$ value is increased by one.

- If the resulting $G$ value exceeds the critical value for this cell, then this value of $G$ is decreased by 4 and the values of $G$ in the north, south, east, and west neighboring cells of this cell are each increased by 1.
• If this redistribution of $G$ values results in a $G$ value in a *neighboring* cell that exceeds its critical value, then another redistribution occurs.

• Otherwise, another cell is chosen at random, its $G$ value is increased by 1, and the process repeats.

• Winslow shows (Figures 2 and 3) that a log-log plot of the avalanche size $C$ versus the frequency of occurrence $N(C)$ of avalanches of size $C$ obeys a power law distribution, where $C$ is the number of cells whose $G$ value is changed as a result of the avalanche.
Pseudo-Code Model of a Sand Pile on an 8× 8 Tabletop

class SandPileAgent {
    int A; // Active (A=1) or Inactive (A=0) Agent
    int G; // G = Average gradient value of agent
    int CV; // CV = Critical value of agent
    int X; // X-coordinate for the agent
    int Y; // Y-coordinate for the agent
    Activate(); // Activation method for the agent
    Rule(); // Behavioral rule for the agent
}

void Rule() {
    G = G+1;
    If (G > CV) { // Does gradient exceed critical value?
        G = G-4; // If yes, 4 particles roll “down hill”
        // Activate north, east, south, west neighbors
        C(X+1,Y).Activate();
        C(X-1,Y).Activate();
        C(X,Y+1).Activate();
        C(X,Y-1).Activate();
    }
    A = 0; // De-activate myself
}

void Activate() {
    A = 1; // Activate myself
    Rule(); // Implement my behavioral rule
}
Main Program for Sand Pile Model

int main () {
    int TMAX = 1000;
    int X;
    int Y;
    int TestActive;
    AgentInit(); // Construct an array C(X,Y) of
        // 64 agents, X=1,...,8; Y=1,...,8,
        // with initial values A=0,G=0,CV=12,
        // and location indicators X and Y
    //If all agents are currently inactive,
    //randomly activate an agent
    For (int T = 0; T < TMAX; ++T) {
        TestAValues(); //Set TestActive=Max Current A Value
        If (TestActive == 0) {
            X = Rand{1,...,8}; //Randomly select agent
            Y = Rand{1,...,8};
            C(X,Y).Activate(); //Invoke its activation method
        }
    }
    Return 0 ;
}
NOTE: Suggested exercise for those with programming background — Try your hand at writing complete compilable code for implementing Bak’s sand pile model!

Winslow (1997) also discusses the difficulties that experimenters have had in trying to get actual sand piles to behave in the idealized way captured in Per Bak’s theory and implemented through simple computer models.

Winslow (1997) cites interesting attempts by Nagel (1992) and Bretz (1992) to conduct experiments with real sand piles.

These researchers were UNABLE to obtain SOC results with ACTUAL sand piles unless the experimental conditions were rather delicately tuned, leading Winslow to question whether actual sand piles can legitimately be said to have self-organizing critical states even when critical slope values are found.