Constant Elasticity of Substitution (CES) formulations

I. One \( \sigma \) methods

Let output in year \( t \) be given by

\[
q_t = [\alpha_t(a_tN_{ct})^\rho + (1 - \alpha_t)(b_tN_{ht})^\rho]^{1/\rho}
\]

\( N_{ct} \) = number of college graduates
\( N_{ht} \) = number of high school graduates
\( a_t, b_t \) skill augmenting technical change for college and high school group, respectively
\( \alpha_t \) time varying shifts in technology or demographics that alter the skill share of production

Elasticities of substitution are \( \sigma = \frac{1}{1-\rho} \). As \( \rho \to -\infty \), \( \sigma \to 0 \) (Leontief). When \( \rho = 0 \), \( \sigma = 1 \) (Cobb Douglas).

As \( \rho \to 1 \) from below, \( \sigma \to \infty \) (linear isoquants).

The marginal product of input \( N_{ct} \) will be of the form

\[
\frac{\partial q_t}{\partial N_{ct}} = \frac{1}{\rho} [\alpha_t(a_tN_{ct})^\rho + (1 - \alpha_t)(b_tN_{ht})^\rho]^{1-\rho} \left\{ \alpha_t \rho (a_tN_{ct})^{\rho-1} a_t \right\} \alpha_t a_t \left( \frac{q_t}{a_tN_{ct}} \right)^{1-\rho}
\]

\[
= \alpha_t a_t \left[ \frac{1}{\rho} N_{ct}^{\rho-1} \right] \left( \frac{a_tN_{ct}}{\rho a_tN_{ct}} \right)^{1-\rho}
\]

\[
= \alpha_t a_t \left( \frac{q_t}{a_tN_{ct}} \right)^{1-\rho}
\]

A similar derivation yields the marginal product of input \( N_{ht} \)

\[
\frac{\partial q_t}{\partial N_{ht}} = (1 - \alpha_t) b_t \left( \frac{q_t}{b_tN_{ht}} \right)^{1-\rho}
\]

Use these marginal products in cost minimization framework,

\[
\mathcal{L} = W_{ct}N_{ct} + W_{ht}N_{ht} + \lambda(q_t - [\alpha_t(a_tN_{ct})^\rho + (1 - \alpha_t)(b_tN_{ht})^\rho]^{1/\rho})
\]

First-order conditions

\[
(3A) \frac{\partial \mathcal{L}}{\partial N_{ct}} = W_{ct} - \lambda \alpha_t a_t \left( \frac{q_t}{a_tN_{ct}} \right)^{1-\rho} \leq 0
\]

\[
(3B) \frac{\partial \mathcal{L}}{\partial N_{ht}} = W_{ht} - \lambda (1 - \alpha_t) b_t \left( \frac{q_t}{b_tN_{ht}} \right)^{1-\rho} \leq 0
\]

\[
(3C) \frac{\partial \mathcal{L}}{\partial \lambda} = q_t - [\alpha_t(a_tN_{ct})^\rho + (1 - \alpha_t)(b_tN_{ht})^\rho]^{1/\rho} = 0
\]

When the first two conditions hold with equality,
Taking logs, we get the Katz-Murphy (1992) formulation for deriving the elasticity of substitution between high and los skilled workers.

\[
\ln \frac{W_{ct}}{W_{ht}} = \ln \left( \frac{\alpha_t}{1 - \alpha_t} \right) + \ln \left( \frac{\alpha_t}{b_t} \right) + (\rho - 1) \ln \left( \frac{\alpha_t}{b_t} \right) + (\rho - 1) \ln \left( \frac{N_{ct}}{N_{ht}} \right)
\]

\[
= \ln \left( \frac{\alpha_t}{1 - \alpha_t} \right) + \rho \ln \left( \frac{\alpha_t}{b_t} \right) + (\rho - 1) \ln \left( \frac{N_{ct}}{N_{ht}} \right)
\]

\[
= \ln \left( \frac{\alpha_t}{1 - \alpha_t} \right) + \rho \ln \left( \frac{\alpha_t}{b_t} \right) - \frac{1}{\sigma} \ln \left( \frac{N_{ct}}{N_{ht}} \right)
\]

II. Sidebar: Other commonly used formulations based on CES cost minimization

A. Use only 1 first order condition 3A or 3B

\[
\ln W_{ct} = \ln \lambda + \ln (\alpha_t) + \rho \ln (\alpha_t) + (1 - \rho) \ln \left( \frac{q_t}{N_{ct}} \right)
\]

Which yields estimates of \( \rho \) and \( \sigma \). Typically, \( \lambda \) is treated as the marginal cost which is equal to price. The dependent variable is then treated as \( \ln \left( \frac{W_{ct}}{P} \right) \), where \( P \) is a measure of the price level.

B. Rearrange the previous formulation to get

\[
\ln \left( \frac{q_t}{N_{ct}} \right) = \text{constant} - \left( \frac{1}{1-\rho} \right) \ln \left( \frac{W_{ct}}{P} \right) = \text{constant} + \sigma \ln \left( \frac{W_{ct}}{P} \right)
\]

III. Two plus \( \sigma \) methods following Card and Lemieux (2001)

What if we want to relax the constraint that the CES form only allows a single elasticity of substitution across all inputs?

To simplify, let \( \alpha_t = b_t = 1 \). Redefine the labor inputs in efficiency units in (1) as

\[
N_{ct} = \left[ \sum_{j=1}^{J} a_j N_{cjt}^\alpha \right]^{\frac{1}{\eta}} \text{ where there are } J \text{ cohorts of college graduates (age groups) in year } t
\]

\[
N_{ht} = \left[ \sum_{j=1}^{J} b_j N_{htj}^\alpha \right]^{\frac{1}{\eta}} \text{ where there are } J \text{ cohorts of high school graduates in year } t
\]

Using the chain rule, the marginal products are of the form

\[
\frac{\partial q_t}{\partial N_{cjt}} = \frac{\partial q_t}{\partial N_{ct}} \cdot \frac{\partial N_{ct}}{\partial N_{cjt}} \quad \text{(5A)}
\]

\[
\frac{\partial q_t}{\partial N_{htj}} = \frac{\partial q_t}{\partial N_{ht}} \cdot \frac{\partial N_{ht}}{\partial N_{htj}} \quad \text{(5B)}
\]
The second term on the right-hand-side would be of the form

$$\frac{\partial N_{ct}}{\partial N_{cjt}} = \frac{1}{\eta} \left[ \sum_{j=1}^{J} a_j N_{cjt}^{\eta} \right]^{1-\eta} \cdot \eta a_j N_{cjt}^{\eta-1}$$

$$= a_j \left[ \sum_{j=1}^{J} a_j N_{cjt}^{\eta} \right]^{\frac{1-\eta}{\eta}} \cdot N_{cjt}^{\eta-1}$$

$$= a_j \left[ \sum_{j=1}^{J} a_j N_{cjt}^{\eta} \right]^{1-\eta} \cdot N_{cjt}^{\eta-1}$$

$$= a_j \left( N_{cjt} \right)^{1-\eta} N_{cjt}^{\eta-1}$$

From (2A), \( \frac{\partial q_t}{\partial N_{ct}} = \alpha_t (q_t)^{1-\rho} (N_{ct})^{\rho-1} \) with \( \alpha_t = 1 \)

Inserting these into the chain rule (5A) yields

(6A) \( \frac{\partial q_t}{\partial N_{cjt}} = \frac{\partial q_t}{\partial N_{ct}} \cdot \frac{\partial N_{ct}}{\partial N_{cjt}} = a_j \alpha_t (q_t)^{1-\rho} N_{cjt}^{\eta-1} (N_{ct})^{\rho-\eta} \)

Similarly,

$$\frac{\partial N_{ht}}{\partial N_{hjt}} = b_j (N_{ht})^{1-\eta} N_{hjt}^{\eta-1}$$

$$\frac{\partial q_t}{\partial N_{ht}} = (1 - \alpha_t) (q_t)^{1-\rho} (N_{ht})^{\rho-1}$$

(6B) \( \frac{\partial q_t}{\partial N_{hjt}} = \frac{\partial q_t}{\partial N_{ht}} \cdot \frac{\partial N_{ht}}{\partial N_{hjt}} = (1 - a_j) b_t (q_t)^{1-\rho} N_{hjt}^{\eta-1} (N_{ht})^{\rho-\eta} \)

This allows different elasticities of substitution between cohorts \( \sigma_{\eta} = \frac{1}{1-\eta} \); and elasticities of substitution between education groups \( \sigma_{\rho} = \frac{1}{1-\rho} \).

The cost minimizing conditions equivalent to (3A-B) will have the form

(7A) \( W_{cjt} = \lambda \frac{\partial q_t}{\partial N_{ct}} \cdot \frac{\partial N_{ct}}{\partial N_{cjt}} = \lambda a_j \alpha_t (q_t)^{1-\rho} N_{cjt}^{\eta-1} (N_{ct})^{\rho-\eta} \)

(7B) \( W_{hjt} = \lambda \frac{\partial q_t}{\partial N_{ht}} \cdot \frac{\partial N_{ht}}{\partial N_{hjt}} = \lambda (1 - a_j) b_t (q_t)^{1-\rho} N_{hjt}^{\eta-1} (N_{ht})^{\rho-\eta} \)
Then the relative wage equation equivalent to (4) is

\[
(8) \ln \frac{W_{ot}}{W_{ht}} = \ln \left( \frac{a_t}{(1-a_t)} \right) + \ln \left( \frac{a_t}{b_t} \right) + (\eta - 1) \ln \left( \frac{N_{ejt}}{N_{ht}} \right) + (\rho - \eta) \ln \left( \frac{N_{ct}}{N_{ht}} \right) + \ln \left( \frac{N_{ct}}{N_{ht}} \right)
\]

\[
= \ln \left( \frac{a_t}{(1-a_t)} \right) + \ln \left( \frac{a_t}{b_t} \right) + (\eta - 1)[\ln \left( \frac{N_{ejt}}{N_{ht}} \right) - \ln \left( \frac{N_{ct}}{N_{ht}} \right)] + (\rho - 1) \ln \left( \frac{N_{ct}}{N_{ht}} \right)
\]

Which allows direct estimation of the parameters of interest \(\eta\) and \(\rho\) yielding \(\sigma_\eta\) and \(\sigma_\rho\).

III. Three Inputs: Hanson (2006)
Suppose that we have two countries, Mexico (o) and the United States (1)

**Production and Labor in Mexico**

\(L_{oht}\) is labor of type \(h\) in Mexico

\[0 < v < 1\] is the elasticity of substitution between \(K\) and \(L\), \(\sigma_{KL} = \frac{1}{1-v}\)

\[0 < k < 1\] is the elasticity of substitution between labor types \(\sigma_{hht}\)

\(L_{ot} = (\sum_h L_{oht}^k)^{1/k}\) is aggregate labor in Mexico

Production is

\[Q_{ot} = (K_{ot}^v + L_{oht}^v)^{1/v} = (K_{ot}^v + (\sum_h L_{oht}^k)^{v/k})^{1/v}\]

The marginal product of Mexican labor of type \(h\) is

\[
(9) \frac{dQ_{ot}}{dL_{oht}} = \frac{1}{v} (K_{ot}^v + L_{oht}^v)^{1-v} \left( \frac{v}{k} \right) \left( \sum_h L_{oht}^k \right)^{v-1} k L_{oht}^{k-1}
\]

\[
= (K_{ot}^v + L_{oht}^v)^{1-v} \left( \sum_h L_{oht}^k \right)^{v-k} \frac{k^{v-k}}{k} L_{oht}^{k-1}
\]

\[
= Q_{ot}^{1-v} L_{oht}^{v-k} L_{oht}^{k-1}
\]

Let the wage for type \(h\) workers in Mexico be given by \(W_{oht}\) and let the output price in Mexico be \(P_{ot}\). First order conditions for firm profit maximization will be

\[W_{oht} = P_{ot} \cdot \frac{dQ_{ot}}{dL_{oht}}\]

Inserting (9) and taking logs,

\[
(10) \ln(W_{oht}) = \ln(P_{ot}) + (1 - v) \ln(Q_{ot}) + (v - k) \ln(L_{ot}) + (k - 1) \ln(L_{oht})
\]

**Production and Labor in the United States**

The U.S. labor force is
\[ L_{1ht} = \left( L_{1ht}^h + l_{1ht}^h \right)^{\frac{1}{\eta}} \] where \( L_{1ht} \) is the domestic workers of type \( h \) and \( l_{1ht} \) is the immigrant labor of type \( h \) in the U.S. The coefficient \( \eta \) will define the elasticity of substitution between Mexican and domestic workers so that \( \sigma_{Lt} = \frac{1}{1-\eta} > 0 \).

Aggregate labor in the U.S. is \( L_{1t} = \left( \sum_h L_{1ht}^k \right)^{\frac{1}{k}} \)

U.S. Production will be
\[ Q_{1t} = \left( K_{1t}^v + L_{1t}^v \right)^{1/v} = \left( K_{1t}^v + \left( \sum_h L_{1ht}^k \right)^{v/k} \right)^{1/v} \]

The marginal product of immigrant labor will be
\[
\begin{align*}
(11) \quad \frac{dQ_{1t}}{dl_{1ht}} &= \frac{1}{v} \left( K_{1t}^v + L_{1t}^v \right)^{v-1} \frac{1}{v} \left( \sum_h L_{1ht}^k \right)^{v-1} \frac{k}{\eta} L_{1ht}^k \eta l_{1ht}^{\eta-1} \\
&= \left( K_{1t}^v + L_{1t}^v \right)^{1-v} \frac{1}{v} \left( \sum_h L_{1ht}^k \right)^{v-k} \frac{k-\eta}{\eta} L_{1ht}^k \eta l_{1ht}^{\eta-1} \\
&= Q_{1t}^{1-v} L_{1t}^{v-k} L_{1ht}^k \eta l_{1ht}^{\eta-1}
\end{align*}
\]

Assume that \( W_{l_{1ht}} = W_{l_{1ht}} = W_{1ht} \) is the wage for type \( h \) labor in the U.S. and that output is paid \( P_{1t} \).

Profit maximization will require that
\[ W_{1ht} = P_{1t} \cdot \frac{dQ_{1t}}{dl_{1ht}} \], assuming that the marginal product of foreign and domestic labor of type \( h \) is equal.

\[ (12) \quad \ln(W_{1ht}) = \ln(P_{1t}) + (1 - v) \ln(Q_{1t}) + (v - k) \ln(L_{1t}) + (k - \eta) \ln(L_{1ht}) + (\eta - 1) \ln(l_{1ht}) \]

Equilibrium
The condition that holds when there is no further incentive to migrate from Mexico to the United States is
\[ (13) \quad \ln(W_{oht}) - \ln(p_{ot}) = \ln(W_{1ht}) - \ln(P_{1t}) - \ln(C_{ht}) \]

Where \( \ln(C_{ht}) \) is the cost of migrating from Mexico to the United States and \( p_{it} \) is the consumer price index (including traded and untraded goods) for country \( i \) in year \( t \). We assume that the cost is proportional to the wage, a reasonable assumption as the primary cost of migrating is time.

Imposing the equilibrium wage condition and substituting in equations (10) and (12), the equilibrium level of immigrant employment of type \( h \) is
\[
\ln \left( \frac{W_{ht}}{W_{oht}} \right) = \ln \left( \frac{p_{1t}}{p_{ot}} \right) + \ln (C_{ht}) \\
= (1 - \nu) \ln \left( \frac{Q_{1t}}{Q_{ot}} \right) + (\nu - k) \ln \left( \frac{L_{1t}}{L_{ot}} \right) + (k - \eta) \ln \left( \frac{L^{1ht}}{L^{1oht}} \right) + (\eta - 1) \ln \left( \frac{L_{ht}}{L_{oht}} \right) + \ln \left( \frac{P_{1t}}{P_{ot}} \right)
\]

which yields the relative demand relationship for immigrant labor of type \( h \) in the U.S. versus in Mexico as

\[
\ln \left( \frac{L_{ht}}{L_{oht}} \right) = \left( \frac{1}{1 - \eta} \right) \left( (1 - \nu) \ln \left( \frac{Q_{1t}}{Q_{ot}} \right) + (\nu - k) \ln \left( \frac{L_{1t}}{L_{ot}} \right) + (k - \eta) \ln \left( \frac{L^{1ht}}{L^{1oht}} \right) + \ln \left( \frac{P_{1t}}{P_{ot}} \right) - \ln \left( \frac{P_{1t}}{P_{ot}} \right) - \ln (C_{ht}) \right)
\]