Show how the long-run elasticity of demand for labor $\theta_{LL}$, is related to the elasticity of demand for output, the elasticity of substitution, $\sigma_{kL}$, and the cost share of labor, $k_L$. Need to allow the firm to have some price setting power to avoid a perfectly elastic output demand curve. Note that in the long-run, all inputs are variable including capital which is fixed in the short-run.

This derivation is based on Ferguson, Chpt. 12, 235-239 or Hamermesh Chpt. 2

Production function is $q = f(K,L)$

Profit is given by $\Pi = pf(K,L) - rK - wL$

Market clearing condition: $q = f(K,L) = h(p); h'(p) < 0$

Endogenous variables are capital, $K$; labor, $L$; and output price, $p$.

Exogenous variables are $r$, rental price of capital; $w$, wage.

We want to derive $\frac{dL}{dw}$ and $\frac{dK}{dw}$ and their associated elasticities $\theta_{LL}$ and $\theta_{KL}$

First-order conditions include derivatives of the profit function with respect to $L$ and $K$ and the market clearing condition:

$$f(K,L) - h(p) = 0$$

$$\frac{\partial \Pi}{\partial K} = pf_K - r = 0$$

$$\frac{\partial \Pi}{\partial L} = pf_L - w = 0$$

To derive the long-run demand, we need to totally differentiate the first-order conditions with respect to $w$:

$$f_K \frac{dK}{dw} + f_L \frac{dL}{dw} - h'(p) \frac{dp}{dw} = 0$$

$$pf_{KK} \frac{dK}{dw} + pf_{KL} \frac{dL}{dw} + f_K \frac{dp}{dw} = 0$$

$$pf_{KL} \frac{dK}{dw} + pf_{LL} \frac{dL}{dw} + f_L \frac{dp}{dw} - 1 = 0$$

To make the problem tractable, assume constant returns to scale which implies that $pq = rK + pL$;

$q = f_K + f_L \rightarrow f_K = f_K + f_{kK}K + f_{kL}L = 0 \rightarrow f_{kK}K + f_{kL}L = 0 \rightarrow f_{kL} = -f_{kK}(K/L)$
\[ q = f_KK + f_LL \rightarrow f_L = f_KL + f_LL + f_L = 0 \rightarrow f_KL + f_LL = 0 \rightarrow f_KL = - f_LL(L/K) \]

\[ \sigma = \frac{f_K f_L}{q f_K L} \]

Formula for \( \sigma \) implies that \( f_KL = (f_K f_L)/q \; \sigma \); \( f_LL = -(L/K)(f_K f_L)/q \; \sigma \)

In addition, the first-order conditions yields \( f_K = r/p \); \( f_L = w/p \)

Finally, define the absolute value of the elasticity of product demand as

\[ \{|\eta|\} = \eta_{L/K} \quad \text{and} \quad |h'| = h' \]

Insert terms for \( f_KL, f_LL, f_K, f_L, \) and \( h' \)

Putting this is matrix form

\[
\begin{bmatrix}
  r & w & q \eta \\
  \frac{-Lw}{K} & w & q \sigma \\
  r & \frac{-Kr}{L} & q \sigma
\end{bmatrix}
\begin{bmatrix}
  \frac{dK}{dw} \\
  \frac{dL}{dw} \\
  \frac{dp}{dw}
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0 \\
  \frac{qp \sigma}{w}
\end{bmatrix}
\]

The determinant of the Hessian matrix is

\[
|H| = 2r w q \sigma + \left( \frac{q \eta r L w}{K L} \right) - 2 r q \eta + \frac{r^2 q \sigma L}{K} + \frac{w^2 q \sigma L}{K}
\]

\[
= 2r w q \sigma + q \eta r w - r w q \eta + \frac{q \sigma L}{K} \left( r^2 K^2 + w^2 L^2 \right)
\]

\[
= 2r w q \sigma + \frac{q \sigma L}{K} \left( r^2 K^2 + w^2 L^2 \right) = \frac{q \sigma L}{K} \left( r^2 K^2 + 2rwLK + w^2 L^2 \right)
\]

\[
= \frac{q \sigma L}{K} (rK + wL)^2 = \frac{q \sigma L}{K} (pq)^2 > 0
\]

And so
\[
\frac{dL}{dw} = \frac{-q^2 \eta \rho \omega w}{wK} \cdot \frac{q^2 \sigma^2 p r}{w} = \frac{-q^2 \eta \rho \omega w}{wK} \cdot \frac{q^2 \sigma^2 p r}{w} = \frac{-\eta L^2 w}{w} \cdot \frac{L \rho \sigma}{pq} = \frac{L}{w} \left( \frac{\eta L w}{pq} + \frac{K \rho \sigma}{pq} \right)
\]

Note that \(k_L = \frac{wL}{pq}, k_K = \frac{rK}{pq} = 1 - k_L\); and so \(\frac{dL}{dw} = \frac{-L}{w} (k_L \eta + (1 - k_L) \sigma)

Rearranging
\[
\frac{dL}{dw} \cdot \frac{wt}{L} = \theta_{LL} = -k_L \eta - (1 - k_L) \sigma < 0 : \text{ Fundamental Law of Derived Demand Scale Substitution}
\]

Cross terms
\[
\frac{dK}{dw} = \frac{q^2 \sigma^2 p w}{w} \cdot \frac{q^2 \sigma p w \eta}{wL} = \frac{wL \sigma (\sigma - \eta)}{wq pq}
\]

Rearranging:
\[
\frac{dK}{dw} \cdot \frac{wt}{K} = \theta_{KL} = \frac{wL}{qp} \cdot (\sigma - \eta) = k_L (\sigma - \eta)
\]

Can be positive or negative depending on whether substitution effect dominates the income effect.

Gross complements if \(\theta_{KL} < 0\) meaning \(\sigma < \eta\)

Gross substitutes if \(\theta_{KL} > 0\) meaning \(\sigma > \eta\)

To derive the Laws of Derived Demand, take the derivatives of the Fundamental Law of Demand with respect to:

\(\sigma\): the role of substitutability of other factors for labor demand elasticity
\(\eta\): the role of the elasticity of demand for output on labor demand elasticity
\(k_L\): the role of labor share of total cost on labor demand elasticity (The Importance of Being Unimportant)

The fourth law requires that we relax the assumption that the firm takes wages as given. In that case, the elasticity of substitutes affects labor demand elasticity similar to the effect of \(\sigma\).