Lecture 12 – Testing for Structural Breaks

I. Known Breakpoint (Chow Tests)
II. Unknown Breakpoint
I. Testing for Structural Change of Known Timing

A. Linear Regression Models

Chow, *Econometrica*, 1960

B. Nonlinear Econometric Models (including, but not limited to, nonlinear regression models)

Consider a linear regression model for which we assume that a break in one or more of the parameters occurred in period $\tau$, e.g.,

$$y_t = x_t'\beta_1 + \varepsilon_t \text{ if } t = 1, \ldots, \tau$$

$$= x_t'\beta_2 + \varepsilon_t \text{ if } t = \tau+1, \ldots, T$$

or, more compactly,

$$(*): y_t = x_t'\beta_1 + 1_t x_t'\gamma + \varepsilon_t, \text{ } t = 1, \ldots, T$$

where

$$1_t = 0 \text{ if } t \leq \tau, 1_t = 1 \text{ if } t > \tau,$$

and

$$\gamma = \beta_2 - \beta_1$$

Assume that the $x$’s are weakly exogenous and the $\varepsilon$’s are spherical (i.e., homoskedastic and serially uncorrelated).

Under additional regularity conditions regarding the joint $x$ and $\varepsilon$ process, the OLS estimator of (*) is consistent, asymptotically normal and asymptotically efficient.

Under the null of parameter stability, i.e., $H_0: \gamma = 0$, the standard F-test is asymptotically valid.

Example – Homework Project 1, Part III
I.B. Andrews and Fair (1988) extended the Chow test to apply in much more general settings –

- Nonlinear regression models with serially correlated and/or heteroskedastic errors (and weakly exogenous regressors).

- A wide class of nonlinear models (other than regression models) of dependent and heterogenous processes estimated by a variety of estimators.

These “Chow tests”, which include Wald-like tests, LM-like tests and LR-like tests depend on the correct specification of the breakpoint.

Notes -
- The paper is not general enough to cover testing for structural change in a GMM setting, although subsequent work by Andrews and others have provided tests that appropriate in a GMM setting when the breakpoint is unknown (the known breakpoint being a special case).
- These tests can easily be modified to only allow for structural break in a subset of the parameters
- Although I call these structural break tests, they are actually test of parameter constancy or homogeneity.
III. Testing for a Structural Break with an Unknown Break Point


A. Linear Regression Model

Suppose

\[ y_t = x_t' \beta_1 + \varepsilon_t, \quad t = 1, \ldots, \tau \]
\[ = x_t' \beta_2 + \varepsilon_t, \quad t = \tau + 1, \ldots, T \]

\( \beta_1, \beta_2, \) and \( x_t \) are \( k \times 1 \). There is a single breakpoint, \( \tau \). Assume the \( x \)'s are stationary and weakly exogenous and the \( \varepsilon \)'s are serially uncorrelated and homoskedastic.

Consider \( H_0: \beta_1 = \beta_2 \)

If \( \tau \) is known the “F-statistic”

\[ F_T(\tau) = \frac{(T-2k) \left[ SSR_{1:T} - (SSR_{1:\tau} + SSR_{\tau+1:T}) \right]}{(SSR_{1:\tau} + SSR_{\tau+1:T})} \]

is asymptotically \( \chi^2(k) \) under \( H_0 \). (Quandt, 1960, showed that if the \( \varepsilon \)'s are normally distributed and the \( x \)'s are strictly exogenous, \( F_T(\tau) \) is the likelihood ratio statistic and is exactly \( \chi^2(k) \) under \( H_0 \).)
In the case where $\tau$ is unknown, Quandt (1960) showed that the likelihood ratio statistic corresponding to $H_0: \beta_1 = \beta_2$ is:

$$QLR_T = \max_{\tau \in \{\tau_{\min}, \ldots, \tau_{\max}\}} F_T(\tau)$$

Quandt realized that the standard assumptions used to show that the LR-statistic is asymptotically $\chi^2$ did not apply in this setting but he did not know how to derive (or even show the existence of) the asymptotic null distribution of $QLR_T$. (Question: If we use the $\chi^2$ distribution do you think that the actual size of the test will be greater than or less than the nominal size? I.e., if the null is correct, do you think we will end up rejecting the null too often or too infrequently? Why?)

This problem was not solved until 30 years later. The keys to solving the problem were developments in econometric theory, particularly the theory of unit root asymptotics, which provided a proper set of tools.

It turned out that the feature of this problem that made it so complicated was that the nuisance parameter $\tau$ is not identified under the null hypothesis. There are a number of other settings in which “a nuisance parameter is only identified under the alternative hypothesis including, e.g., testing for the presence of threshold effects in threshold regression models. Thus, the problem of testing for structural breaks at unknown points has become part of a much larger literature.
Andrews (1993) showed that under appropriate regularity conditions, the QLR statistic, also referred to as a SupLR statistic, has a “nonstandard limiting distribution.” In particular, under H₀,

\[
QLR_T \xrightarrow{D} \sup_{D \in \mathbb{R} \in [r_{\min}, r_{\max}]} \left( \frac{B_k'(r)B_k(r)}{r(1-r)} \right)
\]

where \(0 < r_{\min} < r_{\max} < 1\) and \(B_k(.)\) is a “Brownian Bridge” process defined on \([0,1]\). Percentiles of this distribution as functions of \(r_{\max}, r_{\min}\) and \(k\) are tabulated in Andrews (1993).
Notes –

- The applied researcher has to choose \( r_{\text{min}} \) and \( r_{\text{max}} \) without much guidance. (Think of \( r_{\text{min}} \) as the minimum proportion of the sample that can be in the first subsample and think of \( 1 - r_{\text{max}} \) as the minimum proportion of the sample that can be in the second subsample. \( r_{\text{min}} = 0.1 \) and \( r_{\text{max}} = 0.9 \), e.g.? This kind of problem arises in most of these kinds of tests as we will see when discuss estimating/testing threshold models.

- Andrews (1993) also solved the problem of testing for an unknown break in **any setting in which the GMM estimator can be applied**! I.e., the results for the OLS estimator of the linear regression model is a special case of a much more general set of results which provide the limiting null distribution of the SupLR, SupW and SupLM statistics.

- B. Hansen (1997) provides a numerical procedure to compute asymptotic p-values for these tests.

- These tests can be extended to more than one break point provided the maximum number of possible breakpoints is known. (Bai and various co-authors).

- For more recent developments in this literature: Go to Dr. Bunzel’s seminar!