Review Questions: OLG models
Econ602. Spring 2005. Lutz Hendricks

Question 1. Savings function

Derive the savings function for the utility function \( u(c_t^y, c_{t+1}^o) = \left( (c_t^y)^{1/2} + \beta c_{t+1}^o \right)^{1/2} \). Assume an OLG economy with constant population and endowments \( e_1 \) and \( e_2 \). Hint: A monotonic transformation of the utility function helps.

Answer

Recall that we can replace a utility function by any strictly increasing transformation. This means we can drop the outer square and analyze \( u(c_t^y, c_{t+1}^o) = (c_t^y)^{1/2} + \beta c_{t+1}^o \) instead. The Euler equation is then \( 1/2 \cdot (c_t^y)^{-1/2} = \beta (1 + r_{t+1}) \). Therefore, \( c_t^y = (2\beta [1+r_{t+1}])^{-2} \) and \( s_t = e_1 - (2\beta [1+r_{t+1}])^{-2} \). This utility function is peculiar: young consumption is independent of the endowment. The reason is that utility is linear in old consumption. Therefore, \( c_t^y \) is chosen to equate marginal utility to the constant marginal utility from old consumption, \( \beta (1+r) \). Old age consumption is then chosen residually.

Question 2. A Savings Function

Consider the standard two-period household problem. The household receives a wage \( w_t \) when young and a rate of return \( R_{t+1} \) on savings.

(a) Illustrate the household’s intertemporal budget constraint and the optimal choice of consumption in a diagram. Label it clearly.

(b) If the wage rate rises, what happens to savings? What if the household receives an additional endowment when old? Explain and illustrate in your diagram. No math please.

(c) Derive the consumption function \( c(w_t, R_{t+1}) \) and the savings function \( s(w_t, R_{t+1}) \) for the utility function \( u(c_t^x, x_{t+1}^o) = c_t^x + x_{t+1}^o \), where \( 0 < \alpha < 1 \) is a constant.

(d) Do the same for the utility function \( u(c_t^x, x_{t+1}^o) = A \cdot \ln(c_t^x + x_{t+1}^o) \), \( 0 < A < 1 \). What do you find and why?

Answer

(a) We did that in class.

(b) If both goods are normal, an increase in the wage rate or in the endowment when old leads to higher consumption at both ages. Therefore: \( w \uparrow \Rightarrow s \uparrow \), but \( y_2 \uparrow \Rightarrow s \downarrow \).
(c) The household solves: \( \max c^\alpha + x^\alpha \), subject to the budget constraint \( c + x / R = w \). The first-order condition is \( u_c = R u_x \) or \( \alpha c^{\alpha-1} = R \alpha x^{\alpha-1} \). Thus \( x / R = c R^{\alpha/(1-\alpha)} \). Substituting this into the budget constraint yields \( c = w/(1 + R^{\alpha/(1-\alpha)}) \) and \( s = w(1 - 1/[1 + R^{\alpha/(1-\alpha)}]) \).

(d) The answer is exactly the same. The utility function in (c) is a monotone increasing transformation of the one in (d).

**Question 3. A log utility example**

Consider a standard two period OLG model. Firms rent capital and labor so as to maximize period profits. The production function is \( F(K, L) = K^\alpha L^{1-\alpha} \). Capital does not depreciate. At date \( t \), \((1+n)^t\) households are born. They supply one unit of labor inelastically when young. Preferences are \( \ln(c_t^y) + \beta \ln(c_{t+1}^0) \). The budget constraints are \( w_t = c_t^y + s_t \) and \( c_{t+1}^0 = (1+r_{t+1}) s_t \).

(a) Solve the household’s problem, given prices.

(b) Find the steady state capital-labor ratio.

(c) Is the steady state unique and stable?

(d) If \( \alpha = 0.3 \) and \( \beta = 0.9 \), find the population growth rates for which the economy is dynamically inefficient.

(e) Suppose there is a one-time shock to population in the sense that \( N_{t+1} = (1+n) N_t + \varepsilon \); the population is permanently larger afterwards \( N_{t+2} = (1+n) N_{t+1} \). Describe the effect on interest rates.

**Answer**

(a) We have the standard log-utility solution: \( c_t^y = w_t / (1+\beta) \) and \( s_t = w_t \beta / (1+\beta) \).

(b) Equilibrium requires \( s_t = (1+n) k_{t+1} \). Therefore, the law of motion for \( k \) is \( k_{t+1} = \frac{(1-\alpha)\beta}{(1+\beta)(1+n)} k_t^\alpha \). In steady state: \( (k^*)^{1-\alpha} = \frac{(1-\alpha)\beta}{(1+\beta)(1+n)} \).

(c) To see uniqueness and stability, plot the law of motion for capital.

(d) Dynamic inefficiency requires \( f'(k) = \alpha k^{\alpha-1} = \delta + n \). Inefficiency arises if \( n > 10 \).

(e) The effect of increasing the size of the young generation is analogous to that of destroying some capital. As long as \( k \) is below the steady state, the interest rate is above the steady state, but falling over time.

---

1 Based on a question due to Gregory Smith.
Question 4. Heterogeneous Agents

An economy is populated by two-period lived households with preferences \( c_{h,t}^y \{c_{h,t+1}^o\}^\beta \), where \( h \) indexes the household, \( t \) is the date, and \((y, o)\) represent the two ages. There are 100 individuals born at each date \((h = 1, \ldots, 100)\). The odd-numbered households receive endowments of \([w_{h,t}^y, w_{h,t+1}^o]=[1,1]\), while the even-numbered households receive \([2, 1]\). Endowments cannot be stored.

(a) Define a competitive equilibrium for this economy. There are no financial assets. Instead, households can buy and sell both goods when young at prices \( p_t \) and \( p_{t+1} \), respectively. That is, when a young household is born, it knows its endowments and the two prices relevant for its life: \( p_t \) and \( p_{t+1} \). Its lifetime budget constraint is then
\[
0 = (\beta + 1) (w_{h,t}^y - c_{h,t}^y) + p_{t+1} (w_{h,t+1}^o - c_{h,t+1}^o) = 0.
\]

(b) Define and solve for the competitive equilibrium. That is, solve for the values of consumption and prices that satisfy equilibrium conditions (these will depend on \( \beta \) only). Hint: The price level is indeterminate, but the inflation rate is not. You should find, among others, that
\[
c^y_1 = [1 + 1.5\beta] / [1 + \beta].
\]

(c) Consider a transfer scheme that takes a little bit of the good from each young and gives it to each old. Is the competitive equilibrium for this economy Pareto-superior to the previous economy? Explain. [Hint: The answer depends on parameter values. You need not work out the new equilibrium.]

Answer: Heterogeneous Agents

(a) A competitive equilibrium is a sequence of prices for the consumption good \( \{p_t\} \) and an allocation \( \{c_{h,t}^y, c_{h,t+1}^o\} \) such that (i) every household maximizes utility subject to a budget constraint and (ii) the goods market clears at each date.

(b) To simplify things a bit, take logs of the utility function. The household then solves
\[
\max \ln(c_{h,t}^y) + \beta \ln(c_{h,t+1}^o) \text{ subject to the budget constraint}
\]
\[
p_t (w_{h,t}^y - c_{h,t}^y) + p_{t+1} (w_{h,t+1}^o - c_{h,t+1}^o) = 0.
\]

The Lagrangian can be written as
\[
\Gamma = \ln\left( w_{h,t}^y + (w_{h,t+1}^o - c_{h,t+1}^o) p_{t+1} / p_t \right) + \beta \ln(c_{h,t+1}^o). \]

The first-order condition is therefore \( [p_{t+1} / p_t] / c_{h,t}^y = \beta / c_{h,t+1}^o \) or \( p_{t+1} c_{h,t+1}^o = \beta p_t c_{h,t}^y \). The household spends a constant fraction on young vs. old consumption; a consequence of log utility. We can substitute this back into the budget constraint to obtain the savings or consumption function:
\[
c_{h,t}^y = w_{h,t}^y + p_t w_{h,t+1}^o - \beta c_{h,t}^y,
\]

---

2 This question is taken from the ASU macro prelim, 1996.
where \( \pi_t = p_{t+1} / p_t \). Denote the household’s total wealth by \( W_{h,t} = w_{h,t}^y + \frac{p_{t+1}}{p_t} w_{h,t+1}^o \). Then

\[
\begin{align*}
(1) & \quad c_{h,t}^y = W_{h,t} / (1 + \beta) \quad \text{and} \quad c_{h,t+1}^o = \frac{\beta}{1 + \beta} \frac{W_{h,t}}{\pi_t} .
\end{align*}
\]

Since there is no trade between generations, market clearing becomes \( c_1^y + c_2^y = w_1^y + w_2^y = 3 \) and \( c_1^o + c_2^o = w_1^o + w_2^o = 2 \). From the first-order conditions: \( c_1^y / c_1^y = c_2^o / c_2^y = \beta / \pi \). But then

\[
(2) \quad \frac{c_1^o + c_2^o}{c_1^y + c_2^y} = \frac{\beta}{\pi} = \frac{w_1^o}{w_1^y} = \frac{2}{3}.
\]

which allows to solve for \( \pi = (3/2)\beta \). It is now possible to solve for the entire consumption allocation using (1). The odd wealth level is \( W_1 = 1 + 1.5\beta \), while the even one is \( W_2 = 2 + 1.5\beta \).

Therefore: \( c_1^y = \frac{1 + 1.5\beta}{1 + \beta} \), \( c_2^y = \frac{2 + 1.5\beta}{1 + \beta} \), \( c_1^o = \frac{\beta}{4} c_h^y \), \( c_2^o = \frac{\beta}{4} c_h^y \), \( c_h^o = (2/3) c_h^y \).

(c) This question is closely related to dynamic efficiency. Suppose households are given access to a savings scheme with a zero interest rate. If they choose to save, it will make them better off and the outcome is Pareto-improving (the first generation in the transfer scheme gets a free lunch; the others are better off because they can now save at a higher rate of return than before). Households will only use this opportunity, if the equilibrium interest rate is negative, i.e., if \( \pi = (3/2)\beta > 1 \). A small transfer of the proposed kind would then be Pareto-improving. [You can convince yourself that this is true by working out the new allocation along the lines outlined below and comparing utilities.]

If the transfer was not infinitesimal, such as a unit transfer, we would have to compute the new equilibrium. This amounts to replacing the endowment vectors by \([0, 2]\) and \([1, 2]\). The new interest rate can be derived from (2): \( \beta / \pi = 4 \), so that \( \pi = \beta / 4 \). The new wealth and consumption levels are therefore:

\[
\begin{align*}
W_1 &= \beta / 2, \quad W_2 = 1 + \beta / 2 \\
c_1^y &= \frac{1}{2} \frac{\beta}{1 + \beta}, \quad c_2^y = \frac{1 + \beta}{2} \frac{\beta}{1 + \beta}, \quad c_h^o = 4 c_h^y
\end{align*}
\]

Young consumption of both households fall unambiguously, while old consumption increases (unsurprisingly). Whether a transfer scheme is Pareto-improving therefore depends on \( \beta \). If \( \beta \) is small, giving goods to the old is bad. If \( \beta \) is large, giving goods to the old is good.

**Question 5. BF 3.2**

Consider an OLG endowment economy. At each date a young cohort of size 1 is born. Each household receives an endowment of \( e_t \) when young and \( (1 + g) e_t \) when old. Utility is given by \( \ln(c_t^y) + \beta \ln(c_{t+1}^o) \). Goods can be stored at the constant rate of return \( r \). First period endowments grow at rate \( m \): \( e_t = (1 + m)^t \).

(a) How does an increase in the growth rate \( g \) affect a household’s savings rate?
(b) How does an increase in \( m \) affect the aggregate savings rate? The aggregate savings rate is defined as \( \bar{s}_t = s_t / [e_t + (1 + g) e_{t-1}] \).

(c) How does an increase in \( g \) and \( m \) affect the aggregate savings rate? Assume \( g = m \).

(d) In light of these findings assess the claim that fast growth is responsible for the high Japanese savings rate.

(e) Assess the claim that the lower savings rate in the U.S. in the 1980s is due to slower growth prospects.

**Answer: BF 3.2**

(a) The household solves \( \text{max} \ln(e_t - [c_{t+1}^0 - e_t (1 + g)] / R] + \beta \ln(c_{t+1}^0) \), where \( R = 1 + r \). The Euler equation is \( c_{t+1}^0 = \beta R c_t^y \). Consumption is therefore \( c_t^y = \frac{e_t}{1 + \beta} \left( 1 + \frac{1 + g}{R} \right) \) and savings are given by

\[
(3) \quad s_t = e_t - c_t^y = e_t \left( 1 - \frac{1 + \beta}{1 + (1 + g) / R} \right).
\]

Faster endowment growth reduces savings because the household wishes to consume more at both dates.

(b) From (3), \( m \) does not affect individual savings (given \( e_t \)). Aggregate savings equal savings of the young. The aggregate savings rate is therefore

\[
\bar{s}_t = \frac{s_t / e_t}{1 + (1 + g) e_{t-1} / e_t} = \frac{s_t / e_t}{1 + (1 + g) / (1 + m)}
\]

Faster growth (higher \( m \)) implies a higher savings rate.

(c) The savings rate with \( g = m \) is

\[
\bar{s}_t = \frac{1 - \frac{1 + \beta}{1 + (1 + g) / R}}{1 + (1 + g) / (1 + m)} = \frac{1 - \frac{1 + \beta}{1 + (1 + g) / R}}{2}
\]

Faster growth again reduces the aggregate savings rate.

(d) The point now becomes clear: If fast growth means that earnings grow rapidly with age, savings decline. If fast growth means that the slope of the age-earnings profile does not change, but new generations have higher earnings, savings increase simply because the young are relatively richer.

**Question 6. BF 3.6**

Assume a Cobb-Douglas production function \( F(K, L) = K^\alpha L^{1-\alpha} \) and the simplest two-period OLG model. Population grows at rate \( n \), individuals supply inelastically one unit of labor in the first period of their lives and have log utility \( u(c, x) = \ln(c) + \rho \ln(x) \).

(a) Solve for the steady state capital stock.

(b) Show how the introduction of pay-as-you-go social security, in which the government collects amount \( z \) from each young person and gives \( (1+n)z \) to each old person, affects the steady state capital stock.
**Question 7. Manna**

Consider the following version of the two-period overlapping generations economy we studied in class. In each period, a new cohort of size \( N_t = (1 + n)^t \) is born. There are no firms and there is no production. Instead, in each period the economy receives a fixed endowment \( M_t = mN_t \), which is called “manna” and drops from heaven. M can either be eaten or stored until \( t+1 \). If \( K \) units are stored, they grow to \((1+g)K\) units next period.

(a) Write down the resource constraint a planner faces in this world. Begin by writing down the total amount of goods available if the economy starts out with a “manna stock” of, say, \( K_t \) units of the good in period \( t \). Add to this the new endowment (\( M_t \)), subtract what is eaten; this gives you investment. Investment grows \((1+g)\) fold to yield date \( t+1 \)’s “manna stock.”

(b) Write down the planner’s problem. What are the choice variables? The planner’s utility function is \( v(c_1^o) + \sum_{t=1}^{\infty} \beta^t [u(c_1^y) + \beta u(c_{t+1}^o)] \).

(c) Write down the first-order conditions. Is there something analogous to an interest rate?

**Answer: Manna**

(a) Resource constraint: Start out with \( K_1 \). Receive an endowment of \( M_1 \). Total resources in \( t \) are then \( K_t + M_t \). Therefore: \( K_{t+1} = (1+g)[M_t + K_t - N_t c_t^y - N_{t-1} c_{t}^o] \).

(b) The planner’s problem is to maximize the utility function subject to a sequence of feasibility constraints with the initial condition \( K_1 \) given. It is useful to write the feasibility constraint in per capita terms: \( k_{t+1} (1 + n) = (1 + g)[m_t + k_t - c_t^y - c_t^o / (1 + n)] \)

\[ \Gamma = v(c_1^o) + \sum_{t=1}^{\infty} \beta^t [u(c_1^y) + \beta u(c_{t+1}^o)] + \sum_{t=1}^{\infty} \lambda_t (1 + g)[m_t + k_t - c_t^y - c_t^o / (1 + n)] - k_{t+1} (1 + n) \]

Even nicer is to solve the feasibility constraints for young consumption and substitute into the objective function:

\[ \Gamma = v(c_1^o) + \sum_{t=1}^{\infty} \beta^t \{ u(m_t + k_t - c_t^o / (1 + n) - k_{t+1} (1 + n) / (1 + g)) + \beta u(c_{t+1}^o) \} \]

The FOCs are

\[ \beta^{t-1} u'(c_t^o) = \beta^t u'(c_t^y) / (1 + n) \]

\[ \beta^{t+1} u'(c_{t+1}^y) = \beta^t u'(c_{t+1}^y) (1 + n) / (1 + g) \]

The analogue to the interest is \( g \).

**Question 8. Bequests**

Derive the conditions characterizing the competitive equilibrium for an economy of two-period households with altruistic bequests. That is, the date 1 household solves

\[ \max V_t = u(c_t, x_{t+1}) + \beta V_{t+1} \] subject to the budget constraints (for all \( t \))

\[ c_t - w_t - b_t = (-b_{t+1} - x_{t+1}) / R_{t+1} \]
Show that these conditions are equivalent to the ones characterizing the planner's problem, where
the planner has the objective function
\[ \bar{V}_t = \sum_{t=1}^{\infty} \beta^t u(c_t, x_{t+1}). \]

Follow these steps:
(a) We know the condition characterizing the centrally planned economy. Just write them down to see what you are aiming for.
(b) Write out the household’s problem in full glory. Expand the sums defining the utility function and the present value budget constraint around some value for \( t \) (use a specific number like \( t = 5 \) if that helps).
(c) Set up the Lagrangian. There are now infinitely many constraints!
(d) Take the FOC’s with respect to \( c_t, x_t, b_t \) (yes: \( b_t \) as well!)
(e) Eliminate the multipliers. You should get exactly the same FOCs as for the planner’s problem, except that there are prices instead of marginal products of capital and labor.

Be careful at the beginning of the question. If you get the setup wrong…

**Question 9. The Golden Rule**

The Golden Rule requires \( f'(k) + 1 - \delta = n \).

(a) Explain what this rule means in economic terms. That is, exactly in what sense is consumption maximized when \( k \) is at the Golden Rule level?
(b) How can it be that consumption is lower when \( k \) is above the Golden Rule level? After all, isn’t total output higher?
(c) If the Golden Rule maximizes consumption, how can it be that there are other capital stocks that are Pareto optimal? Couldn’t we make everybody better off by moving to the Golden Rule where everybody’s consumption is higher?
(d) What does it mean for an economy to be dynamically inefficient? You should give an explanation, not just a definition.
(e) If an economy is dynamically inefficient, what policies would you suggest to fix this problem?
(f) Does a planner ever choose a dynamically inefficient allocation? Explain.
(g) Does a market economy ever attain a dynamically inefficient level of \( k \)? What if households are linked by altruistic bequests? Explain.

**Answer**

(a) The Golden Rule maximizes steady state consumption (per capita young). I.e., it maximizes consumption subject to the constraint that constant levels of consumption and capital have to be maintained over time.

(b) Total output is indeed higher when \( k \) is above the Golden Rule level. However, maintaining a constant (per capita) capital stock requires an investment of \( n + \delta \) per unit of capital. If the
marginal product of capital is below $n + \delta$, additional capital does not "pay" for its own maintenance and consumption must be reduced to finance the additional investment.

(c) The key is that in order to get to the golden rule the economy must invest and therefore give up consumption of the earlier generations. The golden rule is the only Pareto optimum only as long as the economy is restricted to hold $k$ constant.

(d) Dynamic inefficiency is essentially overaccumulation. We could eat some $k$ now and still get higher consumption in the future. This is because the rate of return is so low that additional $k$ does not pay for its own maintenance.

(e) A policy would have to reduce savings. Social security (pay-as-you-go) would do that.

(f) The planner has a second policy by which he can transfer resources from young to old ages: transfers across generations. Whenever the rate of return on savings falls below $n$, this is superior. He therefore never chooses a dynamically inefficient $k$.

(g) Nothing prevents a market economy from attaining $k$ above the golden rule because the second policy available to the planner is missing. What determines $k$ is essentially households’ preference for consumption when old versus young. Even if there are altruistic bequests, this remains true. Bequests work only as transfers from old to young, not the other way around. Bequests can increase, but not decrease savings.

**Question 10. Capital externality**

Consider a standard two period olg model with constant population. Households solve

$$\max \ln(c^y_t) + \beta \ln(c^y_{t+1}) \quad \text{s.t.} \quad w_t = c^y_t + s_t, \quad c^y_{t+1} = (1 + r_{t+1})s_t.$$ Firms produce according to

$$\theta K^\alpha_t H_t^{1-\alpha} N^{1-\alpha},$$

where $H$ is an externality. In equilibrium $H = K$, but firms take $H$ as given. There is no depreciation.

(a) Derive the FOCs for profit maximization.

(b) Derive a difference equation for $K$. What is the growth rate of the economy if $\alpha = 0.3, \beta = 0.9, \theta = 3.1$? What is the level of the interest rate? [I do not need to say "in steady state" b/c the model does not have transitional dynamics.]

**Answer**

(a) The FOCs are entirely standard, but the firm views $\overline{\theta} = \theta H^{1-\alpha}$ as the productivity parameter.

(b) In equilibrium, $s_t = w_t \beta/(1 + \beta) = k_{t+1}$. Substituting the FOC for $w$ yields

$$K_{t+1} = \frac{\beta \theta (1 - \alpha)}{1 + \beta} K_t.$$  

Therefore, the growth rate of $K$ is a constant equal to 0.028 for the parameters given. The interest rate is $\alpha \theta = 0.93$.

---

3 Based on a question due to Gregory Smith.
Question 11. Changes in labor supply

Consider a standard two period olg model with constant population. Households solve
\[ \max \ln(c_t^y) + \beta \ln(c_{t+1}^y) \quad \text{s.t.} \quad w_t n_t = c_t^y + s_t, \quad c_{t+1}^y = (1 + r_{t+1})s_t, \] where \( n_t \) is exogenous. Firms produce according to \( K_t^n = n_t^{1-\alpha} \). There is no depreciation.

(a) Solve for the steady state values of \( w \) and \( r \), if \( n_t = 0.5 \), \( \alpha = 0.5 \) and \( \beta = 0.9 \).

(b) Suppose that labor force participation doubles unexpectedly to \( n = 1 \) for dates \( t \) and beyond. Find the steady state effects on \( w \), \( r \), and per capita output.

(c) Describe in words the transitional dynamics of \( w \) and \( r \) after the unexpected increase in \( n \). Note that the young at \( t-1 \) made their savings decisions based on the expectation that \( n = 0.5 \) would last forever.

Answer

(a) \( s = wn\beta/(1+\beta) \). Therefore, \( K^* = 0.028, w^* = 0.118, r^* = 2.11 \).

(b) Output and \( K^* \) both double. Prices remain unchanged.

(c) Initially, \( k \) is below steady state so that \( w \) falls and \( r \) rises. Total output rises. Therefore labor income rises (constant factor shares) and with it savings. Gradually, the capital-labor ratio reverts to the steady state.

---

4 Based on a question due to Gregory Smith.
Fiscal Policy

Question 12. Government bonds in an OLG model

[Final 1998] At each date a measure \( N_t = (1+n)^t \) of households are born who live for 2 periods. Preferences are given by \((1-\beta)\ln(c_t^y) + \beta \ln(c_{t+1}^o)\) with \(0 < \beta < 1\). When young, each household supplies one unit of labor and earns \(w_t\), so that the budget constraint is \(s_t = w_t - c_t^y\). When old, the household is retired and consumes \(c_{t+1}^o = R_{t+1} s_t\), where \(R = 1 + r - \delta\) and \(r\) is the rental price of capital.

The production technology is \(C + K' - (1-\delta)K = F(K, L) = K^\alpha L^{1-\alpha}\) with \(0 < \alpha < 1\).

The government only rolls over debt from one period to the next: \(B_{t+1} = R_t B_t\)

(a) Solve the household problem for a saving function.

(b) Derive the FOCs for the firm.

(c) Define a competitive equilibrium. Make sure the number of variables equals the number of independent equations.

(d) Derive the law of motion for the capital stock: \((b_{t+1} + k_{t+1})(1+n) = \beta(1-\alpha)k_t^o\), where \(b = B / L\).

(e) Derive the steady state capital stock for \(b = 0\). Why does it not depend on \(\delta\)?

(f) Derive the steady state capital stock for \(b > 0\).

(g) Can you show that the capital stock is lower in the steady state with positive debt (crowding out)? Warning: This is tricky. Don’t spend too much time on it.

Answer: Government bonds

(a) The household solves \(\max (1-\beta)\ln(w-s) + \beta \ln(R's)\). The FOC is \(c'/c = R'\beta/(1-\beta)\). Therefore \(s = (w-s)\beta/(1-\beta)\) and thus \(s = \beta w\).

(b) This is standard: \(r = f'(k) = \alpha k^{\alpha-1}\), \(w = f(k) - f'(k) k = (1-\alpha)k^\alpha\), where \(k = K/L\).

(c) A CE is a list of sequences \((c_t^y, c_t^o, s_t, K_t, L_t, b_t, w_t, r_t)\) that satisfies

• the saving function and the 2 household budget constraints
• the 2 firm FOCs
• \(L_t = (1+n)^t\)
• capital market clearing: \(s_t = (1+n)(b_{t+1} + k_{t+1})\)
• goods market clearing: \(N_t c_t^y + N_{t-1} c_t^o + K_{t+1} = F(K_t, L_t) + (1-\delta)K_t\)
• government budget constraint
(d) This follows directly from the capital market clearing condition together with the equilibrium levels of \( w \) and the saving function.

(e) \( k^{1-\alpha} = \beta (1 - \alpha)/(1 + n) \). It does not depend on \( \delta \) because of log utility: households save a constant fraction of earnings.

(f) Now we need to satisfy the law of motion for \( b \): \( b'(1+n) = R b \). In steady state: \( R = 1 + n \). The steady state capital stock therefore satisfies \( \alpha k^{\alpha-1} - \delta = n \) or \( k^{1-\alpha} = \alpha/(n+\delta) \). Note that the steady state satisfies the Golden Rule. There is some concern that this steady state may not be stable. Imagine that \( R > 1 + n \). Then \( b \) rises (\( b' > b \)). This may reduce the capital stocks and drive up \( R \) even further, etc.

(g) Note that from the law of motion derived in (d):

\[
\frac{\beta (1 - \alpha)}{1 + n} k^{\alpha} - k = k \left[ \frac{\beta (1 - \alpha)}{1 + n} k^{\alpha-1} - 1 \right].
\]

Therefore, \( b > 0 \) requires \( \beta (1 - \alpha)/(1 + n) > \alpha/(n + \delta) \), which is exactly what was to be shown.

**Question 13. Labor income taxes in an OLG model**

[Midterm Fall 1998, 40 min] Consider a two-period OLG economy with production.

The representative household has utility \( \ln(c_t^Y) + \beta \ln(c_{t+1}^0) \). Young households supply one unit of labor to firms at wage rate \( w_t \) and rent savings to firms at rental rate \( r_t \). There are no other assets (no bonds). There is no population growth and the size of each cohort is normalized to 1.

Firms produce from capital and labor according to \( Y_t = K_t^0 L_t^{1-\theta} \). There is no depreciation. The resource constraint is therefore \( Y_t + K_t = c_t^Y + c_{t+1}^0 + K_{t+1} \).

(a) Derive the optimal level of savings of the household as a function of \( w \). Briefly, why do savings not depend on \( r \)?

(b) Derive the FOCs for the firm.

(c) Define a competitive equilibrium. Be sure to clearly state the market clearing conditions and to ensure that the number of independent equations equals the number of endogenous variables.

(d) Write down a difference equation for the equilibrium capital-labor ratio \( (k_t = K_t / L_t) \). Sketch a graph of this relationship.

(e) The government now imposes a time-invariant tax \( \tau \) on labor income of the young so that after-tax earnings are \( (1 - \tau) w_t \). The revenues are thrown into the ocean. By how much does this tax lower the savings of the young for given \( w \)? Briefly, what is the intuition for this result?

(f) How does the tax affect the relationship graphed in (d)? What happens to the steady state capital-labor ratio? Sketch a graph.
Question 14. Ricardian Equivalence

Consider the following two-period OLG model. There are \( N = 100 \) households born at each date. The utility function is \( u(.) = c_{h,t}^y \beta c_{h,t+i}^o = c_{h,t}^y (c_{h,t+i}^o)^\beta \). All households receive endowments of \([2, 1]\) when young/old. Now consider alternative ways of financing a transfer scheme that takes 25 units of goods at date 1 from generation 1 and transfers them to generation 0. Compare the following ways of financing this transfer:

(a) The government collects a lump-sum tax of \( \frac{1}{4} \) from each young at date 1.

(b) At date 1 the government sells securities that are titles to date 2 goods. It also announces that members of generation 1 will be taxed equally at \( t = 2 \) to pay off the bonds.

(c) At date 1 the government sells securities that are titles to date 2 goods. It also announces that members of generation 2 will be taxed equally at \( t = 2 \) to pay off the bonds.
Question 15. Government spending and interest rates

Consider a two period OLG model. At each data a cohort of size 1 is born. Preferences are 

\[ \ln(c_{t}^{y}) + \beta \ln(c_{t+1}^{o}) \]. 

The budget constraint is 

\[ c_{t}^{y} + c_{t+1}^{o} / (1 + r_{t+1}) = e_{t} - \tau_{t} \]. 

The endowment grows exogenously at rate \( \mu \): 

\[ e_{t+1} = (1 + \mu) e_{t} \]. 

The government balances its budget in every period: 

\[ g_{t} = \tau_{t} \]. 

(a) Solve for the individual’s consumption function.

(b) Assume \( g_{t+1} = (1 + \mu) g_{t} \). Solve for the equilibrium interest rate.

(c) What is the effect of an unexpected, temporary increase in \( g \) on the interest rate? (At one particular date \( g_{t} \) is increased by \( \varepsilon \).) For simplicity, assume \( \mu = 0 \).

Answer

(a) The result is familiar from log utility: 

\[ c_{t}^{y} = (e_{t} - \tau_{t}) / (1 + \beta) \]. 

\[ c_{t+1}^{o} / (1 + r) = (e_{t} - \tau_{t}) \beta / (1 + \beta) \].

(b) Market clearing requires 

\[ c_{t}^{y} + c_{t+1}^{o} + g_{t} = e_{t} \]. 

Apply the consumption functions to the market clearing condition:

\[ e_{t} = (e_{t} - g_{t}) / (1 + \beta) + (1 + r_{t+1}) (e_{t-1} - g_{t-1}) \beta / (1 + \beta) + g_{t} \].

Therefore

\[ (e_{t} - g_{t}) \{1 - 1 / (1 + \beta) - (1 + r_{t+1}) \beta / (1 + \beta) / (1 + \mu)\} = 0 \]

That gives \( r = \mu \).

(c) Apply the consumption functions to the market clearing condition in the period of the shock:

\[ e = (e - g - \varepsilon) / (1 + \beta) + (1 + r) (e - g) \beta / (1 + \beta) + (g + \varepsilon) \].

Note that consumption of the old is not changed by \( \varepsilon \). Therefore, \( r \) falls in that period. In the next period, the constraint is

\[ e = (e - g) / (1 + \beta) + (1 + r) (e - g - \varepsilon) \beta / (1 + \beta) + g \].

Thus, \( r \) rises in the next period. Thereafter, it returns to \( r = \mu = 0 \).

Question 16. Fully Funded Social Security

This question confirms that fully funded social security has no effect on the equilibrium allocation. Consider the standard two-period OLG model with production. The utility function is 

\[ \ln(c_{t}^{y}) + \rho \ln(c_{t+1}^{o}) \]. 

Households work one unit of time when young.

(a) Derive the household's savings function 

\[ s(w_{t} - \tau_{t}^{y}, \tau_{t}^{o}, R_{t+1}) \] 

for given prices and tax rates.

(b) Assume that the production function is 

\[ F(K_{t}, L_{t}) = K_{t}^{\alpha} L_{t}^{1-\alpha} \], 

where \( 0 < \alpha < 1 \) is a given constant. Solve the firm’s problem.

---

5 Based on a question due to Gregory Smith.
(c) The government imposes a tax/transfer scheme that collects $\tau^y$ from each young household and pays $-\tau^{o}_{t+1} = \tau^y (1 + r_{t+1})$ to each old household. Tax revenues are rented to firms as capital. Define a competitive equilibrium. Be careful with the capital market clearing condition.

(d) Derive the steady state capital stock when taxes are zero.

(e) Derive the steady state capital stock for fully funded social security (positive $\tau^y$). Show that the allocation is independent of the tax rate $\tau^y$, given that $\tau^o$ adjusts as described.

(f) Now suppose that households are heterogeneous. Half of the households have a time endowment of 1 (as before), the others have an endowment of 2. Is fully funded social security still neutral? What if the tax on the young is a labor income tax instead ($\tau^y_h = \mu w_{h,t}$; $0 < \mu < 1$), but transfers ($\tau^o$) remain lump-sum and identical for everybody?

**Question 17. Fully-Funded Social Security**

Fully-funded social security authority taxes households when young, invests the tax revenues, and pays benefits to the old out of the capital income accumulated on their own contributions.

(a) Explain why fully-funded social security does not affect the steady state capital stock, if public and private savings earn the same rate of return.

(b) How would this result change if the public rate of return was lower than the private one?

**Answer**

(a) Households only care about the present value of future tax payments when deciding how much to consume. If the government earns the same rate of return as does the private sector, fully-funded social security does not alter this present value. Thus, consumption does not change and households reduce private savings by exactly the tax revenue. Therefore, total saving (public + private) remains unchanged.

(b) If the public rate of return is lower than the private one, the present value of lifetime resources available to the household declines. Given a rate of return, consumption at all ages is reduced. Therefore, private savings falls by less than the tax revenue and the capital stock increases. An easy way to see this is to note that such a policy is equivalent to a combination of case (a) plus a tax on the old.