CHARACTERISTIC ROOTS AND VECTORS

1. A Digression on Complex Numbers

1.1. **Definition of a complex number.** A complex number is an ordered pair of real numbers denoted by (x_1, x_2). The first member, x_1, is called the real part of the complex number; the second member, x_2, is called the imaginary part. We define equality, addition, subtraction, and multiplication so as to preserve the familiar rules of algebra for real numbers.

1.1.1. *Equality of complex numbers.* Two complex numbers x = (x_1, x_2) and y = (y_1, y_2) are called equal iff

$$x_1 = y_1, \text{ and } x_2 = y_2. \tag{1}$$

1.1.2. *Sum of complex numbers.* The sum of two complex numbers x + y is defined as

$$x + y = (x_1 + y_1, x_2 + y_2) \tag{2}$$

1.1.3. *Difference of complex numbers.* To subtract two complex numbers, the following rule applies.

$$x - y = (x_1 - y_1, x_2 - y_2) \tag{3}$$

1.1.4. *Product of complex numbers.* The product xy is defined as

$$xy = (x_1\, y_1 - x_2\, y_2, x_1\, y_2 + x_2\, y_1). \tag{4}$$

The properties of addition and multiplication defined satisfy the commutative, associative and distributive laws.

1.2. **The imaginary unit.** The complex number (0,1) is denoted by i and is called the imaginary unit. We can show that i^2 = -1 as follows.

$$i^2 = (0, 1)(0, 1) = (0 - 1, 0 + 0) = (-1, 0) = -1 \tag{5}$$

1.3. **Representation of a complex number.** A complex number x = (x_1, x_2) can be written in the form

$$x = x_1 + i\, x_2. \tag{6}$$

Alternatively a complex number z = (x,y) is sometimes written

$$z = x + iy. \tag{7}$$

1.4. **Modulus of a complex number.** The modulus or absolute value of a complex number x = (x_1, x_2) is the nonnegative real number $|x|$ given by

$$|x| = \sqrt{x_1^2 \;+\; x_2^2} \tag{8}$$

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1.5. **Complex conjugate of a complex number.** For each complex number z = x + iy, the number z_- = x - iy is called the complex conjugate of z. The product of a complex number and its conjugate is a real number. In particular, if z = x + iy then

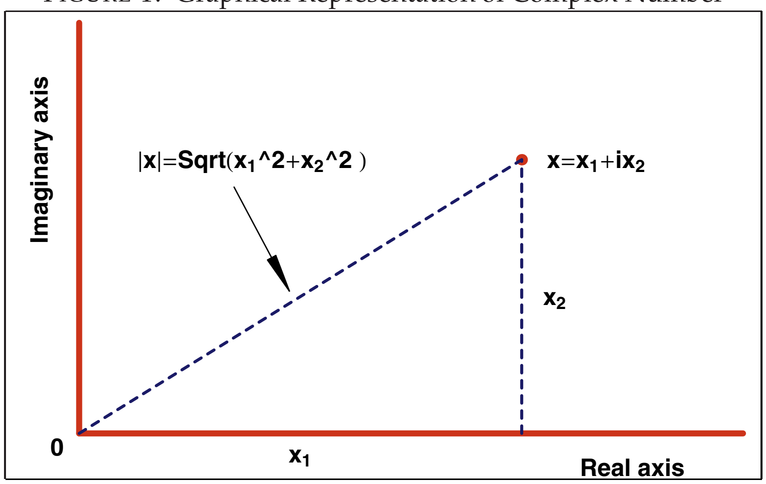
$$z\, z_- = (x, y)(x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = x^2 + y^2 \tag{9}$$

Sometimes we will use the notation $\bar{z}$ to represent the complex conjugate of a complex number. So $\bar{z}$ = x - iy. We can then write

$$z\, \bar{z} = (x, y)(x, -y) = (x^2 + y^2, -xy + yx) = (x^2 + y^2, 0) = x^2 + y^2 \tag{10}$$

1.6. **Graphical representation of a complex number.** Consider representing a complex number in a two dimensional graph with the vertical axis representing the imaginary part. In this framework the modulus of the complex number is the distance from the origin to the point. This is seen clearly in figure 1

FIGURE 1. Graphical Representation of Complex Number



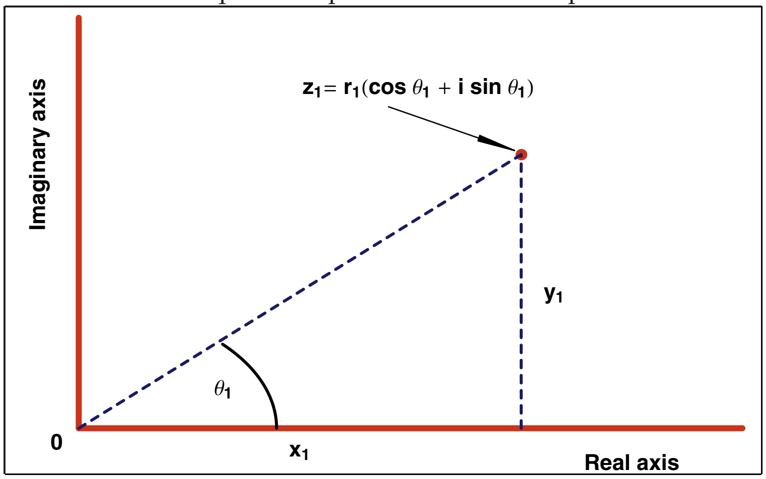
1.7. **Polar form of a complex number.** We can represent a complex number by its angle and distance from the origin. Consider a complex number $z_1 = x_1$ + i y_1. Now consider the angle θ_1 which the ray from the origin to the point z_1 makes with the x axis. Let the modulus of z be denoted by r_1. Then cos $\theta_1 = x_1/r_1$ and sin $\theta_1 = y_1/r_1$. This then implies that

$$\begin{aligned} z_1 &= x_1 + i\, y_1 = r_1 \cos\theta_1 + i\, r_1 \sin\theta_1 \\ &= r_1 (\cos\theta_1 + i \sin\theta_1) \end{aligned} \tag{11}$$

Figure 2 shows how a complex number is represented in polar corrdinates.

1.8. **Complex Exponentials.** The exponential e^x is a real number. We want to define e^z when z is a complex number in such a way that the principle properties of the real exponential function will be preserved. The main properties of e^x, for x real, are the law of exponents, $e^{x_1}e^{x_2} = e^{x_1+x_2}$ and the equation $e^0 = 1$. If we want the law of exponents to hold for complex numbers, then it must be that

FIGURE 2. Graphical Representation of Complex Number



$$e^z \; = \; e^{x+iy} \; = \; e^x \, e^{iy} \tag{12}$$

We already know the meaning of e^x. We therefore need to define what we mean by e^{iy}. Specifically we define e^{iy} in equation 13

Definition 1.

$$e^{iy} \; = \; \cos \, y + i \, \sin \, y \tag{13}$$

With this in mind we can then define $e^z = e^{x+iy}$ as follows

Definition 2.

$$e^z \; = \; e^x \, e^{iy} \; = \; e^x(\cos \, y + i \, \sin \, y) \tag{14}$$

Obviously if x = 0 so that z is a pure imaginary number this yields

$$e^{iy} \; = \; (\cos \, y + i \, \sin \, y) \tag{15}$$

It is easy to show that $e^0 = 1$. If z is real then y = 0. Equation 16 then becomes

$$\begin{aligned} e^z \; &= e^x \, e^{i\,0} \; = \; e^x(\cos\,(0) + i \, \sin\,(0)\,) \\ &= e^x \, e^{i\,0} \; = \; e^x(1 \; + \; 0\,) \; = \; e^x \end{aligned} \tag{16}$$

So e^0 obviously is equal to 1.

To show that $e^x e^{iy} = e^{x \, + \, iy}$ or $e^{z_1} e^{z_2} = e^{z_1 \, + \, z_2}$ we will need to remember some trigonometric formulas.

Theorem 1.

$$\sin(\phi \; + \; \theta) \; = \sin \, \phi \, \cos \, \theta \; + \; \cos \, \phi \, \sin \, \theta \tag{17a}$$

$$\sin(\phi \; - \; \theta) \; = \sin \, \phi \, \cos \, \theta \; - \; \cos \, \phi \, \sin \, \theta \tag{17b}$$

$$\cos(\phi \; + \; \theta) \; = \cos \, \phi \, \cos \, \theta \; - \; \sin \, \phi \, \sin \, \theta \tag{17c}$$

$$\cos(\phi \; - \; \theta) \; = \cos \, \phi \, \cos \, \theta \; + \; \sin \, \phi \, \sin \, \theta \tag{17d}$$

Now to the theorem showing that $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

Theorem 2. *If $z_1 = x_1 + iy_t$ and $z_2 = x_2 + iy_2$ are two complex numbers, then we have $e^{z_1} e^{z_2} = e^{z_1+z_2}$.*

Proof.

$$\begin{aligned} e^{z_1} &= e^{x_1}(cos\ y_1 \ + \ i\ sin\ y_1),\ e^{z_2} \ = \ e^{x_2}(cos\ y_1 \ + \ i\ sin\ y_2), \\ e^{z_1} e^{z_2} &= e^{x_1} e^{x_2}[cos\ y_1\ cos\ y_2 \ - \ sin\ y_1\ sin\ y_2 \\ &\quad + \ i(cos\ y_1\ sin\ y_2 \ + \ sin\ y_1\ cos\ y_2)]. \end{aligned} \tag{18}$$

Now $e^{x_1} e^{x_2} \ = \ e^{x_1+x_2}$, since x_1 and x_2 are both real. Also,

$$cos\ y_1\ cos\ y_2 \ - \ sin\ y_1\ sin\ y_2 \ = \ cos\,(y_1 \ + \ y_2) \tag{19}$$

and

$$cos\ y_1\ sin\ y_2 \ + \ sin\ y_1\ cos\ y_2 \ = \ sin\,(y_1 \ + \ y_2), \tag{20}$$

and hence

$$e^{z_1} e^{z_2} \ = \ e^{x_1+x_2}[cos\,(y_1 \ + \ y_2) \ + \ i\ sin\,(y_1 \ + \ y_2) \ = \ e^{z_1+z_2}. \tag{21}$$

□

Now combine the results in equations 11, 16 and 15 to obtain the polar representation of a complex number

$$\begin{aligned} z_1 &= x_1 \ + \ i\, y_1 \\ &= r_1\,(\cos\ \theta_1 \ + \ i\ \sin\ \theta_1) \\ &= r_1\, e^{i\,\theta_1}, \qquad \text{(by equation 13)} \end{aligned} \tag{22}$$

The usual rules for multiplication and division hold so that

$$\begin{aligned} z_1\ z_2 &= r_1\, e^{i\theta_1} r_2\, e^{i\theta_2} \ = \ (r_1\, r_2)\, e^{i(\theta_1 + \ \theta_2)} \\ \frac{z_1}{z_2} &= \frac{r_1\, e^{i\theta_1}}{r_2\, e^{i\theta_2}} \ = \ \left(\frac{r_1}{r_2}\right)\ e^{i(\theta_1 - \ \theta_2)} \end{aligned} \tag{23}$$

1.9. **Complex vectors.** We can extend the concept of a complex number to complex vectors where z = x + iy is a vector as are the components x and y. In this case the modulus is given by

$$|z| \ = \ \sqrt{x'x \ + \ y'y} \ = \ (x'x \ + \ y'y)^{\frac{1}{2}} \tag{24}$$