1. Introduction

This section extends the two-period model by assuming that the economy lasts forever, but each household still only lives for two periods. At each date, a new generation of households is born. Therefore, two generations are alive at $t$: the young born at $t$ and the old born at $t-1$.

This setup is easily extended to longer-lived households and may seem to be the only sensible way of modeling households. However, it is often more convenient to assume that households are infinitely lived, which greatly simplifies the algebra. One justification is that households may be linked by intergenerational altruism (Barro 1974) — we will come back to this. However, little is known about when infinite horizons are a good approximation for finite horizons.\(^1\)

Finite horizons give rise to an important market imperfection: there cannot be any intergenerational borrowing and lending. For example, the young at $t$ cannot borrow from the old because the old won’t be around at $t+1$ to have their loans repaid. At first, this appears to be an artifact of having just two periods. For example, if each household lives for three periods, generation $t$ can borrow from $t+1$ in the first year of life and repay the debt in the second year. Still, that does not solve the problem if the young typically want to borrow (low earnings) and repay their debt only when old.

The issues studied in this section are:

- How to set up and solve an OLG model
- Dynamic efficiency and the Golden Rule
- “Social security” prevents dynamic inefficiency
- Altruistic bequests convert the households into infinitely lived dynasties (sort of).

2. An OLG Model Without Firms

Demographics: Each period a cohort of mass $N_t = N_0 (1 + n)^t$ is born, which lives for two periods. Therefore, at each date there are $N_t$ young and $N_{t-1}$ old households around.

Markets: Households can issue one period bonds with interest rate $r_{t+1}$.

---

\(^1\) Some of my own research deals with that question, and the results are not encouraging for IH models with human capital.
**Households**

Consider the representative member of the cohort born at date \( t \). It derives utility from consumption when young and when old: \( u(c^y_t) + \beta u(c^o_{t+1}) \). It receives an endowment \( w_t \) when young, which can be stored to yield \( f(s) \) at \( t+1 \), where \( s \) is the amount stored at \( t \). The budget constraints are

\[
w_t = c^y_t + s_{t+1} + b_{t+1}, \quad c^o_{t+1} = f(s_{t+1}) + b_{t+1} (1 + r_{t+1})
\]

The lifetime budget constraint is therefore

\[
w_t - c^y_t - s_{t+1} = [c^o_{t+1} - f(s_{t+1})]/[1 + r_{t+1}]
\]

To characterize the solution, set up the Lagrangean

\[
\Gamma = u(c^y_t) + \beta u(c^o_{t+1}) + \lambda \{[w_t - c^y_t - s_{t+1}] - [c^o_{t+1} - f(s_{t+1})]/[1 + r_{t+1}]\}
\]

and derive the FOCs

\[
u'(c^y_t) = \lambda, \quad \beta u'(c^o_{t+1}) = \lambda/(1 + r_{t+1}), \quad f'(s_{t+1}) = 1 + r_{t+1}
\]

which imply the Euler equation \( \beta f'(s_{t+1}) u'(c^o_{t+1}) = u'(c^y_t) \). Bond holdings follow residually from the period 1 budget constraint. The solution to the household problem consists of \( (c^y_t, c^o_{t+1}, s_{t+1}, b_{t+1}) \) which satisfy 2 FOCs (an EE and the foc for \( s \)) and the 2 budget constraints. This is (unsurprisingly) the same as in the two-period model: the household problem has not changed at all.

**Equilibrium**

An equilibrium is a sequence of prices and quantities \( \{r_t, c^y_t, c^o_t, s_t, b_t\} \) that satisfy

- 4 household conditions
- bond market clearing: \( b_t = 0 \)
- goods market clearing: \( N_t (c^y_t + s_{t+1}) + N_{t-1} c^o_t = N_t w_t + N_{t-1} f(s_t) \)

We have 5 objects in each period and 6 equations, one of which is redundant by Walras’ law. To understand the goods market clearing condition, note that total resources consist of young endowments and old production. These are used for consumption and saving. Since the old cannot trade with the young, there is no trade at all in equilibrium. The households effectively live in autarky and the economy behaves like a sequence of unrelated two period economies. Of course, households *could* trade any quantity of goods or bonds at fixed prices. But the prices adjust so that nobody wants to do so.
3. A Production Economy

The model is modified by adding firms who rent capital and labor from households. The endowment \( w \) is now interpreted as labor earnings. Households supply one unit of labor inelastically to firms when young. Capital depreciates at rate \( \delta \).

**Households**

(1) Budget constraints: 
\[
w_t = c_t^y + s_{t+1} + b_{t+1}, \quad c_{t+1}^o = (s_{t+1} + b_{t+1})(1 + r_{t+1})
\]

There are no profits b/c the technology has constant returns to scale. The lifetime budget constraint is 
\[
w_t - c_t^y = c_{t+1}^o /[1 + r_{t+1}]
\]

Lagrangean: 
\[
\Gamma = u(c_t^y) + \beta u(c_{t+1}^o) + \lambda \{w_t - c_t^y - c_{t+1}^o /[1 + r_{t+1}]\}
\]

(2) FOCs: 
\[
u'(c_t^y) = \lambda, \quad \beta u'(c_{t+1}^o) = \lambda /[1 + r_{t+1}]
\]

Euler: 
\[
\beta(1 + r_{t+1})u'(c_{t+1}^o) = u'(c_t^y)
\]

The solution of the household problem is \((c_t^y, c_{t+1}^o, s_{t+1}, b_{t+1})\) that satisfy 2 budget constraints and 1 EE. We lack one equation! The reason is that the household is indifferent as to the composition of its portfolio, given that both assets have the same rate of return.

**Firms**

Firms maximize current period profits taking factor prices \((r, w)\) as given. The technology, \( Y = F(K, L) \), has constant returns to scale and satisfies Inada conditions.

Objective function: 
\[
\text{max } F(K, L) - wL - qK
\]

FOCs: 
\[
q = F_K(K, L), \quad w = F_L(K, L)
\]

It is almost always convenient to write the production function in **intensive form**, 
\[
F(K, L) = LF(K/L, 1) = L f(k^F), \quad \text{where } k^F = K/L \text{ and } f(k^F) = F(k^F, 1). \text{ This, of course, requires constant returns to scale. Now the factor prices are } F_K = L f'(k^F)(1/L) \text{ and } F_L = f(k^F) + L f''(k^F)(-K/L^2) = f(k^F) - f'(k^F)k^F. \text{ Therefore,}
\]

(3) 
\[
q = f'(k^F), \quad w = f(k^F) - k^F f'(k^F).
\]

Important: \( q \) is the rental price of capital, which may differ from the interest rate. The solution to the firm’s problem is a pair \((K, L)\) so that the 2 FOCs hold.

**Market clearing**

- Capital rental: 
  \[
  K_t = N_{t-1} s_t
  \]
Labor rental: \( L_t = N_t \)
Bonds: \( b_t = 0 \)
Goods: \( F(K_t, L_t) = N_t (c_t^Y + s_{t+1}) + N_{t-1} (c_t^O - s_t [1 - \delta]) \)

The goods market clearing condition is important. The total amount of resources available at \( t \) consists of production \( F \) and of the undepreciated capital stock \( K_t (1 - \delta) \). Resources are used for consumption of the current young and old and for saving. Another way of writing the market clearing condition is therefore: \( K_{t+1} = (1 - \delta) K_t + F(K_t, L_t) - C_t \), which is a way of writing it that we will see often.

Next we need the relationship between \( r \) and \( q \). The household receives \( 1 + r_{t+1} = q_{t+1} + 1 - \delta \) per unit of capital. Therefore, \( q = r - \delta \).

A competitive equilibrium is a sequence of vectors \( (c_t^Y, c_t^O, s_t, b_t, K_t, L_t, q_t, r_t, w_t) \) that satisfy:
- the household EE and budget constraints (3 equations)
- the firm’s FOCs (2 equations)
- the market clearing conditions (4 equations)
- the accounting identity \( q = r - \delta \).

We therefore have 10 equations and 9 unknowns. One equation is redundant b/c of Walras’ law.

**Timing within periods**

Everything we have done so far uses *discrete time*. Time is divided into discrete periods and all events are supposed to happen at discrete points in time. This can be a source of confusion because there are often implicit assumptions about timing “within” periods. For example, young households save some goods in period \( t \), but production takes place only in \( t+1 \), when the new young are around to supply labor. Thus, there is an implicit assumption that the capital \( K_t \) must be in place “before” production can take place at \( t \).

An alternative is to model time as continuous, without discrete periods. This is cleaner and sometimes analytically more convenient. The choice between the two approaches is purely a matter of convenience. We will return to continuous time models in the section on infinite horizon models.

**3.2 Saving Function and Dynamics**

Among the things we’d like to be able to do with models like these is to determine their equilibrium dynamics and to figure out the consequences of exogenous shocks on the equilibrium. In other words, we would like to be able to answer question like these:
1. If the economy starts out with initial condition $K_0$, what happens to the endogenous variables over time? Do they perhaps converge to some constant (over time) levels? In other words, we would like to determine the **transitional dynamics** of the model.

2. If the exogenous parameter $x$ changes, what happens to the endogenous variables in equilibrium? Note that an equilibrium consists of an entire infinite path of all endogenous variables, which has to be recomputed after a shock occurs. This type of analysis is (sometimes) called **comparative dynamics**.

3. There is a static counterpart to the second type of question. If the economy converges to a constant equilibrium (a **steady state**), then we might ask: If the exogenous parameter $x$ changes, what happens to the endogenous variables in steady state? This type of analysis is called **comparative statics**.

How do we figure out the transitional dynamics? First, we need to know what the economy’s **state variables** are. State variables (or states) are those variables that completely determine the future evolution of the economy. A common approach to determine transitional dynamics is to condense the equilibrium conditions into a set of difference equations that contain only the states as endogenous variables. One can then apply results from dynamical systems theory to study the properties of the system of difference equations.

What are the states of our model? We can find this out by elimination. Can $s$, $c^y$, or $c^o$ be state variables? Clearly not. We can figure those out by solving the household problem given prices and endowments. Can the prices be state variables? Clearly not: we can figure out the prices from $K$ and $L$. $L$ and $b$ may be thought of as a state variables (the household’s time endowment and the bonds they bring into the period), but they are constant or exogenous anyway. That leaves only $K$. As a (not very general) rule: State variables are stocks, not flows. But more complicated objects can be state variables as well, such as distributions or technology parameters.

So we should aim at developing a difference equation for $K$. It turns out it is more convenient to use $k = K / L$ instead. We can do that because $L$ is exogenous. Combining the market clearing conditions for capital and labor implies $K_t / N_t = N_{t-1} / N_t \cdot s_t$

\[ (1 + n) k_t = s_t, \]

which is the fundamental law of motion for the capital stock of this economy. In order to make this useful, we need to find out the properties of the saving function $s_t$. And we need to express the right-hand-side of (4) as a function of $k$ only.

Starting from the Euler equation, $\beta (1 + r_{t+1}) u'(c^o_{t+1}) = u' (c^y_t)$, we can substitute in the budget constraints for both ages: $\beta (1 + r_{t+1}) u'( [1 + r_{t+1}] s_{t+1}) = u'(w_t - s_{t+1})$. This implicitly defines a **saving function**
What are the signs of the partials of $s(.)$? The clean way of determining this would be to use the Implicit Function Theorem. But a simple alternative is to totally differentiate (5) and to solve for the “ratio” $ds_{t+1}/dw_t$. This is strictly speaking a sin, but leads to the same result. So we just do it:

$$
\beta (1 + r_{t+1})^2 u''([1 + r_{t+1}]^t_{s_{t+1}}) ds_{t+1} = u''(w_t - s_{t+1})(dw_t - ds_{t+1}).
$$

Since $u'' < 0$, higher wages when young increase savings — simply an income effect. A higher interest rate has opposing income and substitution effects, so that the effect on savings is ambiguous:

$$\begin{align*}
\beta u'([1 + r_{t+1}]^t_{s_{t+1}}) dr_{t+1} + \beta (1 + r_{t+1}) u''(c^o_{t+1})(s_{t+1} dr_{t+1} + [1 + r_{t+1}] ds_{t+1}) = -u''(w_t - s_{t+1}) ds_{t+1} \\
\frac{\partial s_{t+1}}{\partial r_{t+1}} = -\frac{\beta u'(c^o_{t+1}) + \beta (1 + r_{t+1}) u''(c^o_{t+1}) s_{t+1}}{\beta (1 + r_{t+1})^2 u''(c^o_{t+1}) + u''(c^{T}_{t})}
\end{align*}$$

The last step uses the 2nd period budget constraint to replace $(1 + r_{t+1}) s_{t+1}$ by $c^o_{t+1}$. Then it defines

$$
\sigma \equiv -u''(c)/u'(c),
$$

which is known as the coefficient of relative risk aversion. In particular, in the popular CRRA utility function, $u(c) = c^{1-\sigma}/(1-\sigma)$, the coefficient is constant (namely $\sigma$, show this!). It follows that savings respond positively to the interest rate, if $\sigma < 1$ (and not at all with log utility; $\sigma = 1$). In the data, $\sigma$ is most likely greater than one, although its value is highly controversial. The figure illustrates the case where income and substitution effect just cancel.
We can use this savings function to derive a law of motion for capital. Note that \( w \) and \( r \) are both functions of \( k \) and that 
\[
(1+n)k_{t+1} = s_{t+1}.
\]
Therefore, 
\[
(6) \quad (1+n)k_{t+1} = s(f(k_t) - f'(k_t)k_t + f''(k_{t+1}))
\]
which is an implicit difference equation 
\[
k_{t+1} = \phi(k_t).
\]
Implicitly differentiating yields 
\[
\frac{d}{dk_t} k_{t+1} = -s_w f''(k_t) \cdot \frac{1 + n - s_r f''(k_{t+1})}{1 + n - s_r f''(k_{t+1})}.
\]

### 3.3 Dynamics and Steady States

Equation (6) completely determines the dynamic behavior of this economy. Unfortunately, (6) need not be well-behaved at all. We know that \( k_t = 0 \Rightarrow k_{t+1} = 0 \). If \( s_r > 0 \), it is upward sloping, but little else can be said. Why is \( s_r \) so important. Intuitively, if \( s_r < 0 \), then the supply curve in the market for capital is downward sloping, which may give rise to multiple steady states of to steady states that are ill-behaved.

If the law of motion is strictly concave, the economy converges to a steady state level \( k^* \). If the law of motion is not monotone, this leads to oscillations that may or may not converge. We will see by example below that log utility and Cobb-Douglas technology ensure a well-behaved saving function.

What have we learned so far?

- In an OLG model, it is convenient to summarize the entire equilibrium path by a single difference equation which is the capital market clearing condition.
- Even this simple model may not be well-behaved.
3.4 The Golden Rule

We have already seen that there is a missing market in the model: the old cannot trade with the young. A central question is therefore: Is the equilibrium of an OLG model Pareto efficient?

To make progress answering this question, it is useful to first find the level of $k$ that maximizes steady state consumption. At first, the answer appears to be “$k = \infty$”. But the question is not: which $k$ maximizes current consumption? It is: which $k$ maximizes current consumption, given that we must save enough to have the same $k$ next period? It is easy to see that this $k$ must be finite. Even if we set $c = 0$, we cannot sustain an arbitrarily large $k$. With population growth (and depreciation) investment is required to simply maintain the capital stock at a constant (per capita) level. Therefore, if the marginal product of capital is too low, an additional unit of capital may not pay for its own replacement. What is the maximum sustainable capital stock? To find it, set consumption to zero and find the $k$ that satisfies the law of motion: $(1 + n)k' = f(k) + (1 - \delta)k$. At most we can sustain $f(k) = (n + \delta)k$.

So the level that maximizes steady state consumption must be finite. To find it, note that consumption per young household is $c' + c^o/(1 + n) = f(k) + (1 - \delta)k - (1 + n)k'$. Now impose the steady state requirement $k' = k$ and maximize with respect to $k$. The result is the Golden Rule: $f''(k) = n + \delta$. 

\[ f''(k) = n + \delta. \]
What does this mean? If we add another unit of $k$, we increase output by the MPK. But in order to maintain the capital stock constant, we need to invest $n$ units of today’s output, simply because there are more people around tomorrow. In addition, we need $\delta$ units to make up for depreciation. If we have “too much” capital, namely more than the Golden Rule capital stock, the MPK will be so low that an additional unit of $k$ simply doesn’t pay for its own maintenance.

The golden rule is important as an upper bound for efficient levels of capital accumulation. $k^* > k_{GR}$ cannot be Pareto optimal because eating more today allows to eat more at all future dates.

Note that nothing prevents such an outcome in competitive equilibrium. Economies with $k^* > k_{GR}$ are called dynamically inefficient (Diamond 1965). Why is this possible? There is a missing market: the old must finance their consumption out of own saving, even if the rate of return is very low. If a young household saves one unit of consumption, he has $1 - \delta + f'$ units of consumption at old age. However, if all generations entered into a contract where today’s young transfer one unit of consumption (per young household) to the current old, each old household would receive $1 + n$ units. Dynamic inefficiency means over accumulation to the point where the rate of return of saving falls below the “rate of return” of this transfer scheme. We will return to this idea in the section on “social security.”

3.5 Example: Log Utility and Cobb Douglas Production Function

Assume that $u(c) = \ln(c)$. Then the household saves a constant fraction of his earnings. From the Euler equation, $c_{t+1}^e = (1 + r_{t+1})\beta c_t^y$. Substituting into the present value budget constraint yields $w_t - c_t^y = \beta c_t^y$ or $c_t^y = w_t / (1 + \beta)$ and therefore $s_{t+1} = w_t \beta / (1 + \beta)$.

Assume further that $f(k) = k^\theta$. Then $w = (1 - \theta)k^\theta$. The law of motion then becomes

$$(1 + n)k_{t+1} = \frac{\beta}{1 + \beta}(1 - \theta)k_t^\theta,$$

which is strictly concave so that a unique, monotonically stable steady state exists:

$$k^* = \left[\frac{1 - \theta}{1 + n} \frac{\beta}{1 + \beta}\right]^{1/(1 - \theta)}.$$

This is easier to interpret in logs: $\ln(k^*) = \frac{\ln(1 - \theta) + \ln(\beta / (1 + \beta)) - \ln(1 + n)}{1 - \theta}$.

The steady state capital stock increases, if households are more patient (high $\beta$), have fewer children (dilution effect), or if the capital share ($\theta$) is lower (higher wages of the young). To see the effect of $\theta$, differentiate $\ln(k^*)$ w.r.to $(1 - \theta)$ to obtain
Since the argument of the ln(.) is less than one, the numerator is positive and a higher $\theta$ reduces steady state capital.

Under what conditions is this **dynamically inefficient**? The Golden Rule requires $f' (k_{GR}) = 0 k_{GR}^{\theta-1} = n + $ or $ k_{GR} = (\theta / n + \delta) ^{(1-\theta)}$. Dynamic inefficiency arises if

\[
\frac{1 - \theta \beta}{1 + n 1 + \beta} > \frac{\theta}{n + \delta} \quad \text{or} \quad \frac{1 - \theta n + \delta \beta}{\theta 1 + n 1 + \beta} > 1.
\]

Since in the data approximately $\theta = 1/3$, $\beta = 0.05$, and $n = 0.01$, this is not a likely outcome.

Another way of thinking about this is to ask directly: “does $r < n$ hold?” Viewed from this perspective, the answer is much more difficult (especially if the economy grows because then the Golden Rule becomes $f'(k) = g + n + \delta$).

### 3.6 Planning Solution

It is often useful to study a centrally planned economy instead of a competitive equilibrium for two reasons: (a) sometimes the allocations are the same, but the planning problem is much easier to solve than the CE; (b) the planning problem yields conditions for Pareto optimal allocations, which are useful as a point of comparison.

In an OLG model (or any model with heterogeneous agents) the question arises: what is the **objective function** of the planner? With infinite horizons we could simply assume that the planner maximizes utility of the representative agent. Now there is no representative agent. A natural generalization is to assume that the planner maximizes some weighted average of all households’ utilities alive at date 1 or later:

\[
\omega_0 \beta u(c_1^o) + \sum_{t=1}^{\infty} \omega_t [u(c_t^y) + \beta u(c_{t+1}^o)].
\]

By varying the weights ($\omega$), we can obtain all Pareto optimal allocations. To ensure that the objective function is finite, suitable conditions need to be imposed on the weights. The planner only faces feasibility constraints:

\[
(1-\delta)k_t + f(k_t) = c_t^y + c_t^o / (1+n) + (1+n)k_{t+1}.
\]

The choice variables are consumption and capital stocks, where $k_1$ is given by history. The Lagrangean is
\[ \Gamma = \omega_0 \beta u(c^y_t) + \sum_{t=1}^{\infty} \omega_t \left[ u(c^y_t) + \beta u(c^0_{t+1}) \right] + \sum_{t=1}^{\infty} \lambda_t \left[ (1-\delta)k_t + f(k_t) - c^y_t - c^0_t / (1+n) - (1+n)k_{t+1} \right] \]

This is a mess, but not really new. The first term is the objective function. In addition, there is one term for each of the infinitely many constraints, each with its own multiplier \( \lambda_t \). The FOCs are

\[ \omega_t u'(c^y_t) = \lambda_t, \quad \omega_{t-1} \beta u'(c^0_t) = \lambda_t / (1+n), \quad \lambda_t [1-\delta + f'(k_t)] = \lambda_{t-1} (1+n) \]

The trick in deriving the FOCs for \( k \) is to note that \( k_t \) occurs in the constraints for dates \( t \) and \( t-1 \). Combining the FOCs for consumption yields the static optimality condition:

\[ \omega_t u'(c^y_t) = \omega_{t-1} (1+n) \beta u'(c^0_t) \]

The interpretation is that the planner can take away a unit of consumption from the current young and give \((1+n)\) units to each of the current old. If the allocation is optimal, doing this (or the converse) cannot improve the planner’s utility. Substitution marginal utilities into the FOC for \( k \) yields the intertemporal condition:

\[ \omega_t u'(c^y_t) [1-\delta + f'(k_t)] = \omega_{t-1} u'(c^y_{t-1}) (1+n) \]

Again, this has the interpretation that a feasible perturbation of the optimal path cannot increase utility. Consider taking \((1+n)\) units of consumption from the young at date \( t-1 \) and investing them until date \( t \). This yields \([1-\delta + f'(k_t)]\) units of additional consumption for the young at date \( t \).

Using the static condition,

\[ \omega_t u'(c^y_t) / (1+n) = \omega_{t-1} \beta u'(c^0_t), \]

the intertemporal condition can be written as

\[ u'(c^y_t) = \beta u'(c^0_{t+1}) [1-\delta + f'(k_{t+1})] \]

which looks conspicuously like the Euler equation of the household. This is again not surprising: the planner should respect the individual FOCs unless there are externalities.

### 3.6.1 Steady State

Assume that the planner attaches weights to generations that decline at a constant geometric rate: \( \omega_t = \omega^t, \omega < 1 \). In steady state, (7) then becomes

\[ \omega(1-\delta + f'(k_{\text{MGR}})) = (1+n), \]

which is the modified Golden Rule. Clearly, \( k_{\text{MGR}} < k_{\text{GR}} \) because the planner would never choose a Pareto inefficient level of \( k \). How does the planner achieve dynamic efficiency? The key insight is that the planner can always take some consumption from the current young and give it to the current
old.\(^2\) The “rate of return” of this operation is \(n\). That is: \(\Delta c_i^y = -1 \Rightarrow \Delta c_i^o = +(1 + n)\). If the planner desires that each household consumes much more when old than when young (a high \(c_{t+1}^o / c_t^y\) ratio for all generations), he does not “save” a lot, which would drive down the rate of return. Instead he implements a transfer scheme like this:

### 3.7 Social Security

If a market economy is dynamically inefficient, is it possible to implement a tax-transfer scheme that makes everyone better off? Assume the government imposes lump-sum taxes/transfers of \(\tau^y\) and \(\tau^o\).

**Household**

Since the taxes are lump-sum, they do not affect the Euler equation:

\[
\beta(1 + r_{t+1})u'([1 + r_{t+1}] s_{t+1} - \tau^o) = u'(w_t - s_{t+1} - \tau^y)
\]

We can verify that the timing of taxes is irrelevant by computing the effects of a policy change that leaves the present value of tax payments unchanged: \(dt^o = -(1 + r_{t+1}) dt^y\). The household’s response to taxes is as follows:

\[
\frac{\partial s_{t+1}}{\partial \tau^y} = -\frac{u'(c_t^y)}{\beta(1 + r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} < 0
\]

\[
\frac{\partial s_{t+1}}{\partial \tau^o} = \frac{\beta(1 + r_{t+1}) u''(c_{t+1}^o)}{\beta(1 + r_{t+1})^2 u''(c_{t+1}^o) + u''(c_t^y)} > 0
\]

\(^2\) This is a bit misleading. The planner doesn’t need to take anything because he chooses all quantities directly.
(What is the intuition for those signs?) Note that \((1 + r_{t+1})\partial s_{t+1}/\partial \tau^0 - \partial s_{t+1}/\partial \tau^y = 1\). Therefore the change in consumption is zero:

\[
dc_t^y = -(\partial s_{t+1}/\partial \tau^y + 1) d\tau^y - (\partial s_{t+1}/\partial \tau^0) d\tau^0
\]

\[
= -(\partial s_{t+1}/\partial \tau^0)[(1 + r_{t+1}) d\tau^y + d\tau^0]
\]

This is a limited version of Ricardian Equivalence: Policy changes that leave the present value of tax changes for each household unchanged, do not affect consumption.\(^3\)

**Fully Funded Social Security**

In the fully funded system, the government invests the tax revenue \((N_t \tau^y)\), i.e. it supplies the revenue to firms as capital. Think of this as establishing a separate savings account for each household, in which the tax revenues from young age are invested until old age. This implies precisely the special case that leaves consumption unaffected: \(d\tau^o = -(1 + r_{t+1}) d\tau^y\). Moreover, private savings fall by the full tax revenue, leaving aggregate saving unchanged. The policy is therefore entirely neutral. Essentially, the government just relabels some private savings as public. (You can show this rigorously by solving for the CE with such a tax scheme.)

**Pay as you go Social Security**

In this case the government gives the tax revenue to the current old: \(d\tau^o = -(1 + n) d\tau^y\). This reduces current savings (for given \(k_t\) and \(k_{t+1}\)) for two reasons: current income is lower and future income is higher (old age transfer). Write the savings function as

\[
s_{t+1} = s(w(k_t) - \tau^y, - \tau^0, r(k_{t+1})).
\]

Since \(s_1 > 0\) and \(s_2 < 0\), the direct effect of the tax changes is to reduce savings. However, once we move to the law of motion for \(k\) things are more complicated. Totally differentiating (6) yields

\[
[1 + n - s_3 f''(k_{t+1})] dk_{t+1} = -s_1 d\tau^y - s_2 d\tau^o < 0.
\]

Both terms on the right are negative. Therefore, a sufficient condition for \(dk_{t+1} < 0\) is that \(s_3 > 0\). Then the law of motion unambiguously shifts down as illustrated in Figure 3. In that case, social security can alleviate dynamic inefficiency. More realistically, social security distorts the savings decision so as to move an already efficient economy further below the Golden Rule.

---

Note that the argument is not easily reversible: in a dynamically efficient economy, “reverse social security” is not a Pareto improvement: the first generation is harmed. Similarly, if there was a last generation, social security would not be Pareto improving. It relies on leaving debt to future generations.

Figure 3.

4. Bequests

If parents care about utility of their children and leave bequests, dynamic inefficiency is no longer an issue. Essentially, by reducing the bequest to the child the parent can implement the analogue of a social security scheme. However, once the bequest hits zero (parents cannot take from their kids) the possibility of dynamic inefficiency comes back. Thus, altruism does not prevent dynamic inefficiency, but as long as we observe positive bequests, we can be sure that the economy is dynamically efficient.

The first result we will establish is that a household with an operative bequest motive looks very much like one that lives forever (which is precisely the motivation for using infinite horizon models in most of macroeconomics). Suppose each household has \( n \) children that are born when the parent is old (for example, \( n = 1.0124 \) children – let’s assume children are perfectly divisible, or better, let’s ignore population growth and set \( n = 1 \)). A parent derives utility not only from his own consumption, but also from the utility level enjoyed by the child. Own consumption contributes \( u(c^o_t, c^o_{t+1}) \). Let’s call the total utility of a parent born at \( t \) \( V(t) \). Then

\[
V(t) = u(c^o_t, c^o_{t+1}) + \beta V(t+1).
\]
But note that $V(t+1)$ includes utility the child derives from the grand-child, which in turn derives utility from having grand-grand-children, which … you get the idea. By substituting for successive $V(t+j)$’s we get

$$V(t) = u(c^y_t, c^{o_{t+1}}_t) + \beta[u(c^y_{t+1}, c^{o_{t+2}}_{t+1}) + \beta V(t + 2)]$$

$$= u(c^y_t, c^{o_{t+1}}_t) + \beta u(c^y_{t+1}, c^{o_{t+2}}_{t+1}) + \beta^2 [u(c^y_{t+2}, c^{o_{t+3}}_{t+2}) + \beta V(t + 3)]$$

and so on. Expanding infinitely often we get

$$V(t) = \sum_{j=0}^{\infty} \beta^j u(c^y_{t+j}, c^{o_{t+j+1}}_t),$$

which looks exactly like the planner’s utility function. More interestingly, this looks like the utility function of a single household that lives forever.

But what about the budget constraint? An infinitely lived household’s budget constraint could be written as (present value of spending) = (present value of income) + (initial stock of assets). But even with altruism, it is still true that each household has its own present value budget constraint. But what if we allow for bequests?

Suppose each household can leave a bequest of $b_{t+1}$ at the beginning of his old age period $(t+1)$ to each of his $n$ children. The parent himself receives $b_t$ when young. The budget constraints are then $c^y_t + s_t = e_1 + b_t$ when young and $c^{o_{t+1}}_t + n b_{t+1} = e_2 + R_{t+1} s_t$ when old.

We can write a present value budget constraint $b_t = c^y_t - e_1 + (c^{o_{t+1}}_t - e_2 + b_{t+1}) / R_{t+1}$. But $b_{t+1}$ is defined similarly, so we can again expand the successive $b_{t+j}$’s to get a present value budget constraint (try it!)

$$b_t = \sum_{j=0}^{\infty} \frac{c^y_{t+j} - e_1}{D_{t,j}} + \sum_{j=1}^{\infty} \frac{c^{o_{t+j}}_t - e_2}{D_{t,j+1}},$$

where $D_{t,j} = \prod_{i=1}^{j} R_{t+i}$ for $j \geq 1$ is a cumulative discount factor from date $t+j$ to date $t$.

Therefore, introducing a bequest motive transforms the overlapping generations household effectively into an infinitely lived household with a single present value budget constraint. Which unfortunately gives rise to a wonderful, long exercise: Show that the equilibrium conditions coincide with the conditions that characterize the planner’s problem, if the bequest motive is operative (bequests are positive).

We now have the following results:

---

4 For now this is just argued by analogy to the finite horizon problem, but we will show this later on.
• If bequests are positive, the parent faces a present value budget constraint not only over his own lifetime, but over an infinite horizon.

• The parent therefore behaves exactly like an infinitely lived individual maximizing a single utility function over an infinite horizon subject to a single present value budget constraint.

This ignores one potential problem: The parent cannot actually force future generations to behave the way he would like them to behave; he can only indirectly change consumption of future generations by adjusting his own bequest. It turns out that this makes no difference. The parent wants each generation to maximize lifetime utility, given the available resources. Just like the planner, a parent does not want to interfere with how children use their resources. The bottom line: If we believe that bequests are indeed positive, we can simplify the analysis of the household problem substantially. We can simply solve the problem of a single household who lives forever and forget about how generations overlap and so on.

But are bequests positive? A thought experiment clarifies the conditions for that. Let’s assume the economy is in steady state. Suppose we ask a household whether he would like to increase his bequest by \( \varepsilon \). The parent would compare the loss in own utility with the gain of his children. Suppose the household reduces consumption when old in order to finance the increase in the bequest: \( \Delta c^o_{t+1} = -\varepsilon \). The loss in utility is then \( -u_2(t)\varepsilon \), where \( u(t) \) is utility of generation \( t \) (the parent). What does the child gain? Each child receives \( \varepsilon/n \). Suppose it uses this to increase consumption when young: \( \Delta c^y_{t+1} = +\varepsilon/n \). The gain is then \( \beta u_1(t+1)\varepsilon/n \).

The household wants to increase his bequest if \( \beta u_1(t+1)\varepsilon/n > u_2(t)\varepsilon \). This is hard to compare because we know little about \( u_1 \) and \( u_2 \). But we can apply the parent’s FOC to express both gain and loss in terms of \( u_c \). The FOC is \( u_1(t) = (1+r_{t+1})u_2(t) \). Thus the parent increases his bequest if \( \beta u_1(t+1)\varepsilon/n > u_1(t)/(1+r_{t+1})\varepsilon \). But on the steady state, marginal utility is constant and the condition simplifies to \( \beta/n > 1/(1+r_{t+1}) \) or \( 1+r > n/\beta \).

But note that \( 1+r = n/\beta \) is the modified golden rule (the planner’s FOC). What does this mean? A situation where \( R > n/\beta \) can never be an equilibrium. For if it were, every parent would want to increase his bequest. That would increase savings (parents leave bequests when old) and drive down the interest rate. Such a process would continue until the modified golden rule holds with equality and the economy is dynamically efficient.

---

5 One could see this formally as well: solving the problem from date \( t+1 \) onwards yields some behavior (\( c \)'s and \( x \)'s). Now step backwards one period. One can show that the solution from date \( t \) onwards is identical with the date \( t+1 \) solution for periods \( t+1 \) and later. Some models deviate from this assumption; see Becker’s “rotten kid theorem.”
Does that mean the modified golden rule always holds when parents are altruistic? No. Suppose we have $1 + r < n/\beta$ when households leave no bequests. Then parents would want to take resources from their children instead of giving to them. Of course, they have no way of doing that, so they choose $b = 0$ and altruism becomes irrelevant. The economy remains dynamically inefficient (see Figure 4). Thus, the presence of a bequest motive can only increase the capital stock. Since it cannot decrease $k$, dynamic inefficiency remains possible. However, the last result changes if there is two-sided altruism (Kimball): parents care about their kids and kids care about their parents. Then intergenerational transfers can be negative and dynamic efficiency is guaranteed.

All of this has simply assumed that the bequest is financed by reducing old age consumption of the parent and used to increase consumption of the child when young. What if the child decides instead to consume more when old or simply to pass on the bequest to its own offspring? Nothing changes. It doesn’t matter how the child uses the additional resources as long as it behaves optimally, maximizing lifetime utility. It will then be indifferent between all possible uses of the bequest and assuming that all of it is consumed immediately is not restrictive. As an exercise, consider the case where the parent reduces old age consumption (as above), but the child saves the bequest and increases consumption when it is old (not when it is young as above). Show that the condition under which the parent wishes to increase the bequest is again $R > n/\beta$. Alternatively, show that the same rule follows when the parent consumes less when young and the child consumes more when it is old (or young – it doesn’t matter!). The trick here is to use the FOC’s to translate $u_1$ terms into $u_2$ terms and vice versa.
Summary: If the bequest motive is operative \((b > 0)\),

- The economy attains the modified golden rule.
- Therefore it is dynamically efficient.
- The market equilibrium coincides with the planner’s solution (show this!).
- Ricardian equivalence holds even across generations. (We haven’t shown that, but it follows directly from the fact that there is a present value budget constraint that holds across generations.)

If the bequest motive is not operative, altruism makes absolutely no difference. The economy may still be dynamically inefficient.

A final note of caution. The similarity of the OLG model with bequests and the infinite horizon model is often taken to mean that it is harmless to abstract from life-cycle features. In fact, Barro (1974) has derived conditions under which the infinite horizon model arises as a reduced form of an OLG model. However, these conditions are fairly strict and little is known about the quantitative properties of OLG models compared with infinite horizon models. It is therefore advisable to be cautious and to keep in mind that the infinite horizon case is only an approximation.

### 4.1 Reading

The main reference for this section is CM ch. 1-3.

An extensive treatment is MW 1-3. See also SLj ch. 8; BS appendix 3. BF 3.1-3.2.
