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Horizontal and vertical differentiation: The Launhardt model

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Abstract

We show how the spatial duopoly proposed by Launhardt in 1885, where firms have access to different transportation technologies, allows one to model in a simple and elegant way the two major types of product differentiation, i.e. horizontal and vertical. We consider the cases where firms are located near the market end points or near the market center. Launhardt's analysis of price determination is then extended by allowing firms to choose strategically their transportation rates. Subgame perfect Nash equilibria involve minimum (maximum) vertical product differentiation when horizontal product differentiation is large (small) enough.

Keywords: Industrial organization; Market structure; Firm strategy; Market performance; Oligopoly; Other imperfect markets

JEL classification: L13

1. Introduction

Modern theories of product differentiation have been very much influenced by Hotelling (1929) who proposed using a spatial framework to describe product and price competition in oligopolistic industries. Within

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this framework, the location space is considered as the range of potential variants of a product; a consumer's location corresponds to his ideal product; and the transportation cost is interpreted as the decrement of utility from not consuming the ideal product.

The main purpose of this paper is to discuss Launhardt's (1885, chapter 28) contribution. He used a similar paradigm 44 years before Hotelling to study product differentiation.[1] Our discussion does not intend to be historical but, instead, aims at presenting Launhardt's ideas in terms of modern economic theory.

Following Lancaster (1979, chapter 2), it is now common to distinguish between two (polar) cases of product differentiation. Two products are said to be *horizontally differentiated* when both products have a positive demand whenever they are offered at the same price. Neither product dominates the other in terms of characteristics, and heterogeneity in preferences over characteristics explains why both products are present in the market. Two products are said to be *vertically differentiated* if one product captures the whole demand when both are supplied at the same price. One product dominates the other and differences in willingness-to-pay for "quality" across consumers are necessary for the two products to be in the market.

The concept of horizontal differentiation is at the heart of Hotelling analysis. On the other hand, it is only recently that vertical differentiation has been studied (see, e.g. Mussa and Rosen, 1978; Gabszewicz and Thisse, 1979; Shaked and Sutton, 1982). Vertical differentiation has been casted within the spatial framework by Gabszewicz and Thisse (1986) who assume that both firms are located outside of the market but on the same side: one firm is therefore closer to all consumers than its competitor.

Launhardt has modeled both types of differentiation through a simple, but very ingenious, idea: firms locate at different points along a street (as in Hotelling) but have access to different transportation technologies when delivering their products. This assumption allows one to capture the essence of vertical differentiation: when both firms are located together, the firm with the lower transportation cost would supply the whole population of consumers if firms were to charge the same mill price. Put in a different way, two products which are functionally identical may be more or less difficult to carry. Therefore, the product which is easier to transport may be viewed as a product of a higher quality.

Hotelling used his model to analyze the choice of locations in the first stage of a sequential game, the second stage of which is devoted to price competition. Launhardt did not go so far and treated locations as exogen-

[1] Since this paper was written, Launhardt's book has been published in English as *Mathematical Principles of Economics* (1993). Note that Launhardt also made other contributions to location theory in various publications which are reviewed by Pinto (1977).

ous. However, he paid more attention than Hotelling did to the influence of differences in transportation costs and, consequently, was aware of the difference between the two types of differentiation associated respectively with heterogeneity in location and in transportation. Our secondary purpose is to shed further light on the interaction between horizontal and vertical differentiation by giving firms the possibility to choose their transportation technologies, thus extending the Launhardt model.

Launhardt's model is described in Section 2. Section 3 is devoted to the study of price competition between two firms shipping their output and bearing different transportation costs. For simplicity, we emphasize the cases where firms are located near the market endpoints, next to the market center, or have symmetric intermediate locations. The model is extended in Section 4: firms choose, first, their transportation technologies and, then, mill prices. It is assumed that firms are located far apart or close to the center. Some remarks conclude the paper in Section 5.

2. The model

Consider two firms ($i = 1, 2$) located at a_1 and a_2 in the unit interval [0,1]. Without loss of generality, it is assumed that $a_1 \leqslant a_2$; let $a = a_1$ and $a_2 = a + \delta$ with $\delta \geqslant 0$. They produce at no cost a homogeneous product which they deliver to the consumers. Firms do not have access to the same transportation technology; denote by $t_i > 0$ the (constant) transportation rate borne by firm i when shipping its output.[2] Firm 1 is the firm with the lower transportation cost so that $0 < \tau \equiv t_1/t_2 < 1$. Consumers are evenly distributed over [0,1] with density one. Each consumer buys one unit of the product from the firm charging the lower delivered price. Firms choose mill prices p_1 and p_2 (which are the same irrespective of consumers'locations) and pass on to the consumers the total transportation cost. Thus the delivered price charged by firm i to a consumer at $x \in [0, 1]$ is given by $p_i + t_i|a_i - x|$.[3] When $t_1 = t_2$ ($\tau = 1$), the Launhardt model is equivalent to that proposed later by Hotelling.

We now define the firms' demand functions. In order to do so, we must determine the marginal consumers, i.e. the consumers indifferent between buying from either firm. Consider first the marginal consumer situated

[2] A sort of dual model has been studied by Garella and Martinez-Giralt (1989) who suppose that consumers take care of the transportation and that transportation rates are different across consumers.

[3] The model above is a special case of a more general model considered by Launhardt who supposed (i) the firms to be located in the plane, (ii) the consumers to have a finite reservation price or variable individual demands, and (iii) the firms to have unequal unit production costs. See Launhardt (1993, chapters 27–28).

between the two firms: $y \in [a, a+\delta]$. Since the delivered prices at y are equal,

$$p_1 + t_1(y-a) = p_2 + t_2(a+\delta-y)$$

so that

$$y = a + \frac{p_2 + \delta t_2 - p_1}{t_1 + t_2} \text{ with } p_i \leqslant p_j + \delta t_j,\quad i, j = 1, 2 \text{ and } i \neq j\,. \tag{1}$$

The market segmentation where each firm secures its own hinterland, as in Hotelling, is given in Fig. 1 where the slopes of the price lines are the transportation rates.

However, another situation arises in which the low rate firm captures consumers located in the high rate firm's hinterland: there is a marginal consumer situated on the right of firm 2, whose location is given by $z \in [a+\delta, 1]$ such that $p_1 + t_1(z-a) = p_2 + t_2(z-a-\delta)$, leading to

$$z = a + \delta + \frac{p_1 + \delta t_1 - p_2}{t_2 - t_1} \text{ with } p_2 \leqslant p_1 + \delta t_1\,. \tag{2}$$

The following two cases must then be considered according to the position of the other marginal consumer: whether $y \in [a, a+\delta]$ as above or $x \in [0, a]$ such that $p_1 + t_1(a-x) = p_2 + t_2(a+\delta-x)$, which gives

$$x = a - \frac{p_1 - p_2 - \delta t_2}{t_2 - t_1} \text{ with } p_1 \geqslant p_2 + \delta t_2\,. \tag{3}$$

These two cases are illustrated in Figs. 3 and 2 respectively.

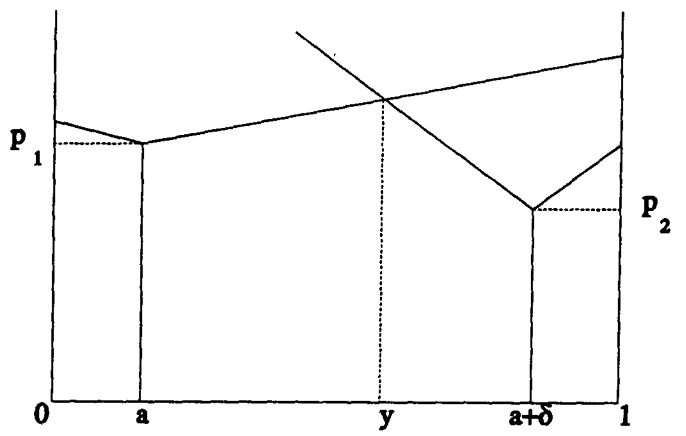

Fig. 1.

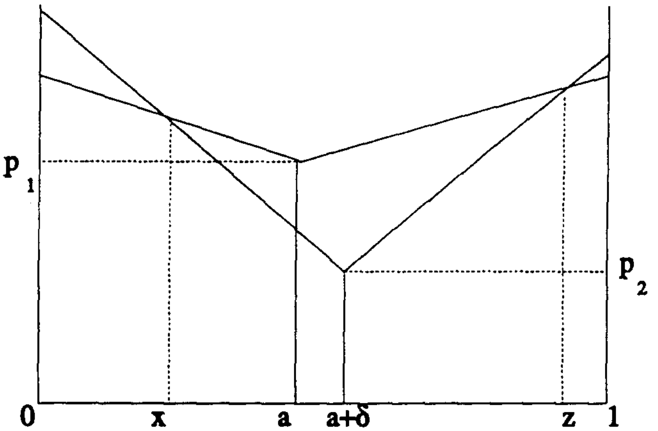

Fig. 2.

Note that firm 2's market area–which is given by $[y, \min\{1,z\}]$ or by $[\max\{0,x\}, \min\{1,z\})]$–is always connected, whereas firm 1's market area may be the union of two disjoint segments–$[0,x]$ and $[z,1]$–as shown in Figs. 2 and 3. Moreover, as long as $t_1 \neq t_2$, market areas vary continuously with prices so that, unlike in Hotelling, *firms'demand functions are continuous in own prices:*

$$D_2(p_1, p_2) = \begin{cases} 0 & \text{if } p_2 \geqslant p_1 + \delta t_1 \\ \min\{1, z\} - y & \text{if } p_1 - \delta t_2 \leqslant p_2 \leqslant p_1 + \delta t_1 \\ \min\{1, z\} - \max\{0, x\} & \text{if } 0 \leqslant p_2 \leqslant p_1 - \delta t_2 , \end{cases}$$

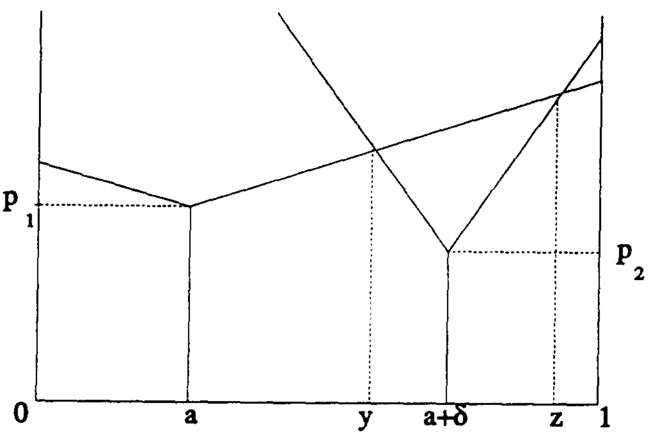

Fig. 3.

and

$$D_1(p_1, p_2) = 1 - D_2(p_1, p_2).$$

The following comments are in order. First, when $\delta = 0$ the two firms are located together but are still differentiated as long as $\tau < 1$. Differentiation is only vertical since all consumers would buy from firm 1 if firms were to choose the same mill price. Second, when $\tau = 1$ the two firms have access to the same technology (as in Hotelling) but are still differentiated as long as $\delta > 0$. Differentiation is now horizontal only since, whatever $\delta > 0$, both firms have a positive demand when they charge the same mill price. Hence *the Launhardt model can be viewed as a model encapsulating both horizontal and vertical differentiation aspects* through the specification of the parameters δ and τ. Roughly speaking, δ can be interpreted as the degree of horizontal differentiation while τ would correspond to the degree of vertical differentiation. The perfectly homogeneous case arises when $\delta = 0$ and $\tau = 1$.

3. Price competition

Consider a noncooperative game in which firms set mill prices simultaneously. Despite the continuity of demands, and hence of profits, a price equilibrium in pure strategies may fail to exist for some parameter values because of the possible lack of quasiconcavity (see Gabszewicz and Thisse (1986) for an example of nonexistence of a price equilibrium when profit functions are continuous but not quasiconcave). In the next two subsections, we will concentrate on two special location configurations where firms are far apart (a close to 0 and δ close to 1) or locate next to the market center (a close to 1/2 and δ close to 0). This choice is motivated by the fact that these two configurations emerge as the equilibrium outcomes of the product game in the two-dimensional characteristics models studied by Neven and Thisse (1990) and Tabuchi (1994). In order to gain more insight into the problem of existence of a price equilibrium, the case of symmetric locations is discussed in the last subsection. However, we will refrain from establishing necessary and sufficient conditions for a price equilibrium to exist because the asymmetric case does not add much here to our understanding of the existence problem.

3.1. Maximum horizontal differentiation

Suppose first that $\delta = 1$ (both firms are located at the endpoints of the market segment). Even when firm j would elect a zero price, firm i can always guarantee to itself a positive demand by charging a mill price lower than t_j. So the equilibrium–if any–must involve the two firms earning positive profits. Firm i $(i = 1, 2)$ has a positive demand if and only if $p_i < p_j + t_j$, for $j = 1, 2$ and $j \neq i$. Under these conditions, the demand functions to firms 1 and 2 are respectively y and $1 - y$, where y is given by (1) in which we set $a = 0$ and $\delta = 1$. First order conditions for profit maximization yield

$$2p_1 - p_2 = t_2 \quad \text{and} \quad -p_1 + 2p_2 = t_1 \tag{5}$$

(they are also sufficient since profit functions are concave when $p_i \leqslant p_j + t_j$). The equilibrium prices can then be obtained from (5):

$$p_1^{\mathrm{H}} = \frac{1}{3}(t_1 + 2t_2) \quad \text{and} \quad p_2^{\mathrm{H}} = \frac{1}{3}(2t_1 + t_2) \tag{6}$$

which were found by Launhardt. Interestingly, these prices are identical to those obtained by Hotelling when $t_1 = t_2$, thus suggesting that the Hotelling model can be viewed as the limiting case of the Launhardt model.

The marginal consumer corresponding to (6) is at $y^{\mathrm{H}} = (t_1 + 2t_2)/3(t_1 + t_2)$ so that the equilibrium profits are

$$\Pi_1^{\mathrm{H}} = \frac{(t_1 + 2t_2)^2}{9(t_1 + t_2)} \quad \text{and} \quad \Pi_2^{\mathrm{H}} = \frac{(2t_1 + t_2)^2}{9(t_1 + t_2)}\,. \tag{7}$$

As expected, the firm which has access to the more efficient technology charges the higher mill price and earns the higher profits.

Similarly, for δ close to 1 (and $a \geqslant 0$),

$$p_1^{\mathrm{H}} = \frac{1}{3}[(t_1 + t_2)(1 + a) + t_2\delta] \quad \text{and}$$

$$p_2^{\mathrm{H}} = \frac{1}{3}[(t_1 + t_2)(2 - a) - t_2\delta]\,. \tag{8}$$

However, profit functions are now piecewise concave only and not necessarily quasiconcave so that (8) may not be an equilibrium. In the Appendix, Section A1, we find a necessary and sufficient condition for p_1^{H} and p_2^{H} to be the equilibrium prices when firms are symmetrically located.[4] This condition shows that existence is guaranteed provided that the degree of horizontal differentiation, expressed by δ, is large enough.

3.2. Minimum horizontal differentiation

Assume now that both firms are located at the market center so that $\delta = 0$. Firm 2 must therefore choose $p_2 < p_1$ in order to get a positive demand. Under this condition, the demand functions to firms 1 and 2 are

[4] These prices are identical to the Hotelling prices when $t_1 = t_2$. As is now well known, there is no price equilibrium in price strategies for a whole range of locations in the Hotelling model.

respectively $1-z+x$ and $z-x$, where z and x are given by (2) and (3) in which $a=1/2$ and $\delta=0$. First order conditions for profit maximization yield

$$4p_1 - 2p_2 = t_2 - t_1 \quad \text{and} \quad p_1 - 2p_2 = 0\,. \tag{9}$$

Again, profit functions are concave when $p_2 < p_1$ so that conditions (9) are sufficient. Solving (9) leads to the equilibrium prices

$$p_1^{\mathrm{V}} = \frac{t_2 - t_1}{3} \quad \text{and} \quad p_2^{\mathrm{V}} = \frac{t_2 - t_1}{6}\,, \tag{10}$$

while the corresponding profits are

$$\Pi_1^{\mathrm{V}} = \frac{2}{9}(t_2 - t_1) \quad \text{and} \quad \Pi_2^{\mathrm{V}} = \frac{1}{18}(t_2 - t_1)\,. \tag{11}$$

Note that demands to firms 1 and 2 at the equilibrium prices are respectively 2/3 and 1/3 and are, therefore, independent of the transportation rates t_1 and t_2. Furthermore, the equilibrium prices reduce to the competitive ones when $t_1 = t_2$ as in the Hotelling case.

As long as the market segmentation is the one depicted in Fig. 2, firms 1 and 2's demands are still given by $1-z+x$ and $z-x$. A quick inspection of (2) and (3) shows that these demands are independent of the location parameters a and δ:

$$D_1(p_1, p_2) = 1 - 2\frac{p_1 - p_2}{t_2 - t_1} \quad \text{and} \quad D_2(p_1, p_2) = 2\frac{p_1 - p_2}{t_2 - t_1}$$

if $\delta t_2 \leqslant p_1 - p_2 \leqslant \min\{a(t_2 - t_1) + \delta t_2,\ (1-a)(t_2 - t_1) - \delta t_2\}$ which, in turn, implies $\delta \leqslant (1-\tau)(1-a)/2$. Hence prices (10) and profits (11) are unaffected by perturbations of a and δ in a neighborhood of (1/2,0). This is, of course, an implication of the assumption of linear transportation costs, yet a somewhat surprising result.

However, since profit functions may not be quasiconcave anymore, conditions (9) are not sufficient. In the Appendix, Section A2, we give a necessary and sufficient condition for (10) to be the equilibrium prices in the case of symmetric locations. In particular, as τ tends to 1, i.e. as vertical differentiation vanishes, the present typical equilibrium subsists only if δ also tends to 0, i.e. when horizontal differentiation disappears. In the limit, when $t_1 = t_2$ (and $\delta = 0$), we get $p_1^{\mathrm{V}} = p_2^{\mathrm{V}} = 0$, the competitive outcome.

3.3. *The symmetric location case*

So far, we have limited ourselves to polar cases. Ideally, the analysis should be developed for any location pair but this turns out to be especially cumbersome. Thus, for the reasons discussed at the beginning of this section, we restrict ourselves to the case of symmetric locations.

When firms are very far from each other, we are in a market situation (called regime H) which has been studied in 3.1. On the contrary, when firms are very close to the center, the market situation (called regime V) has been considered in 3.2. Here, we want to deal with the intermediate case, which gives rise to a new regime (called regime I). It follows from the demand analysis of Section 2 that the firms demands are now $D_1 = 1 - (z - y)$ and $D_2 = z - y$. It is then readily verified that the "candidate" equilibrium prices are given by

$$p_1^{\mathrm{I}} = \frac{1}{3}(t_2 - \delta t_1 - t_1^2/t_2) \quad \text{and} \quad p_2^{\mathrm{I}} = \frac{1}{6}(t_2 + 2\delta t_1 - t_1^2/t_2)\,. \tag{12}$$

In Fig. 4, the region of parameters δ and τ for which these prices are indeed an equilibrium is represented by the shaded area I, between the two dividing curves i and j. These curves are obtained by comparing profits evaluated at prices (12) with the maximum profit each firm (1 and 2, respectively) can earn by unilaterally deviating (the equations for these curves can be found in the Appendix, Section A3). The other two cases,

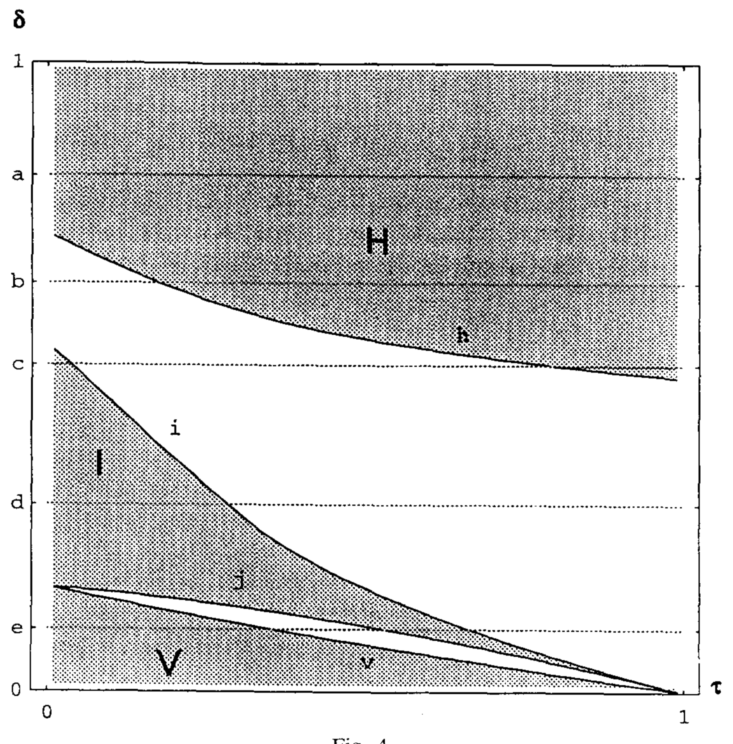

Fig. 4.

maximum and minimum horizontal differentiation, correspond to the shaded areas H and V, respectively. The equations for the dividing curves h and v, which can be similarly interpreted, are given in the Appendix, Sections A1 and A2.

First, an equilibrium exists, with prices given by (8), for all transportation rates provided that firms are far enough from one another (line a in Fig. 4); this corresponds to regime H illustrated in Fig. 1. Second, when firms get closer though we are still in regime H, there is an equilibrium when the discrepancy between the transportation rates is not too large (line b); when this gap is large enough, the firm with the lower rate has an incentive to invade its competitor's hinterland, destroying the equilibrium property.

Third, when the interfirm distance becomes even smaller, an equilibrium turns out to exist for two domains of parameters (line c): when the transportation rates are very similar (regime H) or very different (regime I illustrated in Fig. 3, with prices given by (12)). For intermediate values of τ the lower rate firm has an incentive to deviate in both regimes. The gap in the values of τ which lead to profitable deviations of that firm is due to the discontinuity of its best reply function. For larger or smaller values of τ the two best reply curves still intersect, however, despite the above-mentioned discontinuity.

Fourth, when δ decreases further an equilibrium exists only for very dissimilar transportation rates (line d). Last, when firms are very close (line e) there is a price equilibrium when transportation rates are very different (regime V illustrated in Fig. 2, with prices given by (10)) or different enough (regime I). The nonexistence of an equilibrium in between regimes V and I arises again because of the discontinuity of the best reply function, but now it is the higher rate firm that wants to deviate. Recall also that an equilibrium exists for all values of τ when firms are both located at the center ($\delta = 0$).

4. Quality competition

In this section, we propose to extend Launhardt's analysis by allowing firms to choose strategically their transportation technologies prior to competing in prices. More precisely, we suppose that firms choose, first, transportation rates t_1 and t_2 in a given interval $[\underline{t}, \bar{t}] \subset \mathbb{R}_+$ and, then, prices p_1 and p_2. Since t_i can be viewed as an inverse measure of quality, the equilibrium of the first stage game will provide us with some useful insights about how firms choose their qualities when another form of differentiation–i.e. location–is present.

Since the existence of a price equilibrium does not hold for some location pairs, we will concentrate on the two polar cases of the preceding section: (i) firms are located at the endpoints of the market segment (maximum horizontal differentiation) and (ii) firms are located together at the market center (minimum horizontal differentiation).

Consider the former case ($\delta = 1$). For any given pair (t_1,t_2) such that $t_1 < t_2$, the equilibrium profits of the corresponding subgame are given by (7). It is then readily verified that $\partial \Pi_i^H / \partial t_i > 0$ for $i = 1, 2$. This implies that $t_1^* = t_2^* = \bar{t}$ is the unique equilibrium of the first stage game. *So firms have the same incentive to choose the highest rate, thus resulting in minimum differentiation along the transportation characteristic (quality) if there is maximum differentiation along the geographical characteristic.*[5] Furthermore, both firms end up with the highest transportation rate–the least efficient technology–because they have the same strategic incentive to build up high profits behind the barriers of high transportation costs, thus confirming Hotelling (1929, p.50) for whom "merchants would do well ... to make transportation as difficult as possible".[6]

In the latter case, $\delta = 0$ and $a = 1/2$. Thus, for any pair (t_1,t_2), the equilibrium profits of the resulting subgame are given by (11). It is then obvious that there exists a unique equilibrium (up to a permutation) $t_1^* = \underline{t}$ and $t_2^* = \bar{t}$. Put differently, *firms maximize their differentiation in transport when their geographical differentiation is minimized.* They do so in order to relax price competition.

In both cases, we therefore observe a "max–min" combination of characteristics, a result which is reminiscent of Neven and Thisse (1990), Ansari et al. (1994) and Tabuchi (1994) in their study of a spatial duopoly in a two-dimensional space. However, unlike these authors, we have not been able to deal with an endogeneous determination of the equilibrium locations because a price equilibrium in pure strategies fails to exist for some location pairs.

Comparing profits (7) and (11) at these two configurations, we observe that both firms are better off when they locate at the market endpoints instead of the market center, which might suggest that firms would "prefer" horizontal differentiation to vertical differentiation (see also Lambertini, 1993). However, restricting locations to the quartiles and the choice of the

[5] In a model of monopolistic competition in which firms are equidistantly located on a circle and choose their transportation rate while facing a decreasing cost function of this rate, von Ungern-Sternberg (1988) shows that all firms choose the same rate such that the benefit of an additional customer is equal to the marginal cost associated with the choice of the transportation rate. However, the interpretation he gives to the transportation rate is different from ours.

[6] Of course, this tendency would be mitigated if the total demand would become price-sensitive through the introduction of a binding reservation price, as in Launhardt (1993, chapter 28): "The producer of the goods gains ... only by the perfection of transport facilities if the extension of market areas due to those improvements is not diminished by competition with foreign goods" (p.152).

transportation rates to the pair $(\underline{t},\bar{t})$, it can be verified that, for values of these parameters that ensure the existence of a price equilibrium (in regime H or I), both firms want to select $\bar{t}$. This yields Hotelling's profits which are known to be the same as those earned at the market endpoints. Hence, there is no strict preference for horizontal differentiation.

5. Concluding remarks

By now, it should be clear to the reader that Launhardt deserves to be more widely acknowledged as the founder of oligopoly theory with differentiated products. His model encompasses as special cases both the Hotelling model of horizontal product differentiation and different variants of models of horizontal and vertical differentiation.[7]

Launhardt concentrated on two limit cases of his model: (i) horizontal differentiation with differences in qualities playing a secondary role, and (ii) vertical differentiation as the sole factor for product differentiation. He did not analyze quality competition per se, but he did stress the favorable strategic effects from the firms' viewpoint of high transportation costs (or of low qualities) in the the first case. Launhardt observed that this tendency may be counterbalanced by the unfavorable demand effect generated by high delivered prices.

On the contrary, when horizontal differentiation is negligible, the strategic effect works in the opposite direction, thus generating maximal differentiation. Again, the demand effect may dampen this tendency by yielding a lower bound on quality; however, as this effect does not act symmetrically on the top quality, the tendency toward vertical differentiation persists.

In the Launhardt model, when both types of differentiation tend to vanish, that is, when $(\tau, \delta) \rightarrow (1, 0)$ within domain I or V depicted in Fig. 4, the equilibrium of the price game (see (12) and (10) respectively) tends to the Bertrand equilibrium. This one thus appears as the limit of a sequence of equilibria associated with different degrees of differentiation, and not as an isolated point. In the Hotelling model, we can get to the same conclusion, but the vanishing of horizontal differentiation must then be

[7] Observe that the Launhardt model with $t_2 > t_1$ and $\delta = 1$ (see 3.1.) formally includes the case of horizontal product differentiation in the model of international price competition considered by Neven et al. (1991) in that the demands and hence the equilibrium prices of the latter are identical to those of the former after an appropriate redefinition of variables. This suggests that the Launhardt model might well be a useful framework to unify various product differentiation models developed in industrial organization.

associated with diminishing transportation rates, and not with a decreasing distance between the two firms, leading to inexistence of price equilibrium.

Launhardt has also studied homogeneous oligopolies, but from a completely different perspective. Without warning the reader, and without quoting Cournot, whom he ignored at the time he was writing his book (so he said later), Launhardt proceeded to determine an equilibrium with quantities as strategic variables. This was precisely the approach taken by Cournot.

Much work remains to be done. First, the case of quadratic transportation costs, which assure the existence of a unique price equilibrium for any location pair in the Hotelling model, should be analyzed. A preliminary study shows that this change in the transportation cost is not sufficient to restore the quasiconcavity of the profit function of the low rate firm when the two firms are too close. As a matter of fact, no equilibrium exists when the two locations coincide and when transportation rates differ. This suggests that switching from linear to quadratic transportation costs favors existence of a price equilibrium when horizontal differentiation dominates (see d'Aspremont et al., 1979), but on the contrary makes existence problematic when differentiation is predominantly vertical.

Second, the Launhardt model should be cast in the circular model of monopolistic competition where firms would choose both prices and transportation rates. The result obtained in Section 4 that firms have an incentive to choose the highest rate suggests that, for a given number of firms, profits are higher, thus inviting more entry. The standard discrepancy between the equilibrium and the optimum numbers of firms would therefore be exacerbated.[8]

Last, it would be worth studying an oligopoly model in which firms are supposed to locate together in order to determine how they select their transportation rates along the quality dimension. In particular, the following questions should be studied. Does the "natural oligopoly" property hold in the Launhardt setting? What are the gaps between the transportation rates? Can the incumbents choose rates that deter the entry of new firms? We hope to analyze these issues in future research.

Acknowledgements

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[8] This model should also be reinterpreted within the context of von Ungern-Sternberg's.

Appendix: Existence of price equilibrium with symmetric locations

We assume that firms 1 and 2 are symmetrically located, so that $a = (1 - \delta)/2$.

A1. Maximum horizontal differentiation

For δ close to 1 (maximum horizontal differentiation), we get the regime H (illustrated in Fig. 1), with demands $D_1 = y$ and $D_2 = 1 - y$ and with candidate equilibrium prices given by (8):

$$p_1^{\mathrm{H}} = \frac{1}{6}[(3-\delta)t_1 + (3+\delta)t_2] \text{ and } p_2^{\mathrm{H}} = \frac{1}{6}[(3+\delta)t_1 + (3-\delta)t_2] .$$

The corresponding profits are:

$$\Pi_1^{\mathrm{H}} = \frac{1}{t_1 + t_2}(p_1^{\mathrm{H}})^2 \text{ and } \Pi_2^{\mathrm{H}} = \frac{1}{t_1 + t_2}(p_2^{\mathrm{H}})^2 .$$

First, notice that a price increase and a consequent switch to a regime other than H can never be profitable, either to firm 1 or to firm 2. Indeed, the demand would then be x for firm 1 (in the market segmentation illustrated in Fig. 5) or $z - y$ for firm 2 (in regime I, illustrated in Fig. 3), with a higher elasticity than that of the corresponding demand in regime H. Hence, since the profit functions $p_1 y$ and $p_2(1-y)$ are decreasing in p_1 and p_2 respectively, for prices higher than p_1^{H} and p_2^{H} (by definition of these prices), the profit functions $p_1 x$ and $p_2(z-y)$ are also necessarily decreasing.

Now, by lowering its price, firm 1 can switch to regime I, illustrated in

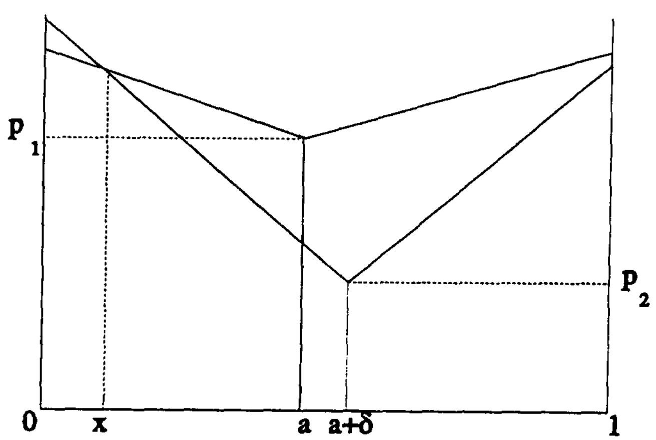

Fig. 5.

Fig. 3, capturing a demand equal to $D_1 = 1 - z + y$. The first order condition for firm 1 then becomes:

$$\frac{\partial}{\partial p_1}[p_1(1-z+y)] = 1 - \frac{2t_2}{t_2^2 - t_1^2}(\delta t_1 - p_2^{\mathrm{H}} + 2p_1) = 0 \,,$$

which yields $\hat{p}_1 = [t_2^2 - t_1^2 + 2t_2(p_2^{\mathrm{H}} - \delta t_1)]/4t_2$. Deviating to regime I is unprofitable for firm 1 if

$$\begin{aligned} \Pi_1^{\mathrm{H}} \geq \hat{p}_1 D_1(\hat{p}_1, p_2^{\mathrm{H}}) \text{ whenever } p_2^H - \delta t_1 \leq \hat{p}_1 \\ \leq p_2^{\mathrm{H}} + t_2(1-\delta)/2 - t_1(1+\delta)/2 \,, \end{aligned}$$

that is, whenever $p_1 D_1(p_1, p_2^{\mathrm{H}})$ has a maximum in regime I. A simple calculation leads to the equivalent condition:

$$\begin{aligned} \delta \geq 3(1+\tau)\frac{2-\tau-\sqrt{2(1-\tau)}}{1+5\tau+(1-\tau)\sqrt{2(1-\tau)}} \equiv h^1(\tau) \\ \text{if } \underline{h}(\tau) \equiv \frac{3\tau(1+\tau)}{1+5\tau} \leq \delta \leq 3\frac{1+\tau+(1-\tau)^2}{7-\tau} \equiv \bar{h}(\tau) \,. \end{aligned}$$

It can be readily verified that $h^1(\tau) < \bar{h}(\tau)$ for any $\tau \in]0, 1[$ and also that $h^1(\tau) < \underline{h}(\tau)$ iff $\tau > \tau^h \simeq 0.418$. Thus, for $\tau \leq \tau^h$ it is unprofitable for firm 1 to deviate to regime I if $\delta \geq h^1(\tau)$. For $\tau > \tau^h$, firm 1 has no incentive to deviate if either $\delta \geq \underline{h}(\tau)$ or $\Pi_1^{\mathrm{H}} \geq \underline{p}_1$ (where $\underline{p}_1 = p_2^H - \delta t_1$ is the price at which firm 1 captures the whole market). It is straightforward to show that the last condition is equivalent to:

$$\delta \geq 3(1+\tau)\frac{-2(1+2\tau)+\sqrt{4(1+2\tau)^2+(1-\tau)^2}}{(1-\tau)^2} \equiv h^2(\tau) \,.$$

Also, as can be easily checked, $h^2(\tau) < \underline{h}(\tau)$ iff $\tau > \tau^h$ (with τ^h as above). Hence, firm 1 will find it unprofitable to switch from regime H by lowering its price when:

$$\begin{aligned} \delta \geq h(\tau), \text{ with } h(\tau) \equiv h^1(\tau) \text{ if } 0 < \tau \leq \tau^h \\ h^2(\tau) \text{ if } \tau^h \leq \tau < 1 \,. \end{aligned}$$

It appears that this condition, necessary to ensure that $(p_1^{\mathrm{H}}, p_2^{\mathrm{H}})$ is a price equilibrium, is also sufficient. First, at these prices, $z \geq 1$ (so that demands are indeed y and $1-y$, as initially assumed) iff $\delta \geq 3(1-\tau)/(5+\tau)$, an inequality implied by the former condition. Second, as will be shown below, the same condition also implies that it is unprofitable for firm 2 to lower its price and to switch to the market segmentation characterized by demands $D_1 = x$ and $D_2 = 1 - x$, as in Fig. 5. This should not come as a surprise, since the firm with the more efficient transportation technology has a stronger incentive to decrease its price in order to attract customers from its rival's hinterland.

The first order condition for firm 2 is:

$$\frac{\partial}{\partial p_2}[p_2(1-x)] = \frac{1+\delta}{2} + \frac{p_1^{\mathrm{H}} - 2p_2 - \delta t_2}{t_2 - t_1} = 0\,,$$

yielding $\hat{p}_2 = [t_2 - t_1 - \delta(t_1 + t_2) + 2p_1^{\mathrm{H}}]/4$. Charging a price lower than $p_1^{\mathrm{H}} - \delta t_2$, thus getting the demand $1-x$, is not profitable for firm 2 if

$$\Pi_2^{\mathrm{H}} \geq \hat{p}_2 D_2(p_1^{\mathrm{H}}, \hat{p}_2) \text{ whenever } p_1^{\mathrm{H}} + t_1(1-\delta)/2 - t_2(1+\delta)/2 \leq \hat{p}_2 \leq p_1^{\mathrm{H}} - \delta t_2$$

or, equivalently,

$$\delta \geq 3(1+\tau)\frac{1-\sqrt{1-\tau^2}}{(1+\tau)(1+2\tau)-(1-\tau)\sqrt{1-\tau^2}} \quad \text{if } 3\frac{2\tau-1}{2\tau+1} \leq \delta \leq 3\frac{\tau}{4-\tau}\,.$$

It can be readily verified that this condition is implied by $\delta \geq h(\tau)$.

If $\hat{p}_2 < p_1^{\mathrm{H}} + t_1(1-\delta)/2 - t_2(1+\delta)/2$, that is, if $p_2(1-x)$ is maximized when $x = 0$, the non-deviation condition for firm 2 is $\Pi_2^{\mathrm{H}} \geq p_1^{\mathrm{H}} + t_1(1-\delta)/2 - t_2(1+\delta)/2$ or, equivalently,

$$\delta \geq 3\frac{-(1+\tau)(1+5\tau) + \sqrt{(1+\tau)^2(1+5\tau)^2 + (3\tau-1)(1+\tau)(1-\tau)^2}}{(1-\tau)^2}\,,$$

an inequality which is again implied by $\delta \geq h(\tau)$.

Thus, the prices $(p_1^{\mathrm{H}}, p_2^{\mathrm{H}})$ are equilibrium prices under the (necessary and sufficient) condition: $\delta \geq h(\tau)$.

A2. Minimum horizontal differentiation

For δ close to 0 (minimum horizontal differentiation), we get regime V (illustrated in Fig. 2), with demands $D_1 = 1 - z + x$ and $D_2 = z - x$ and with candidate equilibrium prices given by (10):

$$p_1^{\mathrm{V}} = \frac{t_2 - t_1}{3} \text{ and } p_2^{\mathrm{V}} = \frac{t_2 - t_1}{6}\,,$$

yielding the profits given by (11):

$$\Pi_1^{\mathrm{V}} = \frac{2}{9}(t_2 - t_1) \text{ and } \Pi_2^{\mathrm{V}} = \frac{1}{18}(t_2 - t_1)\,.$$

First, notice that a price decrease and a consequent switch to a regime other than V is never profitable, either to firm 1 or to firm 2. Indeed, the demand would then be $1 - z + y$ for firm 1 (in regime I, illustrated in Fig. 3) or x for firm 2 (in the market segmentation illustrated in Fig. 5), with a smaller elasticity than that of the corresponding demand in regime V. Since the profit functions $p_1(1 - z + x)$ and $p_2(z - x)$ are increasing in p_1 and p_2 respectively, for prices lower than p_1^{V} and p_2^{V} (by definition of these prices), the profit functions $p_1(1 - z + y)$ and $p_2 x$ are also necessarily increasing.

By increasing its price, firm 2 can switch to regime I, illustrated in Fig. 3, getting a demand $D_2 = z - y$. The first order condition for this firm then becomes:

$$\frac{\partial}{\partial p_2}[p_2(z - y)] = \frac{2t_2}{t_2^2 - t_1^2}(\delta t_1 + p_1^{\mathrm{V}} - 2p_2) = 0 ,$$

which yields $\tilde{p}_2 = (p_1^{\mathrm{V}} + \delta t_1)/2$. Deviating to regime I is unprofitable for firm 2 if

$$\Pi_2^{\mathrm{V}} \geq \tilde{p}_2 D_2(p_1^{\mathrm{V}}, \tilde{p}_2) \text{ whenever } p_1^{\mathrm{V}} - \delta t_2 \leq \tilde{p}_2 ,$$

that is, whenever $p_2 D_2(p_1^{\mathrm{V}}, p_2)$ has a maximum in regime I. This condition can be easily seen to be equivalent to:

$$d \leq \frac{1 - \tau}{3\tau}(\sqrt{1 + \tau} - 1) \equiv v(\tau)$$

whenever $\delta \geq (1 - \tau)/3(2 + \tau) \equiv \underline{v}(\tau)$. If $\delta < \underline{v}(\tau)$, that is, if $\tilde{p}_2 < p_1^{\mathrm{V}} - \delta t_2$, $p_2 D_2(p_1^{\mathrm{V}}, p_2)$ is clearly decreasing in p_2 in regime I, and hence smaller than Π_2^{V}. Since $\underline{v}(\tau) < v(\tau)$, the non-deviating condition for firm 2 is $\delta \leq v(\tau)$, a condition which ensures that both prices p_1^{V} and p_2^{V} lead indeed to regime V and that it is also unprofitable for firm 1 to increase its price and switch to the market segmentation illustrated in Fig. 5. Again, the fact that the non-deviating condition for firm 2 is the strongest one is not surprising, since the firm with the smallest transportation rate is certainly the one which has the largest incentive to set low prices.

If $p_1 > p_2^{\mathrm{V}} + [t_2 - t_1 - \delta(t_1 + t_2)]/2$, $D_1 = x$ (as in Fig. 5) and the first order condition for firm 1 is:

$$\frac{\partial}{\partial p_1}(p_1 x) = \frac{t_2 - t_1 + \delta(t_1 + t_2) + 2p_2^{\mathrm{V}} - 4p_1}{2(t_2 - t_1)} = 0 ,$$

yielding $\tilde{p}_1 = (t_2 - t_1)/3 + \delta(t_1 + t_2)/4$. Setting this price is unprofitable for firm 1 if

$$\Pi_1^{\mathrm{V}} \geq \tilde{p}_1 D_1(\tilde{p}_1, p_2^{\mathrm{V}})$$

or, equivalently,

$$\delta \leqslant \frac{4(\sqrt{2}-1)}{3}\frac{1-\tau}{1+\tau},$$

a condition which is implied by $\delta \leqslant v(\tau)$. We have supposed $\tilde{p}_1 \geqslant p_2^{\mathrm{V}} + [t_2 - t_1 - \delta(t_1 + t_2)]/2$, so that the market configuration corresponding to $(\tilde{p}_1, p_2^{\mathrm{V}})$ is indeed that of Fig. 5. If that were not the case, firm 1's profit would be decreasing when $D_1 = x$ and, hence, smaller than Π_1^{V}.

Thus, $\delta \leqslant v(\tau)$ is a necessary and sufficient condition for prices $(p_1^{\mathrm{V}}, p_2^{\mathrm{V}})$ to be equilibrium prices. If the firms locations are not symmetric, this condition is still necessary, but not sufficient (see Dos Santos Ferreira and Thisse (1992) where the case of asymmetric locations is dealt with).

A3. The intermediate case

For intermediate values of δ, we get regime I (illustrated in Fig. 3), with demands $D_1 = 1 - z + y$ and $D_2 = z - y$ and with candidate equilibrium prices given by (12):

$$p_1^{\mathrm{I}} = \frac{1 - \delta\tau - \tau^2}{3} t_2 \text{ and } p_2^{\mathrm{I}} = \frac{1 + 2\delta\tau - \tau^2}{6} t_2 ,$$

yielding profits:

$$\Pi_1^{\mathrm{I}} = \frac{2}{9}\frac{(1 - \delta\tau - \tau^2)^2}{1 - \tau^2} t_2 \text{ and } \Pi_2^{\mathrm{I}} = \frac{1}{18}\frac{(1 + 2\delta\tau - \tau^2)^2}{1 - \tau^2} t_2 .$$

One should notice that a preliminary condition for existence of a price equilibrium in this regime is that $\max\{-\delta\tau_2, \delta\tau_1 - (1-\delta)(t_2 - t_1)/2\} \leqslant p_2^{\mathrm{I}} - p_1^{\mathrm{I}}$, that is:

$$f(\tau) \equiv \frac{1 - \tau^2}{2(3 + 2\tau)} \leqslant \delta \leqslant \frac{(1-\tau)(2-\tau)}{3-\tau} \equiv g(\tau) .$$

Firm 1 (firm 2) can only switch from regime I by raising (lowering) its price. Two cases must be considered, according to the regimes which are accessible by such price movements. First, if the two firms are far apart, more precisely if $\delta \geqslant (1-\tau)/(3+\tau)$, any firm may, by a price change such that $-\delta t_2 \leqslant p_2 - p_1 \leqslant \delta t_1 - (1-\delta)(t_2 - t_1)/2$, switch to regime H with $D_1 = y$ and $D_2 = 1 - y$. Furthermore, by a price change such that $p_2 - p_1 \leqslant -\delta t_2$, the market segmentation illustrated in Fig. 5, with $D_1 = x$ and $D_2 = 1 - x$, can be attained, at least by firm 1. Since the demands are less elastic in both these regimes, such deviations can only be profitable to firm 1 which increases its price. However, firm 1 will find it unprofitable to charge a price leading to the regime illustrated in Fig. 5. Indeed, for $\delta \geqslant (1-\tau)/(3+\tau)$ and $p_1 \geqslant p_2 + \delta t_2$,

$$\frac{\partial}{\partial p_1}(p_1 x) = \frac{t_2 - t_1 + \delta(t_1 + t_2) + 2p_2^{\mathrm{I}} - 4p_1}{2(t_2 - t_1)}$$
$$\leqslant \frac{(1-\tau)(2-\tau) - \delta(9-\tau)}{6(1-\tau)} < 0\,.$$

As for regime H, the first order condition for firm 1 is:

$$\frac{\partial}{\partial p_1}(p_1 y) = \frac{t_1 + t_2 + \delta(t_2 - t_1) + 2p_2^{\mathrm{I}} - 4p_1}{2(t_1 + t_2)} = 0\,,$$

yielding $\bar{p}_1 = p_2^{\mathrm{I}}/2 + [t_1 + t_2 + \delta(t_2 - t_1)]/4$. Firm 1 will have no incentive to deviate if

$$\Pi_1^{\mathrm{I}} \geqslant \bar{p}_1 D_1(\bar{p}_1, p_2^{\mathrm{I}})$$

for $p_2^{\mathrm{I}} - \bar{p}_1$ in the appropriate interval or, equivalently,

$$\delta \leqslant (1-\tau^2)\frac{4\sqrt{2(1-\tau)} - 4 + \tau}{4\sqrt{2(1-\tau)}\tau + (1-\tau)(3-\tau)} \equiv i^1(\tau)$$
$$\text{if } \delta \geqslant \frac{(1+\tau)(2+\tau)}{9+5\tau} \equiv \underline{i}(\tau) \text{ and } \delta \geqslant \frac{4 - 9\tau - \tau^2}{9+\tau} \equiv \bar{i}(\tau)\,.$$

If $\delta < \bar{i}(\tau)$, $p_1 y$ is decreasing so that the firm 1 will not switch to regime H. Since $\bar{i}(\tau) < i^1(\tau)$, the condition for a deviation to be unprofitable, at this stage, is: $\delta \leqslant i^1(\tau)$ if $\delta \geqslant \underline{i}(\tau)$. This excludes values of τ larger than $\tau^i \simeq 0.299$ since $\underline{i}(\tau) \leqslant i^1(\tau)$ iff $\tau \leqslant \tau^i$.

We must also consider the possibility of having $\delta < \underline{i}(\tau)$, a condition implying that firm 1's profit is increasing in the whole interval corresponding to regime H, so that the condition for an unprofitable deviation is:

$$\Pi_1^{\mathrm{I}} \geqslant (p_2^{\mathrm{I}} + \delta t_2)\frac{1-\delta}{2}$$

or, equivalently, for $\delta \leqslant g(\tau)$ (as required by the preliminary condition):

$$\delta \leqslant (1-\tau^2)$$
$$\frac{15 + 22\tau + 3\tau^2 - \sqrt{(15 + 22\tau + 3\tau^2)^2 - 40(9 + 3\tau - 5\tau^2 - 3\tau^3)}}{4(9 + 3\tau - 5\tau^2 - 3\tau^3)} \equiv i^2(\tau)\,.$$

It can be checked that $i^2(\tau) \geqslant \underline{i}(\tau)$ iff $\tau \leqslant \tau^i$, with τ^i as defined above. Hence, for $\tau \leqslant \tau^i$, firm 1 will have no incentive to deviate if $\delta < \underline{i}(\tau)$ and, for $\tau > \tau^i$, that will also be the case if $\delta \leqslant i^2(\tau)$. Summing up, the condition for unprofitable deviations by firm 1 is:

$$\delta \leqslant i(\tau), \text{ with } i(\tau) \equiv i^1(\tau) \text{ if } 0 < \tau \leqslant \tau^i$$
$$i^2(\tau) \text{ if } \tau^i \leqslant \tau < 1 .$$

Since $i(\tau) < g(\tau)$ and $f(\tau) < (1-\tau)/(3+\tau)$, as can be easily checked, the preceding condition implies that prices $(p_1^{\mathrm{I}}, p_2^{\mathrm{I}})$ lead indeed to the market segmentation characterizing regime I.

Now, let us consider the case where firms are close together: $\delta \leqslant (1-\tau)/(3+\tau)$. In this case, which will appear to be roughly symmetric to the former, regime H is inaccessible. Any of the two firms may on the contrary change its price so that $\delta t_2 \leqslant p_1 - p_2 \leqslant -\delta t_1 + (1-\delta)(t_2 - t_1)/2$, switching to regime V, with $D_1 = 1 - z + x$ and $D_2 = z - x$. A further price change, such that $p_1 - p_2 \geqslant -\delta t_1 + (1-\delta)(t_2 - t_1)/2$, will lead to the market segmentation illustrated in Fig. 5, with $D_1 = x$ and $D_2 = 1 - x$. Demands are more elastic in regime V than in regime I, so that there can be no incentive for firm 1 to switch to regime V by raising its price. However, firm 1 might be tempted to switch to the other regime, where the demand is less elastic. But this will never be the case since, for $\delta \leqslant (1-\tau)/(3+\tau)$,

$$\frac{\partial}{\partial p_1}(p_1 x) = \frac{t_2 - t_1 + \delta(t_1 + t_2) + 2p_2^{\mathrm{I}} - 4p_1}{2(t_2 - t_1)}$$
$$\leqslant \frac{\delta(9 + 7\tau) - (1-\tau)(4+\tau)}{6(1-\tau)} < 0 .$$

Hence, we can concentrate on possible deviations by firm 2, leading to regime V which has a more elastic demand. The first order condition in this regime is:

$$\frac{\partial}{\partial p_2}[p_2(z - x)] = \frac{2(p_1^{\mathrm{I}} - 2p_2)}{t_2 - t_1} = 0 ,$$

so that $\bar{p}_2 = p_1^{\mathrm{I}}/2$ maximizes the profit of firm 2 if $\delta t_2 \leqslant p_1^{\mathrm{I}} - \bar{p}_2 \leqslant [t_2 - t_1 - \delta(t_1 + t_2)]/2$, that is, if the choice of $\bar{p}_2$ leads to regime V. If the first inequality is violated, i.e. if $\delta > (1-\tau^2)/(6+\tau)$, the profit of firm 2 will decrease as it lowers its price. If both inequalities are satisfied, there will be no incentive for firm 2 to deviate if $\Pi_2^{\mathrm{I}} \geqslant \bar{p}_2 D_2(p_1^{\mathrm{I}}, \bar{p}_2)$, that is:

$$\delta \geqslant \frac{1-\tau^2}{\tau} \frac{\sqrt{1+\tau} - 1}{\sqrt{1+\tau} + 2} \equiv j^1(\tau) ,$$

a condition weaker than $\delta > (1-\tau^2)/(6+\tau)$. The condition $\delta \geqslant j^1(\tau)$ is necessary and sufficient for deviations by firm 2 to be unprofitable, if $p_1^{\mathrm{I}} - \bar{p}_2 \leqslant [t_2 - t_1 - \delta(t_1 + t_2)]/2$ or, equivalently,

$$\delta \leqslant \frac{(1-\tau)(2-\tau)}{3+2\tau} \equiv \bar{j}(\tau) .$$

For $p_1^{\mathrm{I}} - \bar{p}_2 \geqslant [t_2 - t_1 - \delta(t_1 + t_2)]/2$, that is, for $\delta \geqslant \bar{j}(\tau)$, the profit of firm 2 has a maximum when $z = 1$, at the price $\underline{p}_2 = p_1^{\mathrm{I}} - [t_2 - t_1 - \delta(t_1 + t_2)]/2$, so that the corresponding deviation is unprofitable if $\Pi_2^{\mathrm{I}} \geqslant \underline{p}_2 D_2(p_2^{\mathrm{I}}, \underline{p}_2)$ or, equivalently,

$$\delta \geqslant (1 - \tau^2)\frac{6 - 2\tau - 3\tau^2 + \sqrt{(6 - 2\tau - 3\tau^2)^2 - (4 - 5\tau)[3(1+\tau)(3+\tau) + 4\tau^2/(1+\tau)]}}{4\tau^2 + 3(1+\tau)^2(3+\tau)} \equiv j^2(\tau)\ ,$$

if this function is well defined, that is, if $\tau \geqslant \underline{\tau} \simeq 0.704$. Otherwise, $\delta \geqslant \bar{j}(\tau)$ is a sufficient condition for an unprofitable deviation.

It can be verified that $j^2(\tau) < j^1(\tau) < \bar{j}(\tau)$ if $\tau < \tau^j \simeq 0.88$ and that $\bar{j}(\tau) < j^2(\tau) < j^1(\tau)$ if $\tau > \tau^j$. It is thus straightforward to conclude that there is no incentive for firm 2 to deviate if (and only if):

$$\delta \geqslant j(\tau), \text{ with } j(\tau) \equiv \begin{cases} j^1(\tau) \text{ if } 0 < \tau \leqslant \tau^j \\ j^2(\tau) \text{ if } \tau^j \leqslant \tau < 1\ . \end{cases}$$

Since $j(\tau) > f(\tau)$ and $g(\tau) > (1 - \tau)/(3 + \tau)$, as it is easy to check, $(p_1^{\mathrm{I}}, p_2^{\mathrm{I}})$ is a price equilibrium in regime I under the necessary and sufficient condition: $j(\tau) \leqslant \delta \leqslant i(\tau)$.

The conditions for existence of a price equilibrium with symmetric locations are represented in Fig. 4 by the shaded regions in the parameter space (τ, δ), defined by the inequalities: $\delta \geqslant h(\tau)$ (regime H), $j(\tau) \leqslant \delta \leqslant i(\tau)$ (regime I) and $\delta \leqslant v(\tau)$ (regime V). The reader will easily verify that these are the only possible equilibrium regimes. Indeed, firm 1's profit function cannot have a maximum such that $z = 1$, since the demand elasticity decreases (discontinuously) at this point. For the same reason, firm 2's profit function cannot have a maximum such that $x = y$ (with $z \neq 1$). Last, in the regime illustrated in Fig. 5, where $D_1 = x$ and $D_2 = 1 - x$, the elasticity of D_1 is larger than the elasticity of D_2 at the same prices, so that there is no pair of admissible prices such that both elasticities are equal to 1, a necessary condition for profit maximization.

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