

Termination of Dynamic Contracts in an Equilibrium Labor Market Model

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Abstract

I construct an equilibrium model of the labor market where workers and firms enter into dynamic contracts that can potentially last forever, but are subject to optimal terminations. Upon a termination, the firm hires a new worker, and the worker who is terminated receives a termination contract from the firm and is then free to go back to the labor market to seek new employment opportunities and enter into new dynamic contracts. The model permits only two types of equilibrium terminations that resemble, respectively, the two kinds of labor market separations that are typically observed in practice: involuntary layoffs and voluntary retirements. The model allows simultaneous determination of its equilibrium turnover, unemployment, and retirement, as well as the expected utility of the new labor market entrants.

Keywords: dynamic contract, termination, labor market equilibrium

JEL Codes: E2, J41, J63

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1 Introduction

I construct an equilibrium model of the labor market where matched workers and firms enter into dynamic contracts that can potentially last forever, but are subject to optimal terminations. Moral hazard is the underlying information friction, that contracts are dynamic and terminations are optimal are both driven by incentive considerations. Upon termination of a contract, the firm hires a new worker, the worker who is terminated receives a termination compensation contract from the firm and is then free to go back to the labor market to seek new matches and enter into new dynamic contracts.

Despite the potentially complex interactions that could take place between the workers and firms in the model, the equilibrium of the model has a simple structure regarding termination. The model permits only two types of equilibrium terminations that resemble, respectively, the two kinds of labor market separations that are typically observed in practice: involuntary layoffs and voluntary retirements. When an involuntary layoff occurs, the firm promises no future payments to the worker, and the expected utility of the worker is strictly lower than that of the new worker the firm hires to replace him. When a voluntary retirement occurs, the worker leaves the firm with a termination compensation that is equal to a sequence of constant payments, and he never goes back to the labor market to seek new employment again.

The model thus allows simultaneous determination of its equilibrium unemployment, retirement, and labor turnover (flows between employment and unemployment, and the flow into retirement), as well as a set of other important labor market variables, including the distribution of current wages and expected utilities the employed workers, the distribution of compensation of the retired, the starting expected utility of a newly hired worker, and the equilibrium expected utility of the new labor market entrants.

Unemployment is involuntary in my model, as in the models of efficiency wages (e.g., Shapiro and Stiglitz 1984). Compared to the efficiency wage models though, my model offers at least three advantages. First, efficiency wage models are often criticized because the employment contracts in these models are not fully optimal. In Shapiro and Stiglitz, for example, because wages are constant, termination (lay-off) is the only incentive device that firms have available to prevent workers from shirking. In the model here, workers and firms enter into fully dynamic contracts where wages vary optimally with the worker's performance history. Second, in the existing models of efficiency wages, in equilibrium *no* workers are actually fired because of shirking (the contract makes effort-making incentive compatible so no one shirks), and the unemployed are a rotating pool of workers who quit for reasons that are exogenous to the model. In the model here, workers are actually

fired involuntarily from their jobs: firing is part of the model's equilibrium path. Third, my model permits simultaneously involuntary unemployment and voluntary retirement as its equilibrium outcome.

The economic logic for the equilibrium voluntary retirement in my model is intuitive. Because of the worker's decreasing marginal utility of consumption, the cost of compensating the worker for a given amount of effort is higher as the worker's expected utility increases. On the other hand, the way that the optimal contract works is that each time the worker delivers a high output, he is rewarded with a higher expected utility. Imagine now the worker produces a sequence of high outputs to make his expected utility sufficiently high. Then it will become too expensive for the firm to compensate for the worker's efforts, and the firm will find it efficient to replace the worker with an unemployed worker whose efforts are less expensive. The worker leaves the firm voluntarily, for his expected utility is not reduced because of the termination. The worker will not go back to the labor market upon termination, because other firms also would find him too expensive to employ.

In the model, retirement is optimal and determined by the worker's history of performance and the cost of the new worker that the firm could hire to replace him. Retirement is an incentive and compensation consideration. It occurs as a consequence of firms efficiently motivating and compensating their workers. Retirement is not a life-cycle consideration, as the workers are "perpetually" young (they die with a constant probability) in my model. Retirement does not depend on the worker's tenure *per se*, although it does depend indirectly on the worker's tenure because it takes time before the worker's expected utility becomes sufficiently high to justify retirement. There is not a unique retirement date in my model. There is a set of performance histories that can all lead to voluntary retirement. This property of my model differentiates it from Lazear's (1979) theory of mandatory retirement, which is based mainly on job tenure. In Lazear, it is imposed that there is a deterministic date T after which the worker's reservation wage exceeds his value of marginal product, and T is the retirement date.¹

This paper extends the existing theories of dynamic contract following

¹Lazear (1979) illustrates an environment where there is a fixed date T of separation which is independent of the labor contract. In order to prevent both the worker and the firm from cheating, especially unilateral termination before T arrives, it is optimal to make the wage scheme back-loaded. The firm then fires the worker after some exogenously given date T which corresponds to the efficient separation. The logic of my story is quite different. In my model, the expected utility of the worker moves up and down to provide incentives for efforts, but if it goes too high, then the worker should be terminated. The optimal date of termination and the optimal compensation contract are solved jointly.

Green (1987) and Srivastava (1987). What this paper does is to put fully dynamic contracts with endogenous termination into an equilibrium framework to allow agents to enter and exit contracting relationships multiple times. This has not been done in the literature. The paper that is closest to the current paper is Spear and Wang (2005). Spear and Wang adds an exogenous external labor market to the otherwise standard model of repeated moral hazard. This external labor market allows the firm to fire the worker and replace him with a new worker. Spear and Wang is a partial equilibrium setup where the unemployed workers' reservation utility is exogenously given, and it is imposed that workers who are terminated are never employed again. In the current paper, workers who are terminated are allowed to go back to the labor market to seek new employment opportunities, and the model makes clear predictions about who actually choose to go back to the labor market and who choose to stay out of the labor market permanently. Being an equilibrium setup, the model here allows me to determine simultaneously the equilibrium aggregate unemployment and retirement, as well as the model's other aggregate variables, including the equilibrium labor turnover and the expected utility of the new labor market entrants. Termination of dynamic contracts is also studied by DeMarzo and Fishman (2003) in a partial equilibrium model of corporate finance with privately observed cash flows. Stiglitz and Weiss (1983) model the incentive effects of termination in a two period environment where there is only one worker and one firm.

An important feature of the dynamic contracts in this paper is that they are required to be renegotiation proof. This not only makes economic sense, but also plays a key role in simplifying the model's equilibrium structure. Specifically, that the contracts must be renegotiation proof implies that all unemployed workers are homogeneous in expected utility. Since workers are identical in ability, that contracts must be renegotiation proof implies that the termination compensation of an involuntarily terminated worker (who after termination goes back to the labor market to seek new employment) must be zero. Otherwise, the firm and the worker can always renegotiate to make both parties strictly better off. This renegotiation simply requires that the worker gives back the termination compensation and the firm hires back the worker.

Section 2 describes the model. Section 3 defines the contracts and labor market equilibrium. Section 4 characterizes voluntary retirement and involuntary layoff. Section 5 concludes the paper.

2 Model

Time is discrete and lasts forever. There is one perishable consumption good in each period. The economy is populated by a sequence of overlapping generations, each of which contains a continuum of workers. The total measure of workers in the economy is equal to one. Each worker faces a time-invariant probability Δ of surviving into the next period. Each new generation has measure $1 - \Delta$, so the number of births and the number of deaths are equal in each period. ²An individual who is born at time τ has the following preferences: $E_{\tau-0} \sum_{t=\tau}^{\infty} (\beta\Delta)^{t-\tau} H(c_t, a_t)$, where $E_{\tau-0}$ denotes expectation taken at the beginning of period τ , c_t denotes period t consumption, a_t denotes period t effort, $H(c_t, a_t)$ denotes period t utility, and $\beta \in [0, 1]$ is the discount factor. Assume $H(c, a) = v(c) - \phi(a)$, for $c \in \mathbf{R}_+$, $a \in \{0\} \cup \mathbf{A}$, where $\mathbf{A} \subseteq \mathbf{R}_+$ is the individual's compact set of feasible effort levels when he is employed. The individual's effort takes the value 0 if he is not employed. Let $\underline{a} \equiv \min\{a \in \mathbf{A}\} > 0$. Finally, the functions v is strictly increasing and concave in c , and the function ϕ is strictly increasing in a with $\phi(0) = 0$.

There are $\eta \in (0, 1)$ units of firms in the model. Firms live forever and maximize expected discounted net profits. For convenience, I assume in each period, each firm needs to employ only one worker. ³ The worker's effort is the only input in the firm's production function, and the worker's effort is observed by himself only. By choosing effort a_t in period t , the worker produces a random output in period t that is a function of a_t . Let θ^t denote the realization of this random output. Assume $\theta^t \in \Theta$, where $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ with $\theta_i < \theta_j$ for $i < j$. Let $X_i(a) = Prob\{\theta^t = \theta_i | a_t = a\}$, for all $\theta_i \in \Theta$, all $a \in \mathbf{A}$ and all t .

The firm and a newly hired worker can enter into a labor contract that is fully dynamic. A component of this dynamic contract is a history dependent plan that specifies whether the worker is terminated at the end of each date (or the beginning of the next date). If the worker is terminated, he is free to go immediately back to the labor market to seek new employment opportunities, and the firm then hires a new worker to replace him. For convenience I assume the process of termination and replacement involves no physical costs to both the firm and the worker. An extension of the current work is to study the effects of a cost of termination which may be imposed by a policy maker.

As part of the model's physical environment, I make three assumptions about the contracts that are feasible between the worker and the firm. First,

²The OLG structure is needed here in order for me to model stationary equilibria with voluntary retirements.

³It would not make a difference if I allow firms to employ more workers, as long as they operate independent production technologies.

contracts are subject to a non-negativity constraint that require that all compensation payments to the worker be non-negative.

Second, contracts are subject to renegotiations, provided that the renegotiations are mutually beneficial and *strictly* beneficial to the firm. This assumption puts an additional restriction on the structure of the dynamic contract that can be signed between the firm and the worker: the contract must be renegotiation-proof (RP). Note that in order for renegotiations to take place, I require that they be *strictly* beneficial to the firm. That is, the firm can commit to the continuation of a dynamic contract if a renegotiation can benefit the worker while leaving the firm indifferent. As will become clear in the analysis of the model, since workers are identical, the requirement that the firm be strictly better off in a renegotiation is needed in order to make involuntary terminations part of the equilibrium contract.⁴

Third, it is feasible for the firm to continue to make compensation payments to the worker after the worker is terminated (i.e., he is replaced by a new worker), but there is a restriction. Post-termination compensations cannot be contingent on the worker's performance and compensation at the new firm the worker works in the future, although these compensations can be made a function of the worker's future employment status. In other words, post termination compensations must be a step function of the worker's employment status after his separation from the current employer.⁵

3 Contracts and Equilibrium

I first define a dynamic contract, taking as given the labor market in which this contract must operate; I then define a labor market equilibrium by requiring the market be consistent with individual firms' optimal contracts.

I take a guess-and-verify approach to find the model's equilibrium. Specifically, when defining the optimal contract, I take as given that the labor market equilibrium has the following property: the workers who are not employed and are looking for jobs at the beginning of a period either have never been employed (these include the new labor market entrants), or are entitled to zero post termination compensations from their former employers. This property implies that an employed worker's compensation from his current

⁴See Wang (2000) and Zhao (2004) for an existing analysis of renegotiation-proof contracts in the model of dynamic moral hazard. In Zhao (2004), a RP contract under the qualification that renegotiations must be strictly beneficial to the principal is called a principal RP contract. Zhao used this concept for a different purpose than mine

⁵This assumption saves me from the difficulty of modelling a potentially complicated dynamic game that can be played between the worker's former and current employers.

employer is his total compensation. I will then verify that this indeed is part of the labor market equilibrium.

3.1 Contracts

Let σ denote a contract between a firm and its newly hired worker. For convenience I use $t = 0$ to denote the time the contract is signed. Let $t(\geq 1)$ denote the t th period into the contract ($t = 1$ is the first period the worker is hired to work for the firm and so on). Let h^t denote the worker's history of output up to the end of period t (or at the beginning of period $t + 1$). Let $h^0 = \emptyset$. Then for each $t \geq 1$, let $h^t = \{\theta^1, \dots, \theta^t\} \in H^t \equiv \Theta^t$. Next, let $\mathbf{H} \equiv \bigcup_{t=1}^{\infty} H^t$. This is the space of all possible histories. With these, I can now define an extensive form contract as

$$\sigma = \left\{ \begin{array}{l} (H_r, H_f), H_r, H_f \subseteq \mathbf{H} \\ (a_t(h^{t-1}), c_t(h^t)), \forall h^{t-1} \in H_r, t \geq 1 \\ g_t(h^{t-1}) : \forall h^{t-1} \in H_f, t \geq 1 \end{array} \right\}.$$

This contract has the following interpretations. First, (H_r, H_f) denotes the firm's termination policy, where the set $H_r(H_f)$ contains all histories of the worker's output upon which the worker is retained (terminated). I require that a termination policy must satisfy the following conditions which have obvious interpretations: $H_r \cup H_f \subseteq \mathbf{H}$; $H_r \cap H_f = \emptyset$; $h^{t-1} \in H_f$ implies $h^t = \{h^{t-1}, \theta^t\} \notin H_r \cup H_f$ for all $\theta^t \in \Theta$; and $h^{t-1} \in H_r$ if and only if $h^t = \{h^{t-1}, \theta^t\} \in H_r \cup H_f$ for all $\theta^t \in \Theta$. Next, consider a worker who is at the beginning of an arbitrary period t , $t \geq 1$. This worker's history is h^{t-1} . Suppose $h^{t-1} \in H_r$ so the worker continues to be employed in the current period, according to the termination policy. The function $a_t : H^{t-1} \rightarrow \mathbf{A}$ then specifies the level of effort the firm wants the worker to make in period t . The worker chooses his effort. The firm's output is realized. The worker's history is updated to become h^t . The function $c_t : H^t \rightarrow \mathbf{R}_+$ then specifies the worker's compensation in period t . Now go back to the beginning of period t . Suppose $h^{t-1} \in H_f$. Then the worker is terminated, leaving the firm with a "termination contract" $g_t(h^{t-1})$ that specifies the future payments he may receive from the firm starting from period t .

Following Green (1987) and Spear and Srivastava (1987), I can use the worker's expected utility as a state variable to summarize the history of the worker's output. This allows me to rewrite the contract recursively as

$$\sigma = \left\{ \begin{array}{l} \Phi = \Phi_r \cup \Phi_f \\ (a(V), c_i(V), V_i(V)), \forall V \in \Phi_r \\ g(V), \forall V \in \Phi_f \end{array} \right\}.$$

Here, V denotes the worker's expected utility at the beginning of a period: the state variable. The set $\Phi = \Phi_r \cup \Phi_f \subseteq R$ is the domain of V : the state space. Φ is partitioned into two subsets Φ_r and Φ_f with $\Phi_r \cap \Phi_f = \emptyset$. This partition of Φ defines the contract's termination rule: If $V \in \Phi_f$, then the worker is terminated; if $V \in \Phi_r$, the worker is retained. Now if the worker is terminated, $g(V)$ denotes the termination contract he receives from the firm in the termination state $V \in \Phi_f$. If the worker is retained, that is, if $V \in \Phi_r$, then $a(V)$ denotes the worker's recommended effort in the current period; $c_i(V)$ denotes the worker's compensation in the current period if his output is θ_i ; and finally, $V_i(V)$ denotes the worker's expected utility at the beginning of the next period, conditional on his output being θ_i in the current period.

The contract σ is said to be feasible if for all $V \in \Phi_r$, $a(V) \in \mathbf{A}$, $c_i(V) \geq 0$, $V_i(V) \in \Phi$; and that for all $V \in \Phi_f$, post termination compensation payments to the worker that are dictated by the termination contract $g(V)$ are all non-negative. Remember a termination contract must be a step function of the worker's employment status after termination. Let \mathcal{G} denote the space of all feasible termination contracts.

The contract must satisfy a promise-keeping constraint. This constraint requires that the structure of σ be consistent with the definition of V being the worker's expected utility at the beginning of a given period, for all $V \in \Phi$. In particular, the termination contract $g(V)$ must be designed to guarantee that the worker who leaves the firm with an expected utility entitlement V is indeed to receive expected utility equal to V . That is, given $g(V)$, and given what the market has to offer to the worker after termination, the worker's expected utility must be equal to V when he leaves the firm. Thus the promise-keeping constraint can be formulated as:

$$V = \sum_i X_i(a(V)) [H(c_i(V), a(V)) + \beta \Delta V_i(V)], \quad \forall V \in \Phi_r, \quad (1)$$

$$M[g(V)] = V, \quad \forall V \in \Phi_f, \quad (2)$$

where equation (1) is familiar from the literature, (2) is not. In equation (1), given that I take as given that the worker was not entitled to any post termination compensation from any previous employers, $c_i(V)$ is just the worker's current consumption. In equation (2), I use $M(x)$ to denote the value of the expected utility that an arbitrary feasible termination contract $x \in \mathcal{G}$ delivers to the worker, given the market that x takes as given. That is, the worker's expected utility is $M(x)$ if he leaves the firm with termination contract x . At this stage, I take the termination value function $M : \mathcal{G} \rightarrow \mathbf{R}$ as given. I will later specify the form of M .

A contract σ is called incentive compatible if

$$\begin{aligned} & \sum_i X_i(a(V))[H(c_i, a(V)) + \beta\Delta V_i(V)] \\ \geq & \sum_i X_i(a')[H(c_i(V), a') + \beta\Delta V_i(V)], \quad \forall V \in \Phi_r, \forall a' \in \mathbf{A}. \end{aligned} \quad (3)$$

Notice that the promise-keeping constraint is defined for all $V \in \Phi$, whereas the incentive constraint need only be defined for all $V \in \Phi_r$.

Given σ , and given the market (where the worker goes back to after termination) that the contract must take as given, I can calculate the firm's expected utility $U(V)$ for each $V \in \Phi$. I then refer to $U : \Phi \rightarrow \mathbf{R}$ as the value function of the contract σ (again, conditional on the market that σ takes as given).

An important component of the market that a contract must take as given is the expected utility of a new labor market entrant which I denote by V_* . Obviously, V_* is also the expected utility of a worker who either was never employed, or was employed but is not entitled to any post termination compensation payments from his former employers. In other words, V_* is the expected utility of all workers who are in the labor market at the beginning of period, as I take as given that the labor market equilibrium has the property that workers who are not employed and looking for jobs are entitled to zero post termination compensations from former employers.

I am now in a position to define renegotiation-proof (RP) contracts. I will call a contract σ RP if it supports a value function that is RP. As is the definition of the contract σ , the definition of the RP-ness of σ is also conditional on the market that σ takes as given. In the following, I first define what it means to say that a value function is RP. I then define what it means to say that a contract supports a RP value function.

A RP value function will be defined as a fixed point of an operator that map from an underlying functional space \mathbf{B} to itself, where

$$\mathbf{B} \equiv \{U : \Psi_r \cup \Psi_f \rightarrow \mathbf{R} \mid \Psi_r, \Psi_f \subseteq \mathbf{R}, \Psi_r \cap \Psi_f = \emptyset; \Psi_r, \Psi_f \subseteq [V_{min}, V_{max}]\}$$

where $V_{min} \equiv \frac{u(0) - \phi(0)}{1 - \beta\Delta}$, $V_{max} = \frac{u(\infty) - \phi(0)}{1 - \beta\Delta}$. Clearly, V_{min} is the minimum expected utility of a worker that is feasible in the model, V_{max} is an upper bound of all feasible expected utilities of the worker.

\mathbf{B} includes all the value functions I need to consider. These value functions each have two components to its domain: one associated with continuation (Ψ_r), one associated with termination (Ψ_f). I say that two value functions in \mathbf{B} are equal if they have the same Ψ_r and Ψ_f and the same

values for each $V \in \Psi_r \cup \Psi_f$. Value functions that have the same graph but not the same domain partition are considered different value functions.

In the following, I will use $U(\Psi_r, \Psi_f)$ to denote a value function in \mathbf{B} whose domain is partitioned into Ψ_r and Ψ_f .

Definition 1 Let $U \in \mathbf{B}$. U is said to be (internally) renegotiation-proof if it satisfies the following functional equation:

$$U = PTU, \quad (4)$$

where T and P , to be defined in the following, are operators that map from \mathbf{B} to \mathbf{B} .

Equation (4) is based on Ray (1994) where the operator T gives the set of all optimal expected utility pairs that are generated by U , and P then gives the subset of the graph of TU such that each utility pair in this subset is not Pareto dominated by any other utility pair in the graph of TU .⁶

I first define the operator T . Let $U(\Phi_r, \Phi_f) \in \mathbf{B}$. I take the following three steps to define the value function $TU(\Phi'_r, \Phi'_f)$.

Step 1. I first define a value function U_r that is associated with the values of the firm when it retains the current worker. To do this, let $\tilde{\Phi}_r$ denote the set of all V such that there exists $\{a, c_i, V_i\}$ that satisfies the following constraints:

$$a \in \mathbf{A}; c_i \geq 0, V_i \in \Phi_r \cup \Phi_f, \forall i, \quad (5)$$

$$\sum_i X_i(a)[H(c_i, a) + \beta \Delta V_i] \geq \sum_i X_i(a')[H(c_i, a') + \beta \Delta V_i], \forall a' \in \mathbf{A}, \quad (6)$$

$$V = \sum_i X_i(a)[H(c_i, a) + \beta \Delta V_i]. \quad (7)$$

Then for each $V \in \tilde{\Phi}_r$, let

$$U_r(V) \equiv \max_{\{a, c_i, V_i\}} \sum_i X_i(a)[\theta_i - c_i + \beta \Delta U(V_i)] + \beta(1 - \Delta) \max_{V' \in \Phi_r, V' \geq V_*} U(V') \quad (8)$$

subject to (5),(6),(7).

⁶There are several other ways to define the sets of renegotiation-proof payoffs for infinitely repeated games. Ray's is a natural extension of the concept of renegotiation-proof payoff sets in finitely repeated games to infinitely repeated games. Ray's concept was used by Zhao (2004) to study renegotiation-proof dynamic contracts with moral hazard.

Here, equation (6) is the incentive constraint, (7) is the promise-keeping constraint. Equation (8) reflects the fact that with probability $(1 - \Delta)$ the existing worker will die, in which case the firm must go back to the labor market to hire a new worker. This new worker has a reservation utility equal to V_* .

Step 2. I define a value function $U_f : [V_*, V_{max}) \rightarrow R$ that is associated with termination. For all $V \in [V_*, V_{max})$,

$$U_f(V) \equiv \max_{g \in \mathcal{G}} \left\{ -C(g) + \max_{V' \in \Phi_r, V' \geq V_*} U_r(V') \right\} \quad (9)$$

subject to

$$M(g) = V, \quad (10)$$

where $C : \mathcal{G} \rightarrow \mathbf{R}_+$, $C(g)$ denotes the cost of the termination contract $g(V)$ to the firm. This is essentially the expected discounted payment that the firm makes to the worker after termination. Note that given non-negative compensation payments after termination, $C(g) \geq 0$ for all $g \in \mathcal{G}$.

Equation (10) is promise-keeping. Again I use $M[g]$ to denote the value of the termination contract g to the worker. When this value is equal to V , then g delivers expected utility V to the worker. Obviously, the functions C and M depend on what is out there for a terminated worker in the market: given $g(V)$, the cost of $g(V)$ to the firm and the expected utility that the worker obtains from $g(V)$ may both depend on the parameters of the market, including when the worker would find new employment and what the terms of the new contract would be.

Step 3. Let $\Phi' = \tilde{\Phi}_r \cup [V_*, V_{max})$. Extend the function $U_r : \tilde{\Phi}_r \rightarrow \mathbf{R}$ to the domain Φ' by letting $U_r(V) = -\infty$ for all $V \in \Phi' - \tilde{\Phi}_r$. Extend U_f from $[V_*, \infty)$ to the domain Φ' by letting $U_f(V) = -\infty$ for all $V \in \Phi' - [V_*, V_{max})$. Then, for each $V \in \Phi'$, let the value of $TU(V)$ be defined by

$$TU(V) = \max\{U_r(V), U_f(V)\}. \quad (11)$$

And then, the sets Φ'_r and Φ'_f are defined by

$$\Phi'_r = \{V \in \Phi' : U_r(V) \geq U_f(V)\}, \quad (12)$$

$$\Phi'_f = \{V \in \Phi' : U_r(V) < U_f(V)\}. \quad (13)$$

Equation (11) says that the firm chooses to retain or fire the worker depending on which action gives the firm a better value. Finally, equations

(12) and (13) gives the partition of the domain of TU by definition. I have now finished the three steps to defining the operator T .

I now move on to define the operator P . I say that a pair of expected utilities (V, Z) is Pareto dominated by another pair of expected utilities (V', Z') , denoted $(V', Z') >_p (V, Z)$, if $V' \geq V$, $Z' > Z$. Here, V and V' denote expected utilities of the worker, Z and Z' denote expected utilities of the firm.

Again, let $U(\Phi_r, \Phi_f) \in \mathbf{B}$. Then $PU : \Phi'_r \cup \Phi'_f \rightarrow R$ is defined by

$$\Phi'_k = \{V \in \Phi_k : \nexists V' \in \Phi \text{ such that } (V', U(V')) >_p (V, U(V))\},$$

for $k = r, f$, and

$$PU(V) = U(V), \forall V \in \Phi'_r \cup \Phi'_f.$$

I have now finished the definition of a renegotiation-proof value function.

Definition 2 Let $U : \Phi (= \Phi_r \cup \Phi_f) \rightarrow R$ be a RP value function. I say that contract $\sigma = \{(a(V), c_i(V), V_i(V)), V \in \Phi_r; g(V), V \in \Phi_f\}$ supports value function $U \in \mathbf{B}$ (and is hence RP) if:

- (i) $\{a(V), c_i(V), V_i(V)\}$ is a solution to the maximization problem (5)-(8) for all $V \in \Phi_r$, and $g(V)$ is a solution to the maximization problem (9)-(10) for all $V \in \Phi_f$; and
- (ii) $V \in \Phi_r$ if and only if $U_r(V) \geq U_f(V)$.

Also by definition, for any RP value function, there is at least one RP contract that supports it.

By definition, if a value function is RP, then it is weakly decreasing.

Now a problem with the concept of the RP-ness of dynamic contracts is that it is difficult to guarantee uniqueness.⁷ To cope with this difficulty, I define the following notation of optimality.

Let Σ denote the set of all RP contracts. Let $\sigma \in \Sigma$. Let $\{U_r^\sigma(V), V \in \Phi_r^\sigma; U_f^\sigma(V), V \in \Phi_f^\sigma\}$ denote the value functions conditional respectively on retention and termination that σ supports. A contract $\sigma^* \in \Sigma$ is said to be optimal if

$$\sigma^* \in \arg \max_{\sigma \in \Sigma} \left\{ \max_{V \in \Phi_r^\sigma} U_r^\sigma(V) \right\}.$$

In other words, a RP contract σ^* is optimal if it allows the firm to achieve the highest possible firm value.

⁷See Pearce (1995) for a discussion of the issue of the non-unique RP value functions in dynamic games.

Notice that given the optimal contract σ^* , suppose the firm has just hired a new worker, and suppose the firm is free to choose a level of expected utility to be promised to this new worker to maximize the value of the firm. Then the firm's optimization problem is

$$\max_{V \in \Phi^{\sigma^*}} U_r^{\sigma^*}(V). \quad (14)$$

I assume the solution to the above problem is unique. Let \underline{V} denote this solution. That is, \underline{V} is the expected utility of the new worker that can give the firm the highest value under the optimal contract σ^* . In fact, \underline{V} is the expected utility of the worker at which the firm can achieve its highest value across all levels of V that are feasible under any RP contract.

Now suppose $\underline{V} \geq V_*$, which I will show to hold in the model's equilibrium (Proposition 3). Then it is feasible for the firm to start a new worker with \underline{V} . Then \underline{V} denotes the unique starting expected utility of a new worker that maximizes the firm's value.

The assumption that the Problem (14) has a unique solution offers an obvious technical convenience. Suppose otherwise. Then the firm's value function $U_r^{\sigma^*}(V)$ is constant over an interval of $[\underline{V}_1, \underline{V}_2]$, where \underline{V}_1 is the minimum and \underline{V}_2 the maximum of V that maximizes the firm's value $U_r^{\sigma^*}(V)$. If this is the case, then it would be natural to assume that the firm starts the worker with \underline{V}_2 .

3.2 Market and Equilibrium

I am now ready to describe the market and then define what constitutes an equilibrium of the market.

Workers in the model are divided into three groups at the beginning of any period: those who are currently employed, those who are unemployed (not employed and looking for employment, including the new labor market entrants), and those who are not in the labor force (not employed and not looking for employment). As the economy moves into the middle of the period (that is after the labor market closes), some of the unemployed will become employed as they match with vacant firms. Then when the period ends, a fraction of the employed workers will be terminated, a fraction of them become unemployed, a fraction of them may decide to stay out of the labor market either temporarily or permanently.⁸

Terminations are divided into two types. I call a termination *involuntary* if the worker's expected utility is strictly below \underline{V} upon termination, i.e.,

⁸As will become clear later, all withdraws from the labor market are permanent in this model.

$V \in \Phi_f$ and $V < \underline{V}$. A termination is called *voluntary* if it is not involuntary, that is, $V \in \Phi_f$ and $V > \underline{V}$. Note that $\underline{V} \notin \Phi_f$. Thus, if an involuntary termination occurs, the worker who is terminated would like to work for a lower expected utility than what is offered by the contract of the new worker the firm hires to replace him. This is not the case in a voluntary termination.

Proposition 1 If $V \in \Phi_f$ and $V < \underline{V}$, then $C[g(V)] = 0$.

Proof. Suppose $C[g(V)] > 0$ for some V that satisfies $V \in \Phi_f$ and $V < \underline{V}$. Then the optimal contract has

$$U(V) = U_f(V) = U_r(\underline{V}) - C[g(V)] < U_r(\underline{V}). \quad (15)$$

This implies $(\underline{V}, U_r(\underline{V})) >_p (V, U(V))$ and so the contract is not renegotiation-proof. A contradiction. Q.E.D.

In equation (15), the left hand side of the inequality is the firm's expected value if the worker is involuntarily terminated; the right hand side is the expected value of the firm if the firm retains the worker, promising him expected utility \underline{V} , and taking back his termination contract $g(V)$. So the firm and the worker can both do strictly better by moving the worker's utility from V to \underline{V} . Thus the contract is not RP.

Because all termination contracts must specify non-negative payments from the firm to the workers in all periods, $C[g(V)] = 0$ holds if and only if the worker receives zero payments from the firm in all future periods after termination. In turn, this implies that upon an involuntary termination, the worker's utility must be equal to V_* . That is,

Corollary 1 If $V \in \Phi_f$ and $V < \underline{V}$, then $V = V_*$.

Proposition 1 confirms the conjecture that in equilibrium, all workers who are involuntarily terminated are entitled to zero compensation payments (current and future) as long as they remain unemployed. Thus in the forward looking sense, all workers who are involuntarily unemployed at the beginning of a period (including the new labor market entrants, workers who were never employed, and workers who were involuntarily terminated) are essentially identical. They each have expected utility V_* , would like to obtain employment, and will be employed in any given period with the same probability and with the same contract.

Let $\pi \in [0, 1]$ denote the equilibrium probability with which a worker who is unemployed at the beginning of a period becomes employed during the period (the rate of hiring out of the pool of the unemployed). So if

$\pi < 1$, then in equilibrium some workers will remain unemployed throughout the period.

Proposition 2 Suppose $\pi < 1$. Then all the voluntarily terminated workers are never re-employed.

Proof. Let V denote a voluntarily terminated worker's expected utility. That the worker was voluntarily terminated implies $U_f(V) = U_r(\underline{V}) - C[g(V)] > U_r(V)$, or $U_r(\underline{V}) > U_r(V) + C[g(V)]$. That is, the firm is strictly better off hiring an involuntarily unemployed worker, who is available given $\pi < 1$, than hiring a voluntarily terminated worker and taking his $g(V)$. So the firm would never hire the voluntarily terminated. Q.E.D.

So once terminated voluntarily, the worker will never go back to the labor market. He is retired.

Propositions 1 and 2 greatly simplify the structure of the termination contract. Suppose $\pi < 1$. Since a voluntarily terminated worker is never reemployed, the termination contract $g(V)$ is simply a sequence of constant consumption equal to $v^{-1}[(1 - \beta\Delta)V]$ paid to the worker until he dies. This implies the following termination conditions for the firm:

$$\text{If } V \in \Phi_f, \text{ then } U(V) = U_r(\underline{V}) - C[g(V)], \quad (16)$$

where

$$C[g(V)] = \begin{cases} 0, & w < \underline{V}, \\ \frac{v^{-1}[(1-\beta\Delta)V]}{1-\beta\Delta}, & w \geq \underline{V}. \end{cases} \quad (17)$$

Propositions 1 and 2 also allow me to specify the worker's termination value function $M(g)$. By the propositions, I need only focus on termination contracts that take the form of a constant stream of compensation pay after termination, denote this stream by $\{c_g\}$ for a given termination contract g . Then I have

$$M(g) = \begin{cases} V_*, & c_g = 0, \\ H(c_g, 0)/(1 - \beta\Delta), & c_g > 0. \end{cases} \quad (18)$$

Notice that for all $V \in \Phi_f$ and $V < \underline{V}$, $U_f(V) = U_r(\underline{V})$. That is, each time a worker is involuntarily terminated, the firm is indifferent between firing him (so the worker will receive expected utility of V_* .) and retaining him and to restart him with the promised utility \underline{V} . This is the reason the model requires that renegotiations be strictly beneficial to the firm in order for them to happen. Otherwise, the firm would face a dilemma which is beyond what I can address in the current paper. Note that this is not a

problem in the case of a voluntary termination, where the firm is always strictly better off ex post to start up with a new worker than to stay with the old worker.

To summarize, if a worker is terminated involuntarily, then he will receive no payments from the firm after termination and hence his expected utility must be equal to V_* . If the termination is voluntary, then the worker will receive in each future period from the firm a constant payment equal to $v^{-1}[(1 - \beta\Delta)V]$ and he never goes back to the market again. Propositions 1 and 2 also imply that if $\pi < 1$, then all new hires will start with the same expected utility \underline{V} . These results greatly simplify the structure of the market for contracts, making it ready now for me to formulate the definition of equilibrium.

I will focus on the model's stationary equilibria in this paper. The first equilibrium condition is the following stationarity condition for V_* :

$$V_* = \pi\underline{V} + (1 - \pi)[H(0, 0) + \beta\Delta V_*]$$

or

$$V_* = \frac{\pi\underline{V} + (1 - \pi)H(0, 0)}{1 - (1 - \pi)\beta\Delta} \quad (19)$$

Since voluntarily terminated workers are never reemployed, I will call these workers the (voluntarily) retired.

Let μ_V denote the measure of the retired workers at the beginning of each period. This number remains constant before and right after the labor market is closed.

Let μ_I denote the measure of the unemployed workers at the beginning of a period but after the labor market is closed. This includes workers who have never been employed and workers who were terminated in a previous period with $C[g(V)] = 0$. Each of these workers have expected utility V_* .

Finally, let $\mu_E : \Phi_r \rightarrow [0, 1]$ denote the distribution of the expected utilities of the employed workers after the labor market is closed but before production occurs: $\int_{\Phi_r} d\mu_E(V) = 1$. Note the total number of these workers is η .

Let ξ denote the aggregate turnover rate: the fraction of employed workers ($V \in \Phi_r$) to flow into unemployment or retirement ($V' \in \Phi_f$) each period,

$$\xi \equiv \int_{\Phi_r} \sum_{\{i: \theta_i \in \Omega(V)\}} X_i(a(V)) d\mu_E(V)$$

where for each $V \in \Phi_r$,

$$\Omega(V) \equiv \{\theta_i : V_i(V) \in \Phi_f\}$$

is the set of all realizations of the current state of the worker's output θ in which the worker with expected utility V will be terminated. Therefore, the aggregate labor market turnover is $\xi\eta$. This is also the number of the newly employed workers in each period (the flow from unemployment to employment).

In addition, for all $V \in \Phi_r$, I let

$$\Omega_I(V) = \{\theta_i : V_i(V) \in \Phi_f, V_i(V) < \underline{V}\}$$

and

$$\Omega_V(V) = \{\theta_i : V_i(V) \in \Phi_f, V_i(V) > \underline{V}\}.$$

So $\Omega_I(V)$ is the set of the realization of θ for which the worker is terminated involuntarily, and $\Omega_V(V)$ is the set of all realizations of θ upon which the worker is terminated voluntarily.

Finally, let

$$\xi_I \equiv \int_{\Phi_r} \sum_{\{i:\theta_i \in \Omega_I(V)\}} X_i(a(V)) d\mu_E(V) \tag{20}$$

$$\xi_V \equiv \int_{\Phi_r} \sum_{\{i:\theta_i \in \Omega_V(V)\}} X_i(a(V)) d\mu_E(V).$$

That is, ξ_I is the fraction of the employed workers to transition to involuntary unemployment each period, and ξ_V is the fraction of the employed workers to transition to retirement each period. Clearly, $\xi = \xi_I + \xi_V$.

Definition 3 A stationary equilibrium of the model is a vector

$$\{\pi, V_*, \underline{V}, \sigma^*, (\xi_I, \xi_V) (\mu_E, \mu_V, \mu_I)\}$$

where

- (i) Given π, V_*, \underline{V} , and (μ_E, μ_V, μ_I) , σ^* is an optimal contract.
- (ii) \underline{V} is the solution to equation (14),
- (iii) V_* is given by (??),
- (iv) π is given by

$$\pi = \frac{\eta(\xi_I + \xi_V)}{(1 - \Delta) + \Delta\mu_I + \Delta\eta\xi_I}$$

- (v) (ξ_I, ξ_V) are given by (20).

- (vi) (μ_E, μ_V, μ_I) satisfy the following stationarity conditions:

$$\mu_I = (1 - \pi)[(1 - \Delta) + \Delta\mu_I + \Delta\eta\xi_I], \tag{21}$$

$$\mu_V = \Delta\mu_V + \Delta\eta\xi_V, \quad (22)$$

$$\mu_E = \Gamma(\mu_E), \quad (23)$$

where the operator Γ maps the distribution of the expected utilities of the employed workers in the current period into that in the next period, as dictated by the law of motion for $V \in \Phi_r$ (i.e., $\{V_i^*(V), V \in \Phi_r\}$), the equilibrium starting expected utility \underline{V} , and the survival rate Δ .

Note that μ_I is the model's equilibrium unemployment measured at the middle of the period. The model's unemployment measured at the beginning of the period and before the labor market opens should then be $\mu_I/(1 - \pi) = (1 - \Delta) + \Delta\mu_I + \Delta\eta\xi_I$.

I now conclude this section with a remark. The set of variables that are determined in the model's equilibrium includes (1) the economy's aggregate unemployment (μ_I), (2) aggregate retirement (μ_V), (3) aggregate labor turnover ($\xi\eta$), the flows from employment to unemployment ($\xi_I\eta$), (4) the flows from employment to retirement ($\xi_V\eta$), (5) the economy's labor force participation rate ($1 - \mu_V$), (6) the distribution of expected utilities of the retired workers, (7) the distribution of current wages and expected utilities of the employed workers, (8) the expected utility of the involuntarily unemployed workers, and (9) the starting expected utility of a newly hired worker. This set includes the majority the labor market variables that are commonly viewed as important. This is a significant advantage my model offers, especially given the model's tight setup of its physical environment that assumes fixed numbers of homogeneous workers and firms.

4 Voluntary and Involuntary Terminations

A necessary condition for the existence of equilibrium involuntary termination and involuntary unemployment is $V_* < \underline{V}$. In addition, if this condition holds, then all the unemployed (if any) are involuntarily unemployed.

Proposition 3 Suppose in equilibrium there is unemployment (i.e., $\pi < 1$). Suppose the equilibrium is not degenerate. That is, suppose $a_i^*(h^{t-1}) > 0$ for some t and h^{t-1} with the equilibrium contract. Then

$$\underline{V} > V_*.$$

Proof. Suppose $\pi = 1$. Then equation (19) implies that it must hold

that $\underline{V} = V_*$. Suppose $\pi < 1$. Then to show $\underline{V} > V_*$ is to show

$$\underline{V} > \frac{\pi \underline{V} + (1 - \pi)H(0, 0)}{1 - (1 - \pi)\beta\Delta},$$

or

$$\underline{V} > H(0, 0)/(1 - \beta\Delta) \equiv V_0.$$

To show $\underline{V} > V_0$, I take two steps.

Step 1. I show $\underline{V} \geq V_0$. In fact, V_0 is the minimum expected utility that can be attained by a feasible and incentive compatible contract. This is easy to see. Given whatever compensation scheme $\{c_t(h^t)\}$, because $c_t \geq 0$ for all t , the worker can always guarantee for himself expected utility $H(0, 0)/(1 - \beta\Delta)$ by following the effort plan $\{a_t = 0\}_{t=1}^\infty$.

Step 2. I show $\underline{V} > V_0$ by showing that V_0 is not a RP expected utility, and therefore \underline{V} , being a RP expected utility, must be strictly greater than V_0 .

There is a unique feasible and incentive compatible contract that delivers V_0 to the worker. To show this, notice first that if a feasible and incentive compatible contract delivers expected utility V_0 to the worker, then it must hold that $c_t = 0$ for all t . For otherwise the worker can always choose the action profile $\{a_t = 0, \forall t\}$ to do strictly better than V_0 . Next, given $c_t = 0$ for all t , clearly the only action profile that is incentive compatible is $a_t = 0$ for all t , and it then follows that $V_t = V_0$ for all $t \geq 1$.

So if V_0 is RP, then all newly employed workers will stay at $\underline{V} = V_0$, and the equilibrium is degenerate. Q.E.D.

Because the expected utilities of all the unemployed workers are equal to V_* , Proposition 3 says that if the equilibrium is not degenerate, then all unemployment is involuntary.

I now proceed to show that involuntary termination is indeed an equilibrium phenomenon: it does occur in equilibrium. More specifically, Proposition 4 shows that in the case of two output and two effort levels, the equilibrium contract has $V_i^*(\underline{V}) = V_*$ for at least some i . That is, the newly higher worker will be terminated in at least some state of the world. I start with a definition and then a lemma.

Definition 4 Let $U : \Phi (= \Phi_r \cup \Phi_f) \rightarrow R$ be a value function. A utility pair (V, Z) is said to be generated by U if either there exists $\{a, (c_i, V_i)\}$ that satisfies (5),(6),(7), and

$$Z = \sum_i X_i(a)[\theta_i - c_i + \beta\Delta U(V_i)] + \beta(1 - \Delta) \max_{V' \in \Phi_r} U(V'); \quad (24)$$

or there exists $g \in \mathcal{G}$ that satisfies equation (10) and

$$Z = -C(g) + \max_{V' \in \Phi_r} U(V').$$

In the following, I will let $G(U)$ denote the set of all utility pairs (V, Z) generated by U . I will let $Graph(U)$ denote the graph of value function U .

Lemma 1 Let $U : \Phi \rightarrow R$. If there exists $(V, Z) \in G(U)$ such that $(V, Z) \notin Graph(U)$ and (V, Z) is not Pareto dominated by any $(V', Z') \in Graph(U)$, then U is not RP.

The proof of Lemma 1 is in the appendix. Lemma 1 provides a sufficient condition for the non-RP-ness of a contract. In order to show that a value function is not RP, I need only construct an utility pair (V, Z) that satisfies the condition in Lemma 1.

I now use Lemma 1 to show that in the case of two income and two effort levels, termination of a new worker occurs with a positive probability in equilibrium. The logic of the proof is that if termination does not occur with a positive probability with the newly hired worker, then the contract would not be RP.

Note that Lemma 1 is more than just being useful for the proof a specific result here, but elaborating on the significance of Lemma 1 is beyond my purpose in this paper.

Proposition 4 Assume $\Theta = \{\theta_1, \theta_2\}$. Assume $\mathbf{A} = \{a_L, a_H\}$ with $a_L < a_H$. Assume $a(\underline{V}) = a_H$ with the optimal contract. Then in equilibrium $\Omega_I(\underline{V}) \neq \emptyset$.

Proof. Suppose the optimal contract has $\Omega_I(\underline{V}) = \emptyset$. Let $\sigma = \{g(V), V \in \Phi_f; (a(V), c_i(V), V_i(V)), V \in \Phi_r\}$ denote the optimal contract. Let $U : \Phi_r \cup \Phi_f \rightarrow R$ be the value function that the optimal contract supports. I have for all $V \in \Phi_r$,

$$a(V) \in \mathbf{A}, c_i(V) \in \mathbf{R}_+, V_i(V) \in \Phi_r \cup \Phi_f,$$

$$\begin{aligned} V &= (1 - X_2(a(V)))[v(c_1(V)) + \beta\Delta V_1(V)] \\ &\quad + X_2(a(V))[v(c_2(V)) + \beta\Delta V_2(V)] - \phi(a(V)), \\ &\geq (1 - X_2(a'))[v(c_1(V)) + \beta\Delta V_1(V)] \\ &\quad + X_2(a')[v(c_2(V)) + \beta\Delta V_2(V)] - \phi(a'), \forall a' \in \mathbf{A} \end{aligned}$$

$$\begin{aligned} U(V) &= (1 - X_2(a(V)))[\theta_1 - c_1(V) + \beta\Delta U(V_1(V))] \\ &\quad + X_2(a(V))[\theta_2 - c_2(V) + \beta\Delta U(V_2(V))] + \beta(1 - \Delta)U_r(\underline{V}). \end{aligned}$$

In the following, I derive a contradiction by constructing an expected utility pair (\hat{V}, \hat{Z}) such that $(\hat{V}, \hat{Z}) \in G(U)$ but $(\hat{V}, \hat{Z}) \notin \text{Graph}(U)$ and (\hat{V}, \hat{Z}) is not Pareto dominated by any $(V', Z') \in \text{Graph}(U)$, and hence, by Lemma 1, U is not renegotiation-proof. To construct (\hat{V}, \hat{Z}) , I first define a tuple $\{\hat{a}, (\hat{c}_i, \hat{V}_i)\}$ that satisfies (5) and (6), and I then use (7) and (24) to define \hat{V} and \hat{Z} . This procedure ensures $(\hat{V}, \hat{Z}) \in G(U)$. There are two cases that I discuss separately.

Case (1). Suppose $c_2(\underline{V}) > \underline{c}$, where \underline{c} is the minimum value in the agent's consumption set. In this case, set

$$\hat{c}_1 = c_1(\underline{V}), \hat{c}_2 = c_2(\underline{V}) - \epsilon, \hat{V}_1 = V_*, \hat{V}_2 = V_2(\underline{V}),$$

where ϵ is chosen to be positive but sufficiently small so that $\hat{c}_2 \geq \underline{c}$ and the following holds:

$$\begin{aligned} & [v(\hat{c}_2) + \beta\Delta\hat{V}_2] - [v(\hat{c}_1) + \beta\Delta\hat{V}_1] \\ & \geq [v(c_2(\underline{V})) + \beta\Delta V_2(\underline{V})] - [v(c_1(\underline{V})) + \beta\Delta V_1(\underline{V})] \\ & \geq \frac{\phi(a_H) - \phi(a_L)}{X_2(a_H) - X_2(a_L)}. \end{aligned} \quad (25)$$

The condition above ensures that $\{\hat{a}, (\hat{c}_i, \hat{V}_i)\}$ satisfies the incentive constraint. Here the first inequality follows the construction of \hat{c}_i and \hat{V}_i , the second inequality follows the assumption that $a(\underline{V}) = a_H$ with the optimal contract. The so constructed $\{\hat{a}, (\hat{c}_i, \hat{V}_i)\}$ obviously also satisfies the feasibility constraint. Next, let

$$\hat{V} = (1 - X_2(a_H))[v(\hat{c}_1) + \beta\Delta\hat{V}_1] + X_2(a_H)[v(\hat{c}_2) + \beta\Delta\hat{V}_2]. \quad (26)$$

$$\begin{aligned} \hat{Z} &= (1 - X_2(a_H))[\theta_1 - \hat{c}_1 + \beta\Delta U(V_*)] \\ &+ X_2(a_H)[\theta_2 - \hat{c}_2 + \beta\Delta U(\hat{V}_2)] + \beta(1 - \Delta)U_r(\underline{V}). \end{aligned} \quad (27)$$

The above construction gives me $(\hat{V}, \hat{Z}) \in G(U)$.

Because $\epsilon > 0$ and $\hat{V}_1 = V_* < \underline{V}$, I have

$$\hat{V} < \underline{V}. \quad (28)$$

Meanwhile, because $U(V_*) = U(\underline{V}) \geq U(V_1(\underline{V}))$ and $\hat{c}_2 < c_2(\underline{V})$, I have

$$\hat{Z} > U(\underline{V}). \quad (29)$$

Because $U(\underline{V}) \geq U(V)$ for all V , I therefore have $(\hat{V}, \hat{Z}) \notin \text{Graph}(U)$, and that (\hat{V}, \hat{Z}) is not Pareto dominated by any $(V', Z') \in \text{Graph}(U)$. U is not RP according to Lemma 1. A contradiction.

Case (2). Suppose $c_2(\underline{V}) = \underline{c} \leq c_1(\underline{V})$. There are two sub-cases here: $c_1(\underline{V}) > \underline{c}$ and $c_1(\underline{V}) = \underline{c}$.

(2i) Suppose $c_1(\underline{V}) > \underline{c}$. Then set

$$\hat{c}_1 = c_1(\underline{V}) - \epsilon, \hat{c}_2 = c_2(\underline{V}), \hat{V}_1 = V_*, \hat{V}_2 = V_2(\underline{V}),$$

where ϵ is positive but sufficiently small so that $\hat{c}_1 \geq \underline{c}$ and (??) is satisfied. I then use (26) and (27) to construct \hat{V} and \hat{Z} to reach a contradiction, just as in case (1).

(2ii) Suppose $c_1(\underline{V}) = \underline{c}$. Then because the optimal contract implements $a = a_H$ at $V = \underline{V}$, incentive compatibility implies $V_2(\underline{V}) > V_1(\underline{V}) \geq \underline{V}$. Therefore, I can set

$$\hat{c}_1 = c_1(\underline{V}), \hat{c}_2 = c_2(\underline{V}), \hat{V}_1 = V_*, \hat{V}_2 = V_2(\underline{V}) - \epsilon,$$

where ϵ is chosen to be sufficiently small to make $\hat{V}_2 \geq \underline{V}$ hold and to satisfy equation (??), and so the incentive constraint is satisfied. Here, to satisfy the incentive constraint, it is sufficient to require $\hat{V}_2 - \hat{V}_1 \geq V_2(\underline{V}) - V_1(\underline{V})$, or $\epsilon \leq V_1 - V_* > 0$. That is, I need only make sure $0 \leq \epsilon \leq \min\{V_2(\underline{V}) - \underline{V}, V_1(\underline{V}) - V_*\}$.

Again, use equation (26) to define \hat{V} . Clearly, $\hat{V} < \underline{V}$. Use equation (27) to define \hat{Z} . Since U is a weakly decreasing function, it holds that $\hat{Z} \geq U(\underline{V})$. By construction, $(\hat{V}, \hat{Z}) \in G(U)$. I also have $(\hat{V}, \hat{Z}) \notin \text{Graph}(U)$. This is true because if $(\hat{V}, \hat{Z}) \in \text{Graph}(U)$, then \underline{V} would not be the unique solution to problem (14). Finally, since $\hat{Z} \geq U(\underline{V})$, (\hat{V}, \hat{Z}) is not Pareto dominated by any $(V', Z') \in \text{Graph}(U)$. Therefore by Lemma 1, U is not RP, a contradiction. Q.E.D.

My next proposition gives a sufficient condition for voluntary termination. It states that voluntary termination must occur if the worker's expected utility becomes sufficiently high. Imagine the worker is promised a high expected utility after a sequence of good draws. When this happens, because of decreasing marginal utility of consumption, the worker's effort becomes too expensive to compensate for, and the firm is then better off replacing this worker with a new worker whose expected utility is lower and so his efforts are less expensive.

The logic for this result is rather intuitive and can be shown in a static compensation problem with no information frictions and no uncertainties. Let c_0 denote the worker's existing consumption. Let the worker's initial effort be zero. Then in order for the firm to fully compensate the worker for a fixed amount of effort a , the firm must pay the worker compensation c to satisfy

$$v(c_0 + c) - v(c_0) = \phi(a) - \phi(0).$$

The right hand side of this equation is constant while the left hand side is increasing in c but decreasing in c_0 . So c increases as c_0 increases. Moreover, given that v is concave, c must be convex in c_0 .

Proposition 5 Assume $(v^{-1})'(x) \rightarrow \infty$ as $x \rightarrow v(\infty)$. Then there exists $\bar{V} \in \left(\underline{V}, \frac{v(\infty) - \phi(0)}{1 - \beta\Delta}\right)$ such that the optimal contract has $V \in \Phi_f$ for all $V \geq \bar{V}$.

Proof. To prove the proposition I need only show that it holds that $U_f(V) > U_r(V)$ as V goes to $\frac{v(\infty) - \phi(0)}{1 - \beta\Delta}$. Suppose otherwise. That is, suppose $U_r(V) \geq U_f(V)$ for all $V > \underline{V}$. I derive a contradiction in the following.

I first define a function $U_r^{fb}(V)$, $V > \underline{V}$. Fix V . Since $V > \underline{V}$, I have $U_r(V) \geq U_f(V)$. Now imagine the following scenario: starting from the current period, the current worker's effort becomes observable to the firm until the termination or death of the current worker; moral hazard resumes when a new worker is employed. Calculate the value of the firm in this scenario and denote it $U_r^{fb}(V)$. Now since the worker is retained at V in the case of moral hazard, he is retained at V in the case of no moral hazard. Indeed, this worker should remain employed with his expected utility constant at V until he dies. This implies that $U_r^{fb}(V)$ must satisfy

$$U_r^{fb}(V) = \bar{\theta}(a^*(V)) - c_r^{fb}(V) + \beta\Delta U_r^{fb}(V) + \beta(1 - \Delta)U(\underline{V}),$$

where $a^*(V)$ denotes the first-best level of effort conditional V being the worker's expected utility, $\bar{\theta}(a^*(V))$ denotes the period expected output conditional on $a^*(V)$, and $c_r^{fb}(V)$ denotes the optimal compensation to the worker,

$$c_r^{fb}(V) = v^{-1}[(1 - \beta\Delta)V + \phi(a^*(V))].$$

I therefore have

$$U_r^{fb}(V) = \frac{\bar{\theta}(a^*(V)) - c_r^{fb}(V)}{1 - \beta\Delta} + \frac{\beta(1 - \Delta)U(\underline{V})}{1 - \beta\Delta}.$$

Now, given

$$U_f(V) = -g(V) + U(\underline{V}) = \frac{-v^{-1}[(1 - \beta\Delta)V]}{1 - \beta\Delta} + U(\underline{V}),$$

I therefore have

$$U_f(V) - U_r^{fb}(V) = \frac{K(V) - \bar{\theta}(a^*(V))}{1 - \beta\Delta} + C,$$

where $C \equiv \frac{1 - \beta}{1 - \beta\Delta}U(\underline{V})$ is constant in V and

$$K(V) \equiv v^{-1}[(1 - \beta\Delta)V + \phi(a^*(V))] - v^{-1}[(1 - \beta\Delta)V].$$

Notice that $U_r^{fb}(V) \geq U_r(V)$ for all V . So, if there exists $\tilde{V} < \frac{v(\infty) - \phi(0)}{1 - \beta\Delta}$ such that

$$U_f(V) > U_r^{fb}(V), \forall V \geq \tilde{V}, \quad (30)$$

then there exists $\hat{V} < \frac{v(\infty) - \phi(0)}{1 - \beta\Delta}$ such that $U_f(V) > U_r(V)$ for all $V \geq \hat{V}$, and then I have a contradiction.

Now because C is constant in V and the value of $\bar{\theta}(a^*(V))$ is bounded in V , in order to show that statement (30) holds, it is sufficient to show

$$K(V) \rightarrow \infty \text{ as } V \rightarrow \frac{v(\infty) - \phi(0)}{1 - \beta\Delta}.$$

But

$$K(V) = \phi(a^*(V))(v^{-1})'[(1 - \beta\Delta)V + \xi]$$

where $\xi \in [0, \phi(a^*(V))]$. Since $\phi(a^*(V)) \geq \phi(\underline{a}) > 0$ (remember $\underline{a} = \min\{a \in \mathbf{A}\} > \mathbf{0}$) for all V , to prove the proposition I need only show that $(v^{-1})'(x) \rightarrow \infty$ as $x \rightarrow v(\infty)$, which is assumed to hold. Q.E.D.

Proposition 5 essentially shows that the equilibrium value functions $U_r(V)$ and $U_f(V)$ must cross each other at some $\bar{V} (> \underline{V})$ and $U_f(V) > U_r(V)$ for all $V \geq \bar{V}$. Note though the proposition does not imply that voluntary termination does occur in equilibrium. Put differently, if I follow a new worker to start out with the optimal starting expected utility \underline{V} , Proposition 5 does not guarantee that with a positive probability the worker will eventually cross \bar{V} to become voluntarily terminated. To prove such a result seems to be a challenging task. However, in a number of partial equilibrium numerical examples that I have computed, the optimal contract does have the feature that the worker is terminated after his expected utility has risen monotonically following a sequence of high outputs.

5 Conclusion

I have constructed an equilibrium model of the labor market where contracts are fully dynamic, job turnover is endogenous, workers separated from their current employers are free to go back to the labor market to look for new employment. At the heart of the model is an optimal termination mechanism that governs the timing and type of the separation of workers and firms. In equilibrium, this optimal termination mechanism appears in two different faces, involuntary layoff and voluntary retirement.

The model has a simple setup, with fixed numbers of homogeneous workers and firms, yet the model is capable of generating in equilibrium a large set

of important labor market variables. Obviously, the model is much deeper than I have been able to characterize. This leaves room for future research. To fully understand the model's implications for compensation and termination dynamics for example, a quantitative approach may prove helpful.

Appendix

Proof of Lemma 2. Suppose U is RP. I take the following steps to construct a contradiction.

1. Because U is RP, I have $Graph(U) = Graph(PTU) = Graph(TU)$.

2. Notice that it is without loss of generality to assume that $(V, Z) \in Graph(TU)$. To show this, let $\hat{Z} = \max\{Z : (V, Z) \in G(U)\}$. Then $(V, \hat{Z}) \in Graph(TU)$, $(V, \hat{Z}) \notin Graph(U)$ and (V, \hat{Z}) is not Pareto dominated by any $(V', Z') \in Graph(U)$. $(V, \hat{Z}) \in Graph(TU)$ because if $(V, \hat{Z}) \in Graph(U)$, then (V, Z) is not Pareto dominated by $(V, \hat{Z}) \in Graph(U)$, a contradiction. And, because (V, \hat{Z}) Pareto dominates (V, Z) and the latter is not Pareto dominated by any $(V', Z') \in Graph(U)$, (V, \hat{Z}) is not Pareto dominated by any $(V', Z') \in Graph(U)$.

3. Because $(V, Z) \notin Graph(PTU) = Graph(U)$, (V, Z) must be dominated by some $(\tilde{V}, \tilde{Z}) \in Graph(TU)$. But since (V, Z) is not Pareto dominated by any $(V', Z') \in Graph(U)$, it must be that $(\tilde{V}, \tilde{Z}) \in Graph(TU) - Graph(U) \neq \emptyset$.

4. Let

$$V^* \equiv \sup\{\tilde{V} : (\tilde{V}, \tilde{Z}) \in Graph(TU) - Graph(U), (\tilde{V}, \tilde{Z}) >_p (V, Z)\}$$

5. V^* belongs to the domain of the function TU , i.e., $V^* \in [V_*, \infty)$. This is straightforward to show. By the definition of V_* , there is a sequence $\{V_n, Z_n\} \subseteq Graph(TU)$ such that $V_n \rightarrow V_*$ as $n \rightarrow \infty$. But $V_n \in [V_*, \infty)$ (the domain of TU), which is a closed set, so $V^* \in [V_*, \infty)$.

6. I can then define $Z^* \equiv TU(V^*)$ and it follows that $(V^*, Z^*) \in Graph(TU)$. So either $(V^*, Z^*) \in Graph(TU) - Graph(U)$ or $(V^*, Z^*) \in Graph(U)$.

7. Notice that $(V^*, Z^*) \geq_p (V, Z)$. (That is, $V^* \geq V$, $Z^* \geq Z$.) This holds because for each n , $V_n \geq V$, $Z_n > Z$, and so $V^* \geq V$ and $Z^* \geq Z$.

8. Suppose $(V^*, Z^*) \in Graph(TU) - Graph(U)$. Notice first that

$$(V^*, Z^*) \geq_p (V, Z) >_p (V', Z'), \forall (V', Z') \in Graph(U).$$

That is, (V^*, Z^*) is not dominated by any $(V', Z') \in Graph(U)$. Second, suppose there exists $(V', Z') \in Graph(TU) - Graph(U)$ such that $(V', Z') >_p (V^*, Z^*)$. Then because $(V^*, Z^*) \geq_p (V, Z)$, I have $(V', Z') >_p (V, Z)$. Now

by the definition of V^* , it holds that $V' \leq V_*$. But $(V', Z') >_p (V^*, Z^*)$ implies $V' \geq V_*$. So it must hold that $V' = V_*$. Therefore

$$Z' = TU(V') = TU(V^*) = Z^*.$$

This is a contradiction to $(V', Z') >_p (V^*, Z^*)$.

9. Suppose $(V^*, Z^*) \in Graph(U)$. Then (V, Z) is Pareto dominated by $(V_*, Z^*) \in Graph(U)$. Again a contradiction. Q.E.D.

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