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Journal of Economic Theory 135 (2007) 357–381
JOURNAL OF Economic Theory
www.elsevier.com/locate/jet

Optimal use of correlated information in mechanism design when full surplus extraction may be impossible

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Received 8 January 2001; final version received 11 May 2006
Available online 1 August 2006

Abstract

We study the mechanism design problem when the principal can condition the agent's transfers on the realization of *ex post* signals that are correlated with the agent's types. Crémer and McLean [Econometrica 53(1985) 345–361; 56(1988) 1247–1257], McAfee and Reny [Econometrica 60(2)(1992) 395–421], and Riordan and Sappington [J. Econ. Theory, 45(1988) 189–199] studied situations where the signals are such that full surplus can be extracted from every agent type. We study optimal utilization of the signals when there are fewer signals than types and the Riordan and Sappington conditions do not always hold. For some special cases, we show the level of surplus that can be extracted, and identify the agent types who obtain rent.

JEL classification: D82; D86

Keywords: Correlated signals; Rent extraction

1. Introduction

In this paper, we study the mechanism design problem when the principal (the mechanism designer) can condition the agent's transfers on the realization of *ex post* signals (i.e., signals that are realized after the agent reports his type) that are correlated with the agent's type. Crémer and McLean [1,2] (CM hereafter), McAfee and Reny [6] (MR), and Riordan and Sappington [8] (RS) provided conditions under which the principal can design a mechanism that *fully* extracts the surplus from every agent type. In particular, CM showed that if the matrix of the conditional

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0022-0531/$ - see front matter
doi:10.1016/j.jet.2006.05.007

probabilities of the signals given the agent types has full rank, then full surplus can be extracted through dominant strategy implementation. Under the condition that no row of the matrix is within the convex hull of the other rows, the same can be attained through Bayesian implementation. MR extended CM to a setting of infinite type space and to many other applications involving asymmetric information. Further, these conditions on the conditional signal distributions are necessary and sufficient for full surplus to be extracted for *all possible* payoff functions of the agents. RS, on the other hand, provided necessary and sufficient conditions on the conditional signal distributions and agents' payoff functions for full surplus extraction when the full rank or convex hull conditions are not satisfied.

In this paper, we study the optimal use of the *ex post* signals to condition the agent's transfers when full surplus extraction *may not* be possible. In particular, how should the signals be utilized? What happens to the optimal allocation profile? How much surplus can still be extracted given the signals? And which agent types will obtain rent in the optimal mechanism? Answers to these questions are valuable, since in many applications full surplus extraction may not be possible either because the set of *ex post* signals is not rich enough (as required in CM–MR), or because the agents' payoff and signal distribution functions do not satisfy the conditions identified in RS. For example, for the results in CM–MR to hold generically, the cardinality of the signal space is required to be at least as large as that of the type space. In many situations—government regulation, provision of public goods, insurance contracts, etc.—it is much more likely that the number of available signals is not as large as the number of types. For instance, the regulatory body may not know the abatement costs in reducing industrial pollution of the firms it regulates, but can observe certain characteristics of the firms. While there are potentially infinite levels of abatement costs (or types), the observable factors (or signals) are finite and typically few (e.g., firm size, industry classification, etc.). In automobile insurance, there are many types of " risky" drivers, but the insurers can observe only limited pieces of information, such as the number of car accidents or the age of the driver. In essence, the signal space represents information that the designer can obtain *without incurring any cost in information gathering*. In reality, at least some signal gathering might be costly and when the type space is large, generating an equally large signal space may involve extremely high costs. It is then natural to assume a relatively small signal space in such applications. [1] With fewer signals than types, determining the optimal mechanism is important when the condition on payoff functions identified in RS does not hold and therefore full surplus cannot be extracted.

We consider mechanisms in which the transfer is partitioned into two components: a (type-dependent) fixed payment and a (type-dependent) lottery that is a function of the realized signals. We first show that the mechanism designer can do no better than to choose lotteries such that, under truth-telling, each type's expected payment from the lottery is zero. That is, the lotteries are used solely to help with the incentive constraints, i.e., to discourage the agent from making false reports about his type. More specifically, if a type's vector of conditional signal probabilities, called a signal vector, lies outside the convex hull of the signal vectors of all the other types, Farkas' Lemma implies existence of lotteries that lead to zero expected payment under truth-telling but arbitrarily large penalties under false-reporting. If a type's signal vector lies within the convex

[1] In applications like auctions where bid of one agent can be used as a signal of another, the full rank assumption might seem natural. However, even here, if bidders' types are drawn from asymmetric distributions with finite supports, there is no reason why the cardinalities of the supports of these different distributions must always be the same. Further, as Parreiras [7] shows, even when the types are drawn from symmetric distributions, the full rank condition may still fail when the principal does not know for sure the bidders' beliefs about each other's types.

hull of those of the other types, then a transfer schedule for this type can always be replaced by an equivalent new schedule that maintains participation and incentive compatibility constraints for all types as in the original, but where the expected payment from the new lottery is zero under truth-telling. Therefore, for all types, the search for optimal lotteries can be restricted to those that have zero expected value under truth-telling. Apart from tying our work to the earlier literature, this result is useful in simplifying the search for the optimal lotteries. [2]

We next consider the nature of the optimal mechanism. By concentrating on special cases that keep the analysis tractable, we are able to highlight the qualitative features of the optimal mechanism and to contrast it with those in CM–RS as well as in the standard mechanism design problems without signals. An important insight is that not only does the presence of the signals reduce the information rent of the agent, but it does so by allowing the principal to *redistribute* the information rents of the types. When designing the lottery for a type, say θ_i, what is important is the *aggregate* (a weighted sum of) incentives of the other types to report θ_i. Hence, a type, say θ_k, who in the absence of the signals has *disincentive* to report θ_i, plays an important role in the presence of the signals; the principal can use the disincentive of types like θ_k to reduce the rents of the other types who would gain from reporting θ_i. We also show that which type obtains rent in the optimal mechanism follows a fairly intuitive condition involving the prior beliefs on the types and a measure of "similarity" among the signal vectors of the various types.

While our objective is to study the impact of less "informationally rich" signals, other researchers have investigated the effect of relaxing other aspects of CM–MR–RS. It has long been recognized that risk-neutrality and the lack of limited liability constraints are crucial for the full surplus extraction results. Robert [9] considers an auction problem similar to that in CM and shows how the presence of risk-aversion or an upper bound on the transfers bidders pay may prevent the auctioneer from extracting full surplus. Kosmopolou and Williams [5] consider a similar model of group decision making but with a continuum of types. They show that the first-best allocation cannot be implemented when the agent types are approximately independent, and either the monetary transfers amongst agents or their *ex post* payoffs have to satisfy a limited liability constraint. Demougin and Garvie [3] study optimal regulation with a continuum of firm types, correlated information and non-negative limited liability constraints. Gary-Bobo and Spiegel [4] [3] study an optimal regulation problem with a continuous type space and finite signals, and assume the same conditions on costs and signals found in RS, but with the restriction that *ex post* payoffs of the agent are not allowed to fall below a certain level in every state. This level is varied to show the impact of the limited liability constraint; in particular when this level is sufficiently high, one obtains the full surplus extraction result of RS.

The rest of the paper is organized as follows. We set up the model in Section 2 and explore a simple example to preview our main results and intuition in Section 3. Section 4 shows that the optimal design of lotteries involves, without any loss of generality, zero expected lottery payments (ELPs) under truth-telling. Next, we study the optimal mechanism by focusing on some special cases. In Section 5, the number of signals is one less than the number of types, while Section 6 extends this to the case when the number of types is two more than the number of signals. Section 7 concludes. Appendix A shows the relation between our results and those in RS. Appendix B contains the proofs.

[2] Neither CM nor MR partitions the transfer into a fixed payment and a lottery in the formal description of their model. However, see page 1253 of CM [2] where this partition is used in the proof of Theorem 2. See also the introduction in MR (pp. 397, 398) for a discussion of the usefullness of this partition.

[3] We thank an Associate Editor for drawing our attention to this paper.

2. Model setup

Consider a mechanism design problem with a principal and an agent where the principal can condition the agent's transfer on a set of *ex post* verifiable signals that are correlated with the agent's type. [4] Because of the revelation principle, we focus on the direct revelation mechanism without any loss of generality. The timing of the moves is as follows. (i) The principal announces a mechanism which consists of a set of schedules, one for each type that specifies the required allocation and a signal-contingent transfer (or payment) from the agent to the principal; (ii) the agent reports a type (which is equivalent to selecting a schedule); (iii) the allocation is undertaken; (iv) a signal is observed; and finally (v) the transfer is made.

The principal has preferences given by $W(x) + T$, where $x \geqslant 0$ is the allocation and T is the transfer. The agent also has quasilinear preferences given by $u(x, \theta) - T$, where θ, the agent's type, is his private information. The types are drawn from a finite type space, $\theta \in \Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$ with $N \geqslant 2$. We use μ_i to denote the commonly known prior that the agent is of type θ_i. To rule out redundancy, we assume $\mu_i > 0$ for all i. The reservation utility of all types is the same and is normalized to zero. We make the following assumptions on $u(\cdot, \cdot)$ and $W(\cdot)$.

Assumption 1. (a) $W(\cdot)$ and $u(\cdot, \theta)$ have derivatives of all orders; (b) $\frac{\partial u(x,\theta)}{\partial x} \geqslant 0$, $\frac{\partial^2 u(x,\theta)}{\partial x^2} < 0$; and (c) $\frac{dW(x)}{dx} < 0$, $\frac{d^2 W(x)}{dx^2} \leqslant 0$, and (d) $u(0, \theta) = 0$ for all θ.

Parts (a), (b) and (c) are standard, and (d) allows all agent types to participate in the mechanism without loss of generality.

The principal can costlessly observe a verifiable signal which is a random variable correlated with the agent's type. The finite signal space is $\{\sigma_1, \sigma_2, \ldots, \sigma_S\}$ with $S \geqslant 2$. Let q_{ik} be the (conditional) probability of observing signal σ_k when the agent's type is θ_i. Given the presence of the signals, the principal can make the transfer to be conditional on the signal realizations. For convenience, we partition transfer T into two components: for a report of type θ_j, the transfer consists of a non-random payment t_j, and a lottery with payment of y_{js} when signal σ_s is realized. Thus, if type θ_i reports θ_j, his total expected payment would be $t_j + \sum_{s=1}^{S} q_{is} y_{js}$.

To facilitate discussion, we use the following notations:

Notation 1. (a) *Let* $\boldsymbol{q}_i = (q_{i1}, q_{i2}, \ldots, q_{iS})$, *called* θ_i*'s* signal vector, *be the row vector of type* θ_i*'s conditional signal probabilities. Let the* $N \times S$ *matrix* $\boldsymbol{Q} = (\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_N)'$, *called the* signal matrix, *be the matrix of the conditional signal probabilities of all types.*

(b) *Let* $\boldsymbol{y}_i = (y_{i1}, \ldots, y_{iS})'$ *be the* payment lottery *when the agent reports type* θ_i.

(c) *Let* $f_j(x_i) \equiv u(x_i, \theta_j) - u(x_i, \theta_i)$ *denote the "intrinsic rent" (alternatively called "intrinsic incentive") of type* θ_j *reporting (or pretending to be) type* θ_i, *measuring the extra utility to* θ_j *relative to* θ_i *if both are given* θ_i*'s allocation* x_i.

(d) *Suppose* $\{r_i\}$ *is an arbitrary collection of vectors from some finite dimensional Euclidean space. Let* $co\{r_i\}$ *denote the* convex hull *of the vectors* $\{r_i\}$, *i.e., it is the smallest convex set that contains all the vectors* $\{r_i\}$.

(e) *Let* $\mathcal{N} = \{1, \ldots, N\}$, $\mathcal{N}_1 = \{i \in \mathcal{N} : \boldsymbol{q}_i \notin co\{\boldsymbol{q}_j, j \in \mathcal{N},\ j \neq i\}\}$ *and* $\mathcal{N}_2 = \mathcal{N} \backslash \mathcal{N}_1$. *Thus,* $\mathcal{N}_1$ *is the collection of the indices of types whose signal vectors are not in the convex hull of those of the other types. If* $i \in \mathcal{N}_k$, $k = 1, 2$, *we say* θ_i corresponds to $\mathcal{N}_k$, *or simply* θ_i is in $\mathcal{N}_k$.

[4] In the concluding section we briefly discuss how the present analysis can be extended to a multiple agent scenario.

(f) *For any arbitrary subset of* $\mathcal{N}$, $\mathcal{K} \subset \mathcal{N}$, *let the matrix* $\boldsymbol{Q}_{\mathcal{K}} = (\boldsymbol{q}_i, i \in \mathcal{K})'$ *be the matrix of the signal vectors of types corresponding to the index set* $\mathcal{K}$.

We follow the convention that vectors and matrices are denoted by bold letters (as in $\boldsymbol{q}$ and $\boldsymbol{Q}$), and sets and collections are represented by calligraphic capital letters (as in $\mathcal{N}$). We make the following assumptions related to the signals:

Assumption 2. (a) $S < N$; (b) $\mathcal{N}_2$ is non-empty; (c) The rank of $\boldsymbol{Q}$ is S. Further, $\boldsymbol{q}_i \neq \boldsymbol{q}_j$ for $i \neq j, i, j, \in \mathcal{N}$.

That is, there are more types than signals, and there is at least one type whose signal vector lies in the convex hull of those of the other types. Assumption 2(c) implies that $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$ contains at least S elements and that a basis for $\mathbb{R}^S$ can always be chosen from the elements of $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$.

Remark 1. Since $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$ forms the extreme points of the set co$\{\boldsymbol{q}_j, j \in \mathcal{N}\}$, every $\boldsymbol{q}_j$, $j \in \mathcal{N}_2$, lies in the convex hull of the signal vectors corresponding to $\mathcal{N}_1$ only. That is $\forall j \in \mathcal{N}_2$, $\boldsymbol{q}_j \in co\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$. (See Corollary 18.5.1 of Rockafellar [11].)

The principal's problem: Given the setup, the principal's problem can be described as:

$$\begin{aligned} &\max_{t_i, x_i, \boldsymbol{y}_i, i \in \mathcal{N}} \sum_{i \in \mathcal{N}} \mu_i [W(x_i) + t_i + \boldsymbol{q}_i \boldsymbol{y}_i] \\ &\text{s.t.} \quad (\text{PC-}i) \quad u(x_i, \theta_i) - t_i - \boldsymbol{q}_i \boldsymbol{y}_i \geqslant 0 \quad \forall i \\ &\qquad (\text{ICC-}i) \quad u(x_j, \theta_j) - t_j - \boldsymbol{q}_j \boldsymbol{y}_j \geqslant u(x_i, \theta_j) - t_i - \boldsymbol{q}_j \boldsymbol{y}_i \quad \forall j, \ \forall i, \end{aligned} \tag{P}$$

where (PC-*i*) is the participation constraint of type θ_i, and (ICC-*i*) are the incentive compatibility constraints of reporting θ_i. Note that (ICC-*i*) refers to the constraints that no other type should report θ_i.

Definition 1. (i) Given an allocation profile $\{x_i, i \in \mathcal{N}\}$, we say the principal extracts *full rent* if (PC-*i*) is binding for all types.

(ii) The *full information allocation profile* $\{x_i^{FI}, i \in \mathcal{N}\}$ is the allocation profile when (P) is solved in the absence of the incentive compatibility constraints (ICC-*i*) for all *i*.

(iii) We say the principal extracts *full surplus* if in (P), full rent is extracted for the allocation $\{x_i^{FI}, i \in \mathcal{N}\}$.

(iv) We say a type θ_j *has incentive to report* θ_i if $u(x_i, \theta_j) - t_i - \boldsymbol{q}_j \boldsymbol{y}_i > 0$.

The literature on mechanism design has focused mainly on two polar cases: the *independent case*, when there are *no ex post* signals, and the case of CM–MR, where for all $i \in \mathcal{N}$, $\boldsymbol{q}_i \notin co\{\boldsymbol{q}_j, j \in \mathcal{N}, j \neq i\}$ (i.e., $\mathcal{N}_1 = \mathcal{N}$). RS, like us, falls between the two polar cases. However, as mentioned before, their objective is to study joint restrictions on payoffs and signals that allow full surplus extraction, whereas ours is to study situations when full surplus extraction is not possible.

3. An example

Before presenting the formal analysis, we first analyze a simple example to highlight our main results and to explain the intuition. Consider the example of a single buyer and a seller, with the seller being the principal and the buyer being the agent. Let $x \geqslant 0$ be the units of the good

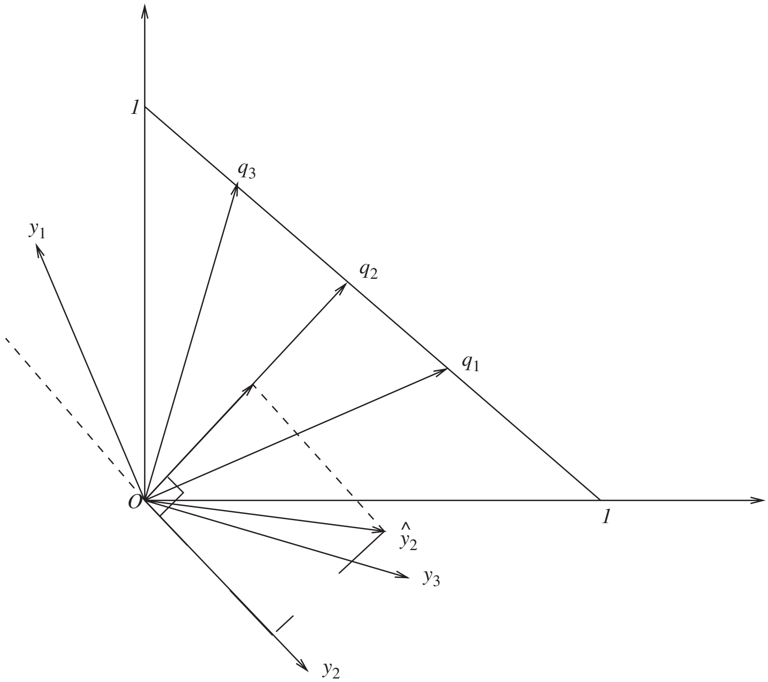

Fig. 1. The use of perpendicular lotteries.

purchased, and $u(x, \theta)$ the buyer's utility from consumption. Let there be three types, $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Suppose that the buyer and the seller can costlessly observe a verifiable binary signal (for example, whether the state of demand in some other market is high or low) that is correlated with the buyer's type and can be used to condition the buyer's payment to the seller. Throughout, we consider the direct revelation mechanism.

Fig. 1 illustrates one possible layout of the signal vectors $\boldsymbol{q}_1$, $\boldsymbol{q}_2$ and $\boldsymbol{q}_3$. In this case, $\boldsymbol{q}_2 \in co\{\boldsymbol{q}_1, \boldsymbol{q}_3\}$, i.e., $\boldsymbol{q}_2 = \lambda_1 \boldsymbol{q}_1 + \lambda_3 \boldsymbol{q}_3$, with $\lambda_1 > 0$, $\lambda_3 > 0$ and $\lambda_1 + \lambda_3 = 1$. Thus, the CM convex hull conditions fail, and it is not possible to guarantee full surplus extraction for all payoff functions $u(\cdot)$.

We start by exploring the nature of the lotteries for the three types. Consider first the types corresponding to $\mathcal{N}_1 = \{1, 3\}$ only. Since neither $\boldsymbol{q}_1$ nor $\boldsymbol{q}_3$ belongs to the convex hulls of other signal vectors, Farkas' Lemma implies existence of lotteries $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$ such that $\boldsymbol{q}_i \boldsymbol{y}_i = 0$ and $\boldsymbol{q}_j \boldsymbol{y}_i > 0$ for $j \in \{1, 2, 3\}$, $i \in \{1, 3\}$ and $j \neq i$.[5] Such lotteries $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$ are shown in Fig. 1. Similar to CM, by making $\|\boldsymbol{y}_1\|$ and $\|\boldsymbol{y}_3\|$ sufficiently large, we ensure that *no* type has incentive to falsely report θ_1 or θ_3. Thus, in problem (P), the incentive compatibility constraints of reporting

[5] This application of Farkas' Lemma is somewhat different from that in CM where $\mathcal{N}_2$ is empty, i.e., there are only types θ_1 and θ_3, and the Lemma is used to show that $\boldsymbol{q}_i \boldsymbol{y}_i = 0$ and $\boldsymbol{q}_j \boldsymbol{y}_i > 0$ for $j \in \{1, 3\}$ only. However, having $\boldsymbol{q}_2$ does not affect applying the Lemma to find $\boldsymbol{y}_1$ and $\boldsymbol{y}_3$; as long as $\boldsymbol{q}_i$ does not belong to the cone generated by the other vectors $\boldsymbol{q}_j$, $j \neq i$, regardless of how many other vectors there are, the Lemma guarantees existence of a hyperplane $\boldsymbol{y}_i$ that separates $\boldsymbol{q}_i$ from the cone.

θ_1 and θ_3, (ICC-1) and (ICC-3), are made slack and can be ignored in searching for the optimal mechanism.

The remaining, and more interesting, question concerns the lottery for θ_2. It is still possible to have a lottery $\boldsymbol{y}_2$ such that $\boldsymbol{q}_2\boldsymbol{y}_2 = 0$; however, it is not possible to have both $\boldsymbol{q}_1\boldsymbol{y}_2$ and $\boldsymbol{q}_3\boldsymbol{y}_2$ to be positive, and one might wonder whether we will need to search for other kinds of lotteries to utilize the signals optimally. We show (in Proposition 1) no need to do so: without any loss of generality, we can consider only lotteries that have zero expected value under truth-telling, i.e., consider only $\boldsymbol{y}_2$ such that $\boldsymbol{q}_2\boldsymbol{y}_2 = 0$.

To see this, consider a mechanism $\{\hat{t}_i, x_i, \hat{\boldsymbol{y}}_i, i = 1, 2, 3\}$ satisfying all the participation and incentive compatibility constraints in (P), and where the lottery $\hat{\boldsymbol{y}}_2$ is such that the expected lottery payment (ELP) $\boldsymbol{q}_2\hat{\boldsymbol{y}}_2 \neq 0$ (as shown in Fig. 1). Below we illustrate an alternative mechanism that gives the same expected payoff to the principal and all agent types as in the original one, while also satisfying the participation and incentive compatibility constraints for all types in (P); however, the lottery in the new mechanism is perpendicular to vector $\boldsymbol{q}_2$.

The new mechanism preserves the schedules for θ_1 and θ_3 in the original mechanism, $\{\hat{t}_1, x_1, \hat{\boldsymbol{y}}_1\}$ and $\{\hat{t}_3, x_3, \hat{\boldsymbol{y}}_3\}$, and the same allocation for θ_2, x_2. The *only* change made is in the transfer part of the schedule for θ_2, which is changed to $\{t_2, \boldsymbol{y}_2\}$ with $t_2 = \hat{t}_2 + \boldsymbol{q}_2\hat{\boldsymbol{y}}_2$ and $\boldsymbol{y}_2 = \hat{\boldsymbol{y}}_2 - \boldsymbol{q}_2\hat{\boldsymbol{y}}_2\mathbf{1}$, where $\mathbf{1} = (1, 1)'$ is the unit vector in $\mathbb{R}^2$. Since $\boldsymbol{q}_2\mathbf{1} = 1$, it follows that $\boldsymbol{q}_2\boldsymbol{y}_2 = 0$.

In the original mechanism, the total transfer (i.e., combining the fixed and the lottery components) of any type reporting θ_2, contingent on the signals is represented by vector $\hat{t}_2\mathbf{1} + \hat{\boldsymbol{y}}_2$. This vector can be rewritten as $(\hat{t}_2 + \boldsymbol{q}_2\hat{\boldsymbol{y}}_2)\mathbf{1} + (\hat{\boldsymbol{y}}_2 - \boldsymbol{q}_2\hat{\boldsymbol{y}}_2\mathbf{1}) = t_2\mathbf{1} + \boldsymbol{y}_2$, the total contingent transfer of reporting θ_2 in the new mechanism. Therefore, for *every* type, including θ_2, its total contingent transfer upon reporting θ_2 is preserved in the new mechanism, and hence its expected total transfer upon reporting θ_2 remains unaltered as well. Since the schedules of types θ_1 and θ_3 as well as the allocation for all types have not been changed, if the original mechanism satisfies all the participation and incentive compatibility constraints, so must the new mechanism. Also, since every agent type receives the same expected payoff as before, the principal's expected payoff must also remain unchanged.

Essentially, in constructing the new schedule for θ_2, we remove a certain amount from the lottery part of the transfer (i.e., subtract an equal amount for every signal realization) and add the same amount to the non-random part. By setting this amount to be θ_2's ELP under the original schedule, the resulting lottery $\boldsymbol{y}_2$ satisfies $\boldsymbol{q}_2\boldsymbol{y}_2 = 0$. Furthermore, doing this does not change the total payment of any type reporting θ_2, so that the participation and incentive compatibility constraints remain satisfied as well.

We now turn to the characterization of the optimal mechanism. In particular, when full surplus cannot be extracted, we describe the condition that identifies the type(s) who obtains rent. We also note interesting differences and similarities between the optimal mechanism here and the one that is obtained when there are no correlated signals.

The crucial element in the subsequent analysis is not the lottery $\boldsymbol{y}_2$, but the amounts types θ_1 or θ_3 expect to pay upon reporting θ_2. Rather than working with vector $\boldsymbol{y}_2$, we find it more convenient to work with the ELP z_i where $z_i = \boldsymbol{q}_i\boldsymbol{y}_2$, $i = 1, 3$. Note that since $(\boldsymbol{q}_1 \;\; \boldsymbol{q}_3)'\boldsymbol{y}_2 = (z_1 \;\; z_3)'$ and $\{\boldsymbol{q}_1, \boldsymbol{q}_3\}$ forms a basis for $\mathbb{R}^2$, each $\boldsymbol{y}_2$ uniquely defines $\{z_1, z_3\}$ and vice versa. Since $\boldsymbol{q}_2 = \lambda_1\boldsymbol{q}_1 + \lambda_3\boldsymbol{q}_3$, the condition $\boldsymbol{q}_2\boldsymbol{y}_2 = 0$ implies the (only) restriction on z_1 and z_3: $\lambda_1 z_1 + \lambda_3 z_3 = 0$.

Given any allocation, since all types other than θ_2 lie in $\mathcal{N}_1$, type θ_2 cannot obtain any rent and t_2 should optimally be set equal to $u(x_2, \theta_2)$. In fact, since no type can obtain rent by reporting types in $\mathcal{N}_1$, the only source of rent comes from some type(s) having incentive to report θ_2. Therefore in (P), we only need to consider (PC-1), (PC-3), and (ICC-2):

$$
\begin{aligned}
&u(x_i, \theta_i) - t_i \geqslant 0; \quad i = 1, 3, \\
&u(x_i, \theta_i) - t_i \geqslant u(x_2, \theta_i) - t_2 - z_i; \quad i = 1, 3.
\end{aligned} \tag{1}
$$

Substituting in $t_2 = u(x_2, \theta_2)$, and recalling that $f_i(x_2) = u(x_2, \theta_i) - u(x_2, \theta_2)$, we can combine the two constraints in (1) as $t_i \leqslant u(x_i, \theta_i) - \max\{0, f_i(x_2) - z_i\}, i = 1, 3$. Since the principal prefers higher payments from the agent, we can write the inequality as an equality:

$$t_i = u(x_i, \theta_i) - \max\{0, f_i(x_2) - z_i\}; \quad i = 1, 3. \tag{2}$$

From this expression and (1), we see that θ_i obtains rent if $z_i < f_i(x_2)$, and its incentive compatibility constraint of reporting θ_2 is slack when $z_i > f_i(x_2)$. This representation also allows us to write the expected transfer the principal receives from the agent, $\sum_{i=1,2,3} \mu_i t_i$, as

$$\sum_{i=1,3} \mu_i \left[u(x_i, \theta_i) - \max\{0, f_i(x_2) - z_i\}\right] + \mu_2 u(x_2, \theta_2). \tag{3}$$

From (3), the principal has incentive to raise the z_i's in order to reduce the rent, if any, to the agent types. But since the z_i's have to satisfy $\lambda_1 z_1 + \lambda_3 z_3 = 0$, when one of the z_i's is increased, the other has to be reduced accordingly. While a higher z_i reduces the rent to θ_i if $f_i(x_2) - z_i > 0$, a lower z_j does not affect the zero rent to θ_j if $f_j(x_2) - z_j < 0$. Thus, the principal can choose the z_i's based on, among other factors, the relative signs and magnitudes of $f_1(x_2)$ and $f_3(x_2)$. This observation leads to the crucial insight—one on which the remainder of the analysis rests—that the presence of the correlated signals allows the principal to redistribute rents amongst the types. As in the case without the signals, different values of x_2 result in different amount of "intrinsic" rents, $f_i(x_2)$. However, now the principal can choose z_1 and z_3 based on the values of $f_1(x_2)$ and $f_3(x_2)$ to determine the actual amount of rent each type may obtain in the optimal mechanism.

We now show the first feature of the optimal solution that departs from that when there are no signals. Suppose that in the optimal solution there is a type θ_1 or θ_3 who obtains rent. Then, the incentive compatibility constraints of *both* θ_1 and θ_3 reporting θ_2 must bind. (Without signals, in general, only the incentive compatibility constraint of the type that obtains rent binds.) To see why, suppose θ_3 obtains rent, i.e., $f_3(x_2) - z_3 > 0$, but the incentive compatibility constraint of θ_1 reporting θ_2 is slack, i.e., $f_1(x_2) - z_1 < 0$. This cannot be optimal since by reducing z_1 slightly, z_3 can be increased while still satisfying $\lambda_1 z_1 + \lambda_3 z_3 = 0$, allowing the principal to reduce the rent of θ_3 while θ_1 continues to receive no rent.

The above argument helps identify the type, if any, who obtains rent. Suppose full rent cannot be extracted, so both incentive compatibility constraints (of θ_1 and θ_3) in (1) bind. Starting from a situation where both θ_1 and θ_3 obtain rent, if z_1 is increased, the principal's payoff in (3) is raised at a rate of μ_1. However, z_3 has to be decreased by λ_1/λ_3 to satisfy $\lambda_1 z_1 + \lambda_3 z_3 = 0$, reducing (3) by $\mu_3\lambda_1/\lambda_3$. Raising z_1 benefits the principal if and only if $\mu_1 > \mu_3\lambda_1/\lambda_3$, or $\mu_1/\lambda_1 > \mu_3/\lambda_3$. Furthermore, as long as $\mu_1/\lambda_1 > \mu_3/\lambda_3$, this adjustment of z_1 and z_3 should continue until all rent accruing to θ_1 is eliminated. Thus, when full rent cannot be extracted, which type obtains rent depends on the relative size of μ_1/λ_1 and μ_3/λ_3: θ_3 (or θ_1) obtains rent if $\mu_1/\lambda_1 >$ (or $<$)μ_3/λ_3.

In a sense, $\frac{\mu}{\lambda}$ reflects the "effective cost" to the seller of giving rent. Other things equal, giving one unit of rent to θ_3 rather than to θ_1 has lower expected cost if $\mu_3 \leqslant \mu_1$. On the other hand, λ_i

relates to the extent of similarity between the signals generated by θ_i and θ_2. Hence, the higher λ_i is, the more similar θ_i is to θ_2 in terms of their signals, and therefore the more difficult it is to separate out θ_i from θ_2 through the signals, or to punish θ_i for reporting θ_2.

It is easy to check whether full surplus can be extracted. Given the full information allocation, x_i^{FI}, $i = 1, 2, 3$, if full surplus can be extracted then $t_i = u(x_i^{FI}, \theta_i)$ for all i. But from (2), $t_1 = u(x_1^{FI}, \theta_1)$ and $t_3 = u(x_3^{FI}, \theta_3)$ implies $f_1(x_2^{FI}) \leqslant z_1$ and $f_3(x_2^{FI}) \leqslant z_3$. Since $\lambda_1 z_1 + \lambda_3 z_3 = 0$, $\lambda_1 \geqslant 0$ and $\lambda_3 \geqslant 0$, these inequalities imply

$$\lambda_1 f_1(x_2^{FI}) + \lambda_3 f_3(x_2^{FI}) \leqslant 0. \tag{4}$$

On the other hand, if (4) is satisfied, z_1 and z_3 can be found that satisfy $\lambda_1 z_1 + \lambda_3 z_3 = 0$ and $f_i(x_2^{FI}) \leqslant z_i$, $i = 1, 3$, guaranteeing full surplus extraction. Thus, (4) is the necessary and sufficient condition for full surplus extraction; it is also equivalent to the condition identified in RS. (See Appendix A for more detailed discussion.)

Finally, note that in this example $\mathcal{N}_2 = \{2\}$. Suppose that the types are ordered such that $u(x, \theta_i)$ is increasing in θ_i. If $\mathcal{N}_2 = \{1\}$, then since $f_3(x_1) > 0$ and $f_2(x_1) > 0$, (4) is always violated and full surplus can never be extracted. On the other hand, if $\mathcal{N}_2 = \{3\}$, then since $f_1(x_3) < 0$ and $f_2(x_3) < 0$, full surplus can always be extracted.

4. Utilization of signals

To relate our results to the literature, we first briefly discuss the intuition of the results in CM. In CM, there is *no* type θ_i such that $\boldsymbol{q}_i \in co\{\boldsymbol{q}_j, j \in \mathcal{N}, j \neq i\}$. Farkas' Lemma then guarantees existence of payment lotteries $\boldsymbol{y}_i$, for all $i \in \mathcal{N}$, such that $\boldsymbol{q}_i\boldsymbol{y}_i = 0$ and $\boldsymbol{q}_j\boldsymbol{y}_i > 0$ for $j \neq i$. That is, the principal can construct a lottery for each type θ_i, such that the expected cost of the lottery is zero for θ_i but positive for other types reporting to be θ_i. Then by setting the length of $\boldsymbol{y}_i$, $\|\boldsymbol{y}_i\|$, to be sufficiently large, the expected cost of any other type reporting θ_i is made prohibitively high. All the incentive compatibility constraints can thus be satisfied as non-binding constraints. Consequently, the optimal allocations are at the full-information levels, and the optimal non-random transfer is $t_i = u(x_i^{FI}, \theta_i)$ so that the principal achieves full surplus extraction.

For our purposes, it is useful to observe that an important property of the lotteries in CM is that under truth-telling, the expected payment from the lotteries is zero. Put differently, the lotteries are not used to transfer payment from the agent to the principal; rather, their sole purpose is to help satisfy the incentive compatibility constraints, i.e., to discourage types from making false reports. The main result of this section shows that this feature of the lotteries carries over to situations beyond the world of CM. Even when the set of signals is not rich enough to enable full surplus extraction, we can still utilize the signals optimally by considering only lotteries that have zero expected value under truth-telling and whose entire purpose, therefore, is to help with the incentive compatibility constraints.

Since the lotteries for types θ_i, $i \in \mathcal{N}_1$, and θ_j, $j \in \mathcal{N}_2$, are determined differently, it is convenient to define them formally:

Definition 2. A *perpendicular lottery* (PL) for type θ_i is a lottery $\boldsymbol{y}_i \in \mathbb{R}^S$ such that $\boldsymbol{q}_i\boldsymbol{y}_i = 0$. A *separating lottery* (SL) for type θ_i is a PL $\boldsymbol{y}_i$ such that $\boldsymbol{q}_j\boldsymbol{y}_i > 0$ for all $j \in \mathcal{N}\backslash i$.

For every type θ_i corresponding to $\mathcal{N}_1$, it is possible to have lotteries that are SLs. In other words, for these types we can have *exactly* the lotteries of CM.

Lemma 1. *For all $i \in \mathcal{N}_1$, a SL $\boldsymbol{y}_i \in \mathbb{R}^S$ exists for θ_i.*

The proofs are in Appendix B. This lemma implies the following important corollary.

Corollary 1. *For all $i \in \mathcal{N}_1$, by using a SL $\boldsymbol{y}_i$ and setting $\|\boldsymbol{y}_i\|$ to be sufficiently large, the incentive compatibility constraints of reporting θ_i, (ICC-i), in (P) can be satisfied as strict inequalities, i.e., $u(x_j, \theta_j) - t_j - \boldsymbol{q}_j\boldsymbol{y}_j > u(x_i, \theta_j) - t_i - \boldsymbol{q}_j\boldsymbol{y}_i$, $j \in \mathcal{N}$, $j \neq i$, $i \in \mathcal{N}_1$.*

Thus, whenever a SL exists for a type θ_i, the principal can make all types θ_j, $j \neq i$, *strictly* prefer not to report θ_i. When solving the problem (P), we can therefore ignore all the incentive compatibility constraints (ICC-*i*) for $i \in \mathcal{N}_1$.

For types θ_i, $i \in \mathcal{N}_2$, there do not exist SLs. However, the following lemma shows that θ_i's schedule can always be chosen such that the lottery for θ_i is a PL.

Lemma 2. *For any $i \in \mathcal{N}_2$, consider two schedules $(\hat{t}_i, x_i, \hat{\boldsymbol{y}}_i)$, and $(t_i, x_i, \boldsymbol{y}_i)$ with $\boldsymbol{q}_i\hat{\boldsymbol{y}}_i \neq 0$, $t_i = \hat{t}_i + \boldsymbol{q}_i\hat{\boldsymbol{y}}_i$ and $\boldsymbol{y}_i = \hat{\boldsymbol{y}}_i - \boldsymbol{q}_i\hat{\boldsymbol{y}}_i\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)'$ is the $S \times 1$ unit vector in $\mathbb{R}^S$. Then, the expected total transfers of every type reporting θ_i are the same under the two schedules. Further, $\boldsymbol{y}_i$ is a PL: $\boldsymbol{q}_i\boldsymbol{y}_i = 0$.*

Essentially, starting from the schedule $(\hat{t}_i, x_i, \hat{\boldsymbol{y}}_i)$, the schedule $(t_i, x_i, \boldsymbol{y}_i)$ for θ_i is constructed such that the ELP of *every* type reporting θ_i is reduced by an equal amount of $\boldsymbol{q}_i\hat{\boldsymbol{y}}_i$, while the non-random payment is increased by $\boldsymbol{q}_i\hat{\boldsymbol{y}}_i$. Thus, the expected *total* payment from reporting θ_i, truthfully or falsely, is the same under the original and the new schedule, ensuring that the two schedules are equivalent. Further, since the ELP of θ_i under truth-telling in the original lottery is $\boldsymbol{q}_i\hat{\boldsymbol{y}}_i$, the ELP becomes zero in the new lottery, i.e., $\boldsymbol{q}_i\boldsymbol{y}_i = 0$, or $\boldsymbol{y}_i$ is a PL.

Corollary 1 and Lemma 2 imply that the principal can do no better in utilizing the signals than to choose the lotteries to be PLs, and whenever possible, SLs. In summary,

Proposition 1. *Consider a mechanism $\{\hat{t}_i, x_i, \hat{\boldsymbol{y}}_i, i \in \mathcal{N}\}$ that satisfies all the participation and incentive compatibility constraints in (P). Suppose $\boldsymbol{q}_j\hat{\boldsymbol{y}}_j \neq 0$ for some $j \in \mathcal{N}$. Then there exists another mechanism $\{t_i, x_i, \boldsymbol{y}_i, i \in \mathcal{N}\}$, also satisfying all the participation and incentive compatibility constraints in (P), with $t_i = \hat{t}_i + \boldsymbol{q}_i\hat{\boldsymbol{y}}_i$, and $\boldsymbol{q}_i\boldsymbol{y}_i = 0, \forall i \in \mathcal{N}$, that implements the same allocation profile $\{x_i, i \in \mathcal{N}\}$, and in which the principal's expected payoff is at least as high as in the original mechanism.*

From Corollary 1, Lemma 2, and Proposition 1, we can simplify problem (P) to the following one, denoted as (P1).

$$
\begin{aligned}
&\max_{\{t_i, x_i\}_{i \in \mathcal{N}}, \{\boldsymbol{y}_i\}_{i \in \mathcal{N}_2}} \sum_{i \in \mathcal{N}} \mu_i [W(x_i) + t_i] \\
&\text{s.t.} \quad (\text{PC-}i) \quad u(x_i, \theta_i) - t_i \geqslant 0 \quad \forall i \in \mathcal{N}, \\
&\quad (\text{ICC-}i) \quad u(x_j, \theta_j) - t_j \geqslant u(x_i, \theta_j) - t_i - \boldsymbol{q}_j\boldsymbol{y}_i, \quad j \in \mathcal{N} \ \forall i \in \mathcal{N}_2, \\
&\quad\quad \boldsymbol{q}_i\boldsymbol{y}_i = 0, \quad i \in \mathcal{N}_2.
\end{aligned}
\tag{P1}
$$

Notice that unlike (P), the objective function, the (PC-*i*) constraint, and the left-hand sides of (ICC-*i*) in (P1) include only the non-random payment t_i but not $\boldsymbol{q}_i\boldsymbol{y}_i$. Further, the (ICC-*i*) constraints are for types corresponding to $\mathcal{N}_2$ only. Implicit in (P1), SLs with sufficiently large lengths are used for types corresponding to $\mathcal{N}_1$.

5. Optimal contract when $N = S + 1$

Despite the simplification of (P1) over (P), the analysis is still complicated. We concentrate on a set of special cases to illustrate the qualitative features of the optimal mechanism. We first consider the situation when the number of types is one more than the number of signals ($N = S + 1$); this represents the simplest possible departure from the full rank condition. The analytical tractability of this case allows us to characterize the solution completely while making it possible to illustrate the essential differences (and similarities) between the mechanism design problem here and that in the independent case. In the following section, we consider the case when $N = S+2$ to discuss some of the additional complications that will arise, especially when there are multiple types in $\mathcal{N}_2$.

When $N = S + 1$, Assumptions 2(b) and (c) imply that $\mathcal{N}_2$ is a singleton and $\mathcal{N}_1$ has exactly S elements. Let the single element in $\mathcal{N}_2$ be denoted by $\tilde{n}$. Since $\boldsymbol{q}_{\tilde{n}} \in co\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$, there exist S scalars $\lambda_i \geqslant 0, i \in \mathcal{N}_1$, with $\sum_{i\in\mathcal{N}_1} \lambda_i = 1$, such that $\boldsymbol{q}_{\tilde{n}} = \sum_{i\in\mathcal{N}_1} \lambda_i \boldsymbol{q}_i$. Since the rank of matrix $\boldsymbol{Q}$ is S, and $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$ are the extremal vectors of $\{\boldsymbol{q}_i, i \in \mathcal{N}\}$, the set $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$ forms a basis for $\mathbb{R}^S$. Thus, $\lambda_i, i \in \mathcal{N}_1$, are unique. Let $\boldsymbol{\lambda} = (\lambda_i, i \in \mathcal{N}_1)$ be the $1 \times S$ row vector of the λ_i's. For the rest of the section, whenever we refer to λ_i and $\boldsymbol{\lambda}$, we will mean this particular choice of λ_i and $\boldsymbol{\lambda}$.

Since in the optimal mechanism $\theta_{\tilde{n}}$ has no incentive to report any other types (the associated lotteries of the other types are all SLs), the principal should not leave $\theta_{\tilde{n}}$ any rent. Therefore, $t_{\tilde{n}} = u(x_{\tilde{n}}, \theta_{\tilde{n}})$. Then the constraints (ICC-$\tilde{n}$) in (P1) can be rewritten as

$$u(x_j, \theta_j) - t_j \geqslant u(x_{\tilde{n}}, \theta_j) - u(x_{\tilde{n}}, \theta_{\tilde{n}}) - \boldsymbol{q}_j\boldsymbol{y}_{\tilde{n}} = f_j(x_{\tilde{n}}) - \boldsymbol{q}_j\boldsymbol{y}_{\tilde{n}}, \quad j \in \mathcal{N}_1. \qquad \text{(ICC-}\tilde{n}\text{)}$$

The constraints (PC-i), $i \in \mathcal{N}_1$, in (P1) and (ICC-$\tilde{n}$) can be combined into one set of constraints: $t_i \leqslant u(x_i, \theta_i) - \max\left\{0, f_i(x_{\tilde{n}}) - \boldsymbol{q}_i\boldsymbol{y}_{\tilde{n}}\right\}, i \in \mathcal{N}_1$. Since the principal would want to choose as high a t_i as possible, the inequalities should be equalities in the optimal mechanism:

$$t_i = u(x_i, \theta_i) - \max\left\{0, f_i(x_{\tilde{n}}) - \boldsymbol{q}_i\boldsymbol{y}_{\tilde{n}}\right\}, \quad i \in \mathcal{N}_1. \qquad (5)$$

The crucial element of the subsequent analysis is not the lottery $\boldsymbol{y}_{\tilde{n}}$ itself but the *expected payment* from the lottery that a type expects to make upon reporting $\theta_{\tilde{n}}$. We therefore introduce notation, z_i, called the ELP of θ_i:

Notation 2. *Let* $z_i = \boldsymbol{q}_i\boldsymbol{y}_{\tilde{n}}, i \in \mathcal{N}_1$, *be the expected lottery payment ELP of* θ_i. *Let* $\boldsymbol{z} = (z_i, i \in \mathcal{N}_1)'$.

Since $\boldsymbol{Q}_{\mathcal{N}_1}\boldsymbol{y}_{\tilde{n}} = \boldsymbol{z}$, and the $S \times S$ matrix $\boldsymbol{Q}_{\mathcal{N}_1}$ has full rank, there is a one-to-one correspondence between $\boldsymbol{y}_{\tilde{n}}$ and $\boldsymbol{z}$. Furthermore, since $\boldsymbol{\lambda}\boldsymbol{Q}_{\mathcal{N}_1} = \boldsymbol{q}_{\tilde{n}}$ and thus $\boldsymbol{\lambda}\boldsymbol{Q}_{\mathcal{N}_1}\boldsymbol{y}_{\tilde{n}} = \boldsymbol{q}_{\tilde{n}}\boldsymbol{y}_{\tilde{n}}$, the constraint $\boldsymbol{q}_{\tilde{n}}\boldsymbol{y}_{\tilde{n}} = 0$ in (P1) is satisfied if and only if $\boldsymbol{\lambda}\boldsymbol{z} = 0$, which becomes a constraint on $\boldsymbol{z}$.

Using (5), we can simplify problem (P1) and further write it in terms of $\boldsymbol{z}$ as

$$\begin{aligned} \max_{\{t_i, z_i\}_{i\in\mathcal{N}_1}, \{x_i\}_{i\in\mathcal{N}}} \; & \sum_{i\in\mathcal{N}_1} \mu_i[W(x_i) + t_i] + \mu_{\tilde{n}}[W(x_{\tilde{n}}) + u(x_{\tilde{n}}, \theta_{\tilde{n}})] \\ \text{s.t.} \quad & \text{(C-}i\text{)} \quad t_i = u(x_i, \theta_i) - \max\{0, f_i(x_{\tilde{n}}) - z_i\}, \quad i \in \mathcal{N}_1, \\ & \boldsymbol{\lambda}\boldsymbol{z} = 0, \end{aligned} \qquad \text{(P2)}$$

where (C-i) stands for θ_i's *combined* participation constraint and its incentive compatibility constraint of reporting $\theta_{\tilde{n}}$.

The expression $\max\{0, f_i(x_{\tilde{n}}) - z_i\}$ represents θ_i's expected rent. It is useful to observe that if $z_i < f_i(x_{\tilde{n}})$, θ_i obtains rent $f_i(x_{\tilde{n}}) - z_i$, and the incentive compatibility constraint of θ_i reporting $\theta_{\tilde{n}}$ is binding. If $z_i > f_i(x_{\tilde{n}})$, θ_i earns no rent and the incentive compatibility constraint is slack. If the principal can find $\mathbf{z}$ such that $z_i \geqslant f_i(x_{\tilde{n}})$ for all $i \in \mathcal{N}_1$, full rent can be extracted. Even when full rent cannot be extracted, the transfer t_i can be increased—and therefore the rent obtained by type θ_i decreased—by raising z_i. The only constraint for raising the z_i's is $\boldsymbol{\lambda z} = 0$. This point underlies almost all the major results in this section.

Our first result shows that as long as there is *some* type θ_i, $i \in \mathcal{N}_1$, who obtains rent, *none* of the incentive compatibility constraints in (P2) can be slack. As discussed in the previous paragraph, (C-i) only implies that if a type θ_i gets rent, its *own* incentive compatibility constraint of reporting $\theta_{\tilde{n}}$ must be binding.

Proposition 2. *Suppose in the optimal mechanism there exists a type θ_i, $i \in \mathcal{N}_1$, that obtains rent. Then the incentive compatibility constraints (of reporting $\theta_{\tilde{n}}$) of all types in $\mathcal{N}_1$ whose λ is non-zero should bind, including those who receive no rent. That is, if $\exists i \in \mathcal{N}_1$, such that $t_i < u(x_i, \theta_i)$, then $z_j \leqslant f_j(x_{\tilde{n}})$ for all $j \in \mathcal{N}_1$ with $\lambda_j \neq 0$.*

To understand the intuition, note that if a type, say θ_i, gets rent, it must be that $z_i < f_i(x_{\tilde{n}})$, and this rent is reduced if z_i can be raised. If a type θ_j who does not get rent has a slack incentive compatibility constraint, i.e., if $z_j > f_j(x_{\tilde{n}})$, z_j can be reduced a little without affecting the (zero) rent of θ_j. Then the principal can improve the mechanism since z_i can be raised a little through a suitable reduction of z_j without violating the constraint $\boldsymbol{\lambda z} = 0$.

Next we characterize the type(s) who obtains rent in the optimal mechanism.

Proposition 3. *Given an allocation profile $\{x_i, i \in \mathcal{N}\}$, if full rent cannot be extracted, type θ_i, $i \in \mathcal{N}_1$, obtains rent in the optimal mechanism only if $\frac{\mu_i}{\lambda_i} = \min\left\{\frac{\mu_j}{\lambda_j}, j \in \mathcal{N}_1\right\}$. Therefore, generically at most one type obtains rent in the optimal mechanism.*

In a sense, $\frac{\mu_i}{\lambda_i}$ measures the effective cost of giving rent to type θ_i. For two types, say θ_i and θ_j, other things equal, giving one unit of rent to θ_i than to θ_j has (strictly) lower expected cost to the principal if $\mu_i \leqslant (<) \mu_j$. On the other hand, λ_i relates to the extent of similarity between the signals generated by types θ_i and $\theta_{\tilde{n}}$. The higher λ_i is, the more similar is type θ_i to type $\theta_{\tilde{n}}$ in terms of their signals, and the more difficult it is to separate out type θ_i from type $\theta_{\tilde{n}}$ through these signals. Put differently, penalizing type θ_i is more difficult than penalizing θ_j through the utilization of the signals if λ_i is greater than λ_j.

As Proposition 3 indicates, which type obtains rent is independent of the allocation profile, or the extent of the "intrinsic incentive" $f_i(x_{\tilde{n}})$, $i \in \mathcal{N}_1$, to report $\theta_{\tilde{n}}$. However, as we show next, the allocation profile does affect the *magnitude* of the rent.

Proposition 4. *Let $\eta = \min\left\{\frac{\mu_k}{\lambda_k}, k \in \mathcal{N}_1\right\}$. The agent's expected rent in the optimal mechanism is given by*

$$\max\left\{0,\ \eta \sum_{k \in \mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}})\right\}. \tag{6}$$

Furthermore, this rent is continuous in $\boldsymbol{\lambda}$.

In the independent case (i.e., without signals), if $f_k(x_{\tilde{n}}) \leqslant 0$, i.e., if θ_k has no intrinsic incentive to report $\theta_{\tilde{n}}$, the degree of θ_k's disincentive to report $\theta_{\tilde{n}}$, i.e., how negative $f_k(x_{\tilde{n}})$ is, does not affect the total expected rent. In contrast, in Proposition 4, it is the weighted "aggregate" incentive to report $\theta_{\tilde{n}}$—the weights being the degree of signal similarity λ_k—that determines the rent that can be obtained by the agent. An intrinsic disincentive to report $\theta_{\tilde{n}}$ (i.e., a negative $f_k(x_{\tilde{n}})$) is useful in bringing down the rents of types who obtain rent, and the magnitude of the disincentive affects the total expected rent.

A corollary of Proposition 4 is that, in general, the total expected rent is *strictly lower* than in the independent case.[6] This is due to two reasons. First, under the standard independent case, whenever a type, say θ_k, gets rent, it is determined by the rent he would have obtained by reporting the type that is the "best" for him to report. In other words, the rent obtained by θ_k is the maximum among $f_k(x_j)$, $j \in \mathcal{N}\backslash k$. In (P2), type θ_k can only report type $\theta_{\tilde{n}}$. Except for special cases, one would not expect $\max_j f_k(x_j)$ to be equal to $f_k(x_{\tilde{n}})$ *for all* k. Second, even if $\max_j f_k(x_j) = f_k(x_{\tilde{n}})$ for all k, the rent would still, in general, be strictly lower than that without signals. This is because

$$\sum_{k\in\mathcal{N}_1} \eta\lambda_k f_k(x_{\tilde{n}}) \leqslant \sum_{k\in\mathcal{N}_1} \eta\lambda_k \max\{0, f_k(x_{\tilde{n}})\}$$
$$\leqslant \sum_{k\in\mathcal{N}_1} \left(\frac{\mu_k}{\lambda_k}\right) \lambda_k \max\{0, f_k(x_{\tilde{n}})\} = \sum_{k\in\mathcal{N}_1} \mu_k \max\{0, f_k(x_{\tilde{n}})\},$$

where the first inequality is strict unless $f_k(x_{\tilde{n}}) \geqslant 0$ for *all* $k \in \mathcal{N}_1$, while the second inequality is strict unless $\frac{\mu_k}{\lambda_k}$ is the same for all $k \in \mathcal{N}_1$. Thus, the presence of the correlated signals helps the principal in two ways: first, it prevents many types from getting rent at all, and second, even for types who get rent, their rent is reduced because through judicious choice of the lotteries, intrinsic disincentives of some types are used to reduce rent of the types who obtain rent.

Proposition 4 also leads to an easy way to check whether or not full surplus can be extracted.

Proposition 5. *Given allocation $x_{\tilde{n}}$, no type gets any rent if and only if $\sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}}) \leqslant 0$.*

Corollary 2. *Full surplus can be extracted if and only if $\sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}}^{FI}) \leqslant 0$.*

The condition in Corollary 2 is equivalent to the necessary and sufficient condition in RS. (See Appendix A for further details.)

Given the expected rent in (6), we can express the principal's problem as maximizing the "virtual surplus":

$$\max_{x_i, i\in\mathcal{N}} \left[\sum_{k\in\mathcal{N}} \mu_k \left[u(x_k, \theta_k) + W(x_k)\right] - \max\left\{0, \eta \sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}})\right\}\right]. \tag{7}$$

Consequently, we know

Proposition 6. *The optimal allocation x_i for type θ_i, $i \in \mathcal{N}_1$, is at its full-information level x_i^{FI}. If any type θ_i, $i \in \mathcal{N}_1$ obtains rent, the optimal allocation $x_{\tilde{n}}$ for $\theta_{\tilde{n}}$ is distorted away from $x_{\tilde{n}}^{FI}$.*

[6] It is obvious that the rent with signals has to be weakly lower than the rent without the signals.

As in other mechanism design problems, the allocations for types corresponding to $\mathcal{N}_1$ are at the full information level since no type has incentive to report these types. It is only $\theta_{\tilde{n}}$ whose allocation $x_{\tilde{n}}$ may be distorted away from the full information level. Note that even under "standard" ordering assumptions used in the independent case (single crossing, etc.), the optimal allocations may not be monotonic with types.

6. Optimal contract when $N = S + 2$

The analysis becomes more complicated, and the algebra more arduous as we move to the general case of $N = S + K$, $K \geqslant 2$. To see why, note that in a standard model (i.e., without signals) with N types, there are in general $N(N-1)$ incentive compatibility constraints to satisfy. However, useful additional assumptions, for example the single crossing property, and a "regularity" condition on the distribution of types allow the analysis to be simplified tremendously. A certain type (a worst-off type) can be identified exogenously whose participation constraint is known to bind in the optimal solution. Furthermore, because of the additional assumptions, satisfying incentive compatibility constraints locally guarantees satisfying them globally. The principal's optimization problem can be solved as a first-order difference equation (involving the $N - 1$ local incentive compatibility constraints) without having to use $N(N - 1)$ Lagrangian multipliers.

In our model, when $N = S + K$, $K \geqslant 2$, it is not necessarily possible to identify exogenously any "worst" type who cannot obtain rent in the optimal solution (i.e., the one whose participation constraint will bind in the optimal solution). More importantly, the subset of types who might get rent by being able to report a type *depends* on the allocation chosen. [7] Below, we discuss the case when $N = S + 2$, a situation that allows us to highlight the complexities of the general case while still keeping the analysis tractable. From Assumptions 2(b) and (c), this case is further divided into two scenarios: one where $\mathcal{N}_2$ is a singleton and the other where $\mathcal{N}_2$ contains two elements. In the first case, even though the algebra can be tedious, the problem is still simple conceptually since there is only one type (corresponding to $\mathcal{N}_2$) that other types can report. In the second scenario, when $\mathcal{N}_2$ has multiple elements, the complexities mentioned above comes into full force.

Since the mathematical derivation, though straightforward, is rather tedious, we present only certain qualitative features of the optimal solution and discuss the intuition behind them. In particular, we show that parallel to the case of $N = S + 1$ when full rent cannot be extracted, (i) types that obtain rent can be characterized using the priors and the parameters describing the similarity of the signals, (ii) generically no more than two types can obtain rent, and (iii) in such situations, only a few other types can have slack incentive compatibility constraints. The detailed derivation of the results is available from the authors upon request.

6.1. $\mathcal{N}_2$ is a singleton

Let $\widetilde{n}$ denote the single element in $\mathcal{N}_2$. Since the rank of $\boldsymbol{Q}$ is S, and $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$ are the extremal elements of the set $\{\boldsymbol{q}_i, i \in \mathcal{N}\}$, there exists an S element subset of $\boldsymbol{Q}_{\mathcal{N}_1}$ that forms a basis for $\mathbb{R}^S$ and the convex hull of which contains the signal vector of $\theta_{\widetilde{n}}$. We denote the indices

[7] A similar difficulty arises in the mechanism design problem with multidimensional types where the subsets of incentive compatibility constraints that bind at the optimal solution depend on the allocation chosen. As noted by Rochet and Stole [10], "(M)ultiple-dimension problems are difficult precisely when they give rise to endogenous ordering over the types..."

of the types belonging to this subset as $\mathcal{N}_b \subset \mathcal{N}_1$. In other words, the matrix $\boldsymbol{Q}_{\mathcal{N}_b}$ contains the S signal vectors such that $\{\boldsymbol{q}_i, i \in \mathcal{N}_b\}$ forms a basis for $\mathbb{R}^S$ and $\boldsymbol{q}_{\tilde{n}} \in co\{\boldsymbol{q}_i, i \in \mathcal{N}_b\}$.[8] Let θ_m be the type such that $m = \mathcal{N}_1 \backslash \mathcal{N}_b$. Since $\boldsymbol{q}_{\tilde{n}}$ and $\boldsymbol{q}_m$ lie in the convex and linear span of $\{\boldsymbol{q}_i, i \in \mathcal{N}_b\}$, respectively, there exist unique sets of $\lambda_i \geqslant 0$ and γ_i, $i \in \mathcal{N}_b$, such that $\boldsymbol{q}_{\tilde{n}} = \sum_{i \in \mathcal{N}_b} \lambda_i \boldsymbol{q}_i$ and $\boldsymbol{q}_m = \sum_{i \in \mathcal{N}_b} \gamma_i \boldsymbol{q}_i$. As before, we find it more convenient to work with the ELP than with the lottery $\boldsymbol{y}_{\tilde{n}}$. Let $z_m = \boldsymbol{q}_m \boldsymbol{y}_{\tilde{n}}$ be the ELP of θ_m, and define $\boldsymbol{z} = (z_i, i \in \mathcal{N}_b)' = (\boldsymbol{q}_i \boldsymbol{y}_{\tilde{n}}, i \in \mathcal{N}_b)'$, $\boldsymbol{\lambda} = (\lambda_i, i \in \mathcal{N}_b)$, and $\boldsymbol{\gamma} = (\gamma_i, i \in \mathcal{N}_b)$. Since $\boldsymbol{\lambda} \boldsymbol{Q}_{\mathcal{N}_b} = \boldsymbol{q}_{\tilde{n}}$, $\boldsymbol{\gamma} \boldsymbol{Q}_{\mathcal{N}_b} = \boldsymbol{q}_m$, and $\boldsymbol{Q}_{\mathcal{N}_b} \boldsymbol{y}_{\tilde{n}} = \boldsymbol{z}$, there is a one-to-one correspondence between $\boldsymbol{y}_{\tilde{n}}$ satisfying $\boldsymbol{q}_{\tilde{n}} \boldsymbol{y}_{\tilde{n}} = 0$ and $\{\boldsymbol{z}, z_m\}$ satisfying $\boldsymbol{\lambda z} = 0$ and $\boldsymbol{\gamma z} = z_m$.

Again, by using SLs, we ensure that no type has incentive to report types corresponding to $\mathcal{N}_1$. Hence if the agent obtains rent, it must be due to some type(s) in $\mathcal{N}_1$ having incentive to report $\theta_{\tilde{n}}$. Since type $\theta_{\tilde{n}}$ obtains no rent, its non-random transfer $t_{\tilde{n}} = u(x_{\tilde{n}}, \theta_{\tilde{n}})$. Thus, similar to the last section, we can re-write the principal's problem (P) as

$$
\begin{aligned}
&\max_{\{t_i, z_i\}_{i \in \mathcal{N}_1}, \{x_i\}_{i \in \mathcal{N}}} \sum_{i \in \mathcal{N}_1} \mu_i \left[W(x_i) + t_i \right] + \mu_{\tilde{n}} \left[W(x_{\tilde{n}}) + u(x_{\tilde{n}}, \theta_{\tilde{n}}) \right] \\
&\text{s.t. (C-}i\text{)} \quad t_i = u(x_i, \theta_i) - \max\{0, f_i(x_{\tilde{n}}) - z_i\}, \quad i \in \mathcal{N}_1, \\
&\qquad \boldsymbol{\lambda z} = 0, \quad \boldsymbol{\gamma z} = z_m.
\end{aligned}
\tag{P3}
$$

The basic optimization problem is similar to that in the last section except that now there are two constraints on the choice of $\boldsymbol{z}$. For $i \in \mathcal{N}_b$, when the principal raises z_i to reduce the rent of this type, not only will z's for some other types in $\mathcal{N}_b$ have to be reduced so as to satisfy $\boldsymbol{\lambda z} = 0$, but z_m may also decrease, causing the rent of θ_m, if any, to increase. Thus, which type in $\mathcal{N}_b$ obtains rent in the optimal mechanism depends on whether θ_m also obtains rent. The next proposition characterizes the type θ_i, $i \in \mathcal{N}_b$, that obtains rent when full rent cannot be extracted.

Proposition 7. *Suppose allocation $x_{\tilde{n}}$ is given and full rent cannot be extracted. In the optimal mechanism, generically there is at most one type in $\mathcal{N}_b$ who can obtain rent. In particular,*

(a) *If θ_m has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$, i.e., if $z_m > f_m(x_{\tilde{n}})$, then type θ_i, $i \in \mathcal{N}_b$, can obtain rent (i.e. $z_i < f_i(x_{\tilde{n}})$) only if*

$$
\frac{\mu_i}{\lambda_i} = \min \left\{ \frac{\mu_k}{\lambda_k}, k \in \mathcal{N}_b \right\}.
$$

(b) *If θ_m obtains rent, i.e., $z_m < f_m(x_{\tilde{n}})$, then θ_i, $i \in \mathcal{N}_b$, can obtain rent only if*

$$
\frac{\mu_i}{\lambda_i} + \mu_m \frac{\gamma_i}{\lambda_i} = \min \left\{ \frac{\mu_k}{\lambda_k} + \mu_m \frac{\gamma_k}{\lambda_k}, k \in \mathcal{N}_b \right\}.
$$

When $z_m > f_m(x_{\tilde{n}})$, type θ_m obtains no rent and has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$. In that case, if z_m is reduced by a small amount, θ_m still obtains no rent and so this change in z_m has no impact on the principal's expected payoff. The incentive compatibility constraint of θ_m can be ignored (locally) and we are essentially back to the situation with $N = S + 1$. Note the similarity between Propositions 7(a) and 3. When θ_m obtains rent, reallocating

[8] Such S vectors may not be unique. However, we can verify that choosing different basis vectors does not affect the optimal mechanism, since different sets of basis vectors result in the same two restrictions on the ELPs in (P3) below.

rents of types in $\mathcal{N}_b$ (i.e., varying the z_i's for $i \in \mathcal{N}_b$) also affects the rent of θ_m (since $z_m = \boldsymbol{\gamma z}$). A higher γ_k reflects a higher degree of similarity between $\boldsymbol{q}_k$ and $\boldsymbol{q}_m$, and hence giving rent to θ_k increases $\theta'_m s$ rent more than if the rent had been given to some other type with lower γ_k. Therefore, when θ_m and some type from $\mathcal{N}_b$ gets rent in the optimal mechanism, the "ideal candidate" type in $\mathcal{N}_b$ to give rent to is one with a small prior (low μ_k), whose signals are "highly similar" with $\boldsymbol{q}_{\tilde{n}}$ (high λ_k), but distinctive from $\boldsymbol{q}_m$ (low γ_k).

Whether type θ_m obtains rent or has a slack incentive compatibility constraint depends on the allocation $x_{\tilde{n}}$. Thus, unlike the case of $N = S + 1$, the rent structure (i.e., which of the type(s) obtains rent) may now depend on the allocation profile.

Recall that in the case of $N = S + 1$, generically there can be at most one type that obtains rent, and when this occurs, all incentive compatibility constraints of reporting $\theta_{\tilde{n}}$ must be binding. Now with two constraints on the values of the ELP z_i's, we have

Proposition 8. *Generically, in the optimal mechanism,*

(a) *There cannot be more than two types who obtain rent.*

(b) *If two types obtain rent, then there cannot be a type whose incentive compatibility constraint of reporting $\theta_{\tilde{n}}$ is slack.*

(c) *If full rent cannot be extracted, there can be at most one type who has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$.*

As in the last section, the key to understanding the Proposition lies in the principal's incentive to raise the z_i's so as to reduce the rents to the agent types. Since there are two restrictions on the z_i's, the principal can adjust the values of any three of the z_i's while keeping the values of the others fixed. (Note that when $N = S + 1$ and there is a single equation restriction on the z_i's, the principal can adjust any two z's while keeping the others unchanged.) Since μ_i/λ_i or $\mu_i/\lambda_i + \mu_m\gamma_i/\lambda_i$ reflects the relative cost to the principal of giving rent to type θ_i, if three (or more) types obtain rent, unless a non-generic situation occurs such that these relative costs of adjusting the z_i's are the same across the three types, the principal is always able to reduce the rent of the type whose relative cost is the highest, thereby reducing the total expected rent. This observation underlines Proposition 8(a). Statements (b) and (c) can be understood in a similar fashion: whenever these conditions fail to hold, the principal has room to adjust the z_i's to reduce the expected rent.

Similar to the case of $N = S + 1$, the total expected rent of the agent depends on the weighted sum of the intrinsic incentives of types in $\mathcal{N}_1$ to report $\theta_{\tilde{n}}$, $f_i(x_{\tilde{n}})$. Also, in the optimal mechanism, the allocation of all types in $\mathcal{N}_1$ is at the full information level and only $x_{\tilde{n}}$ can be different from $x_{\tilde{n}}^{FI}$.

6.2. $\mathcal{N}_2$ contains two elements

Let $\theta_{\widetilde{n}_1}$ and $\theta_{\widetilde{n}_2}$ be the two types corresponding to $\mathcal{N}_2$. Since $\mathcal{N}_1$ has exactly S elements, $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$ form a basis for $\mathbb{R}^S$. Thus there exist unique non-negative $1 \times S$ vectors $\boldsymbol{\lambda}(\widetilde{n}_1) = (\lambda_i(\widetilde{n}_1), i \in \mathcal{N}_1)$ and $\boldsymbol{\lambda}(\widetilde{n}_2) = (\lambda_i(\widetilde{n}_2), i \in \mathcal{N}_1)$ with $\sum_{i\in\mathcal{N}_1} \lambda_i(j) = 1$, $j = \widetilde{n}_1, \widetilde{n}_2$, such that $\boldsymbol{q}_{\widetilde{n}_1} = \sum_{i\in\mathcal{N}_1} \lambda_i(\widetilde{n}_1)\boldsymbol{q}_i$ and $\boldsymbol{q}_{\widetilde{n}_2} = \sum_{i\in\mathcal{N}_1} \lambda_i(\widetilde{n}_2)\boldsymbol{q}_i$. As before, we work with the ELP rather than with the lotteries $\boldsymbol{y}_{\widetilde{n}_1}$ and $\boldsymbol{y}_{\widetilde{n}_2}$ themselves. Since each type in $\mathcal{N}_1$ can report either $\theta_{\widetilde{n}_1}$ or $\theta_{\widetilde{n}_2}$, each is associated with two ELPs. Define $\boldsymbol{z}(\widetilde{n}_1) = (z_i(\widetilde{n}_1), i \in \mathcal{N}_1)' = (\boldsymbol{q}_i\boldsymbol{y}_{\widetilde{n}_1}, i \in \mathcal{N}_1)'$ and $\boldsymbol{z}(\widetilde{n}_2) = (z_i(\widetilde{n}_2), i \in \mathcal{N}_1)' = (\boldsymbol{q}_i\boldsymbol{y}_{\widetilde{n}_2}, i \in \mathcal{N}_1)'$ as the ELPs of reporting $\theta_{\widetilde{n}_1}$ and $\theta_{\widetilde{n}_2}$, respectively. Since $\boldsymbol{\lambda}(\tilde{n}_j)\boldsymbol{Q}_{\mathcal{N}_1} = \boldsymbol{q}_{\tilde{n}_j}$, $\boldsymbol{Q}_{\mathcal{N}_1}\boldsymbol{y}_{\tilde{n}_j} = \boldsymbol{z}(\tilde{n}_j)$, $j = 1, 2$, and the matrix $\boldsymbol{Q}_{\mathcal{N}_1}$ has rank S, there is a one-to-one correspondence between lotteries $\boldsymbol{y}_{\widetilde{n}_1}$ and $\boldsymbol{y}_{\widetilde{n}_2}$ satisfying $\boldsymbol{q}_{\tilde{n}_1}\boldsymbol{y}_{\tilde{n}_1} = 0$ and $\boldsymbol{q}_{\tilde{n}_2}\boldsymbol{y}_{\tilde{n}_2} = 0$ and the ELP vectors $z(\widetilde{n}_1)$ and $z(\widetilde{n}_2)$ satisfying $\boldsymbol{\lambda}(\widetilde{n}_1)z(\widetilde{n}_1) = 0$ and $\boldsymbol{\lambda}(\widetilde{n}_2)z(\widetilde{n}_2) = 0$.

Since each type in $\mathcal{N}_2$ (e.g., $\theta_{\tilde{n}_1}$) can report to be of the other type (e.g., $\theta_{\tilde{n}_2}$), each is associated with an ELP as well. Define the scalars $z_{\widetilde{n}_1}$ and $z_{\widetilde{n}_2}$ as $z_{\widetilde{n}_1} = \boldsymbol{q}_{\widetilde{n}_1}\boldsymbol{y}_{\widetilde{n}_2}$ and $z_{\widetilde{n}_2} = \boldsymbol{q}_{\widetilde{n}_2}\boldsymbol{y}_{\widetilde{n}_1}$. Again, since $\boldsymbol{\lambda}(\tilde{n}_j)\boldsymbol{Q}_{\mathcal{N}_1} = \boldsymbol{q}_{\tilde{n}_j}$ and $\boldsymbol{Q}_{\mathcal{N}_1}\boldsymbol{y}_{\tilde{n}_j} = z(\tilde{n}_j)$, $j = 1, 2$, we know $z_{\widetilde{n}_2} = \boldsymbol{q}_{\tilde{n}_2}\boldsymbol{y}_{\tilde{n}_1} = \boldsymbol{\lambda}(\tilde{n}_2)\boldsymbol{Q}_{\mathcal{N}_1}\boldsymbol{y}_{\tilde{n}_1} = \boldsymbol{\lambda}(\widetilde{n}_2)z(\widetilde{n}_1)$, and similarly $z_{\widetilde{n}_1} = \boldsymbol{\lambda}(\widetilde{n}_1)z(\widetilde{n}_2)$.

Using SLs for θ_i, $i \in \mathcal{N}_1$, no type has any incentive to report types θ_i, $i \in \mathcal{N}_1$. The agent's information rent can only arise because of types corresponding to $\mathcal{N}_1$ having incentive to report $\theta_{\tilde{n}_1}$ and/or $\theta_{\tilde{n}_2}$, and from $\theta_{\tilde{n}_1}$ and $\theta_{\tilde{n}_2}$ having incentives to report each other. Since a type in $\mathcal{N}_1$ can report either $\theta_{\tilde{n}_1}$ or $\theta_{\tilde{n}_2}$, its rent has to be the *maximum* of the rents required to prevent it from reporting either one of $\theta_{\tilde{n}_1}$ and $\theta_{\tilde{n}_2}$. For the types in $\mathcal{N}_2$, note that $\theta_{\tilde{n}_1}$ and $\theta_{\tilde{n}_2}$ cannot both obtain rents from reporting each other; as in standard mechanism design problems (i.e., without correlated signals), if a type, say $\theta_{\tilde{n}_1}$, obtains rent from having incentive to report $\theta_{\tilde{n}_2}$, the incentive compatibility constraint of $\theta_{\tilde{n}_2}$ reporting $\theta_{\tilde{n}_1}$ must be slack.

It is possible that in the optimal mechanism, incentive compatibility constraints of both types in $\mathcal{N}_2$ are slack (i.e., $\theta_{\tilde{n}_1}$ strictly prefers not to report $\theta_{\tilde{n}_2}$ and $\theta_{\tilde{n}_2}$ strictly prefers not to report $\theta_{\tilde{n}_1}$). Then the optimal ELPs $z(\tilde{n}_1)$ and $z(\tilde{n}_2)$ can be found in exactly the same way as in $N = S + 1$, giving rents, if any, to the type(s) in $\mathcal{N}_1$ according to the criterion in Proposition 3. That is, a type $\theta_i, i \in \mathcal{N}_1$, is offered rent in order for it not to report $\theta_{\tilde{n}_1}$ only if $\mu_i/\lambda_i(\tilde{n}_1) = \min\{\mu_k/\lambda_k(\tilde{n}_1), k \in \mathcal{N}_1\}$, and a type θ_j, $j \in \mathcal{N}_1$ is offered rent in order for it not to report $\theta_{\tilde{n}_2}$ only if $\mu_j/\lambda_j(\tilde{n}_2) = \min\{\mu_k/\lambda_k(\tilde{n}_2), k \in \mathcal{N}_1\}$. For each type in $\mathcal{N}_1$ the final rent is the maximum of the rents that needs to be given such that this type reports truthfully rather than reporting either $\theta_{\tilde{n}_1}$ or $\theta_{\tilde{n}_2}$.

Of course, it is possible that one type in $\mathcal{N}_2$ obtains rent in the optimal mechanism, and for the remainder of this section, we focus on this case. Without loss of generality, suppose $\theta_{\tilde{n}_1}$ obtains rent (from reporting $\theta_{\tilde{n}_2}$), i.e., $z_{\tilde{n}_1} < f_{\tilde{n}_1}(x_{\tilde{n}_2})$, and thus the incentive compatibility constraint of $\theta_{\tilde{n}_2}$ reporting $\theta_{\tilde{n}_1}$ is slack: $z_{\tilde{n}_2} > f_{\tilde{n}_2}(x_{\tilde{n}_1})$. Since $\theta_{\tilde{n}_2}$ obtains no rent, we know

$$\begin{aligned} t_{\tilde{n}_2} &= u(x_{\tilde{n}_2}, \theta_{\tilde{n}_2}), \\ t_{\tilde{n}_1} &= u(x_{\tilde{n}_1}, \theta_{\tilde{n}_1}) - (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}). \end{aligned} \tag{8}$$

Since types corresponding to $\mathcal{N}_1$ can report either $\theta_{\tilde{n}_1}$ or $\theta_{\tilde{n}_2}$, we have

$$\begin{aligned} t_i &= u(x_i, \theta_i) - \max\left\{0,\ f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}),\ f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2)\right\}, \\ &\quad i \in \mathcal{N}_1. \end{aligned} \tag{9}$$

Note that since $\theta_{\tilde{n}_1}$ obtains rent, in order to prevent θ_i from reporting $\theta_{\tilde{n}_1}$, the rent of $\theta_{\tilde{n}_1}$ has to be added to that of θ_i. Thus, the rent needed to prevent θ_i from reporting $\theta_{\tilde{n}_1}$ is the sum of the two "direct rents," $f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)$ and $f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}$.

From (9), if θ_i in $\mathcal{N}_1$ has incentives to report both $\theta_{\tilde{n}_1}$ and $\theta_{\tilde{n}_2}$, its final rent is determined by the greater of these incentives. The smaller of the two incentives does not matter in determining θ_i's rent or the optimal mechanism. Thus, θ_i can have a *slack* incentive compatibility constraint of reporting, say, $\theta_{\tilde{n}_2}$, even when it *can* obtain rent from reporting $\theta_{\tilde{n}_2}$ if θ_i can obtain still higher rent by reporting $\theta_{\tilde{n}_1}$. We adopt the following definition to make this point clear.

Definition 3. (a) A type θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_1}$ (or has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_1}$) if

$$(f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}) > (\text{or} <) \max\left\{0,\ f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2)\right\}.$$

(b) A type θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_2}$ (or has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_2}$) if

$$f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2) > (\text{or} <) \max\left\{0, (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1})\right\}.$$

Similar to the previous sections, we can write the principal's problem as

$$\max_{\{t_i,x_i\}_{i\in\mathcal{N}},z(\tilde{n}_1),z(\tilde{n}_2),z_{\tilde{n}_1},z_{\tilde{n}_2}} \sum_{i\in\mathcal{N}_1} \mu_i\left[W(x_i)+t_i\right]+\mu_{\tilde{n}_1}\left[W(x_{\tilde{n}_1})+t_{\tilde{n}_1}\right]+\mu_{\tilde{n}_2}\left[W(x_{\tilde{n}_2})+t_{\tilde{n}_2}\right] \tag{P4}$$
$$\text{s.t.}\quad \boldsymbol{\lambda}(\tilde{n}_1)\boldsymbol{z}(\tilde{n}_1) = 0;\quad \boldsymbol{\lambda}(\tilde{n}_2)\boldsymbol{z}(\tilde{n}_2)=0,\quad \boldsymbol{\lambda}(\tilde{n}_2)\boldsymbol{z}(\tilde{n}_1)=z_{\tilde{n}_2},\quad \boldsymbol{\lambda}(\tilde{n}_1)\boldsymbol{z}(\tilde{n}_2)=z_{\tilde{n}_1},$$

where t_i, $i \in \mathcal{N}_1$, are given in (9), and $t_{\tilde{n}_1}$ and $t_{\tilde{n}_2}$ are given in (8).

The next proposition characterizes the type(s) in $\mathcal{N}_1$ who obtains rent when full rent extraction is impossible.

Proposition 9. *Suppose allocations $x_{\tilde{n}_1}$ and $x_{\tilde{n}_2}$ are given. In the optimal mechanism,* (a) *If θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_1}$, then it must be that θ_i has the lowest $\mu/\lambda(\tilde{n}_1)$:*

$$i = \arg\min_{j\in\mathcal{N}_1}\left\{\frac{\mu_j}{\lambda_j(\tilde{n}_1)}\right\}.$$

(b) *If θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_2}$, then it must be that*

$$i = \arg\min_{j\in\mathcal{N}_1}\left\{\frac{\mu_j}{\lambda_j(\tilde{n}_2)} + \left(\mu_{\tilde{n}_1} + \sum_{k\in\mathcal{N}_1(\tilde{n}_1)} \mu_k\right)\frac{\lambda_j(\tilde{n}_1)}{\lambda_j(\tilde{n}_2)}\right\},$$

where $\mathcal{N}_1(\tilde{n}_1) \subseteq \mathcal{N}_1$ is the index set of the types who obtain rent from having incentive to report $\theta_{\tilde{n}_1}$.

The interpretation of Proposition 9(a) is similar to that of Propositions 3 and 7(a). Since $\theta_{\tilde{n}_2}$ does not obtain rent and its incentive compatibility constraint of reporting any other type is slack, $\theta_{\tilde{n}_2}$ becomes irrelevant (locally) in the optimization problem of choosing the type in (a), and we are essentially back to the case of $N = S + 1$. In (b), since $\theta_{\tilde{n}_1}$ obtains rent, it plays a role similar to θ_m in the previous section when θ_m obtains rent, and Proposition 9(b) is thus parallel to Proposition 7(b). As in the previous case, a higher $\lambda_j(\tilde{n}_2)$ means that it is harder to separate θ_j from $\theta_{\tilde{n}_2}$ through the signals, making it more difficult to use lotteries to reduce the rent of θ_j. Further, since $\theta_{\tilde{n}_1}$ obtains rent from having incentive to report $\theta_{\tilde{n}_2}$, reallocating rents of types in $\mathcal{N}_1$ has to take into consideration the effects on the rent to $\theta_{\tilde{n}_1}$. A lower $\lambda_j(\tilde{n}_1)$ reflects a lower degree of similarity between $\boldsymbol{q}_j$ and $\boldsymbol{q}_{\tilde{n}_1}$, and thus a lower amount of rent to $\theta_{\tilde{n}_1}$ when giving rent to θ_j. Different from Proposition 7(b), however, since there could be types in $\mathcal{N}_1$ obtaining rent from having incentive to report $\theta_{\tilde{n}_1}$, increasing the rent to $\theta_{\tilde{n}_1}$ raises the rent to these types as well. Thus, the "cost coefficient" of giving rent to $\theta_{\tilde{n}_1}$ includes not only $\mu_{\tilde{n}_1}$ but also $\sum_{k\in\mathcal{N}_1(\tilde{n}_1)} \mu_k$.

As in the previous sections, there cannot be too many types who obtain rent, and when some types obtain rent, there cannot be too many incentive compatibility constraints that are slack.

Proposition 10. *Generically in the optimal mechanism for $k = 1, 2$,* (a) *there is at most one type in $\mathcal{N}_1$ who can obtain rent from having incentive to report $\theta_{\tilde{n}_k}$;*

(b) *if a type obtains rent from having incentive to report* $\theta_{\tilde{n}_k}$, *there cannot be a type* (*with a non-zero* $\lambda(\tilde{n}_k)$) *whose incentive compatibility constraint of reporting* $\theta_{\tilde{n}_k}$ *is slack.*

When $k = 1$, this Proposition is similar to Propositions 2 and 3 of the case $N = S + 1$. Since $\theta_{\tilde{n}_2}$ obtains no rent and has a slack incentive compatibility constraint, it can be ignored in optimally choosing the $z(\tilde{n}_1)$'s, and the optimization becomes exactly similar to that of the case of $N = S + 1$. In particular, as in the case of $N = S + 1$, if there are two or more types obtaining rent or having slack incentive compatibility constraints, there is always room for the principal to adjust the $z(\tilde{n}_1)$'s to reduce the rent. When $k = 2$, Proposition 10 is parallel to the special case of Proposition 8 where θ_m obtains rent. Essentially $\theta_{\tilde{n}_1}$ plays the same role here that θ_m plays there. Recall that in the previous section when there are two or more types obtaining rent or having slack incentive compatibility constraints, there is room for the principal to adjust the $z(\tilde{n}_2)$'s to reduce the rent. That intuition exactly applies to Proposition 10 when $k = 2$: since $\theta_{\tilde{n}_1}$ already obtains rent by assumption, there can at most be one other type in $\mathcal{N}_1$ who can obtain rent or have a slack incentive compatibility constraint.

From Definition 3, if a type θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_2}$ (or $\theta_{\tilde{n}_1}$), it must have a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_1}$ (or $\theta_{\tilde{n}_2}$). Then Proposition 10 immediately implies that

Corollary 3. *If there is a type* θ_i, $i \in \mathcal{N}_1$, *who obtains rent from having incentive to report* $\theta_{\tilde{n}_1}$ (*or* $\theta_{\tilde{n}_2}$), *then generically no type in* $\mathcal{N}_1$ *with* $\lambda(\tilde{n}_1) \neq 0$ (*or with* $\lambda(\tilde{n}_2) \neq 0$) *obtains rent from having incentive to report* $\theta_{\tilde{n}_2}$ (*or* $\theta_{\tilde{n}_1}$).

Corollary 3 implies that in Proposition 9(b), generically, the term $\sum_{k \in \mathcal{N}_1(\tilde{n}_1)} \mu_k$ vanishes. This is because if θ_i obtains rent from having incentive to report $\theta_{\tilde{n}_2}$, generically no type obtains rent from having incentive to report $\theta_{\tilde{n}_1}$, which means $\mathcal{N}_1(\tilde{n}_1) = \emptyset$. Proposition 10 and Corollary 3 imply the following important corollary.

Corollary 4. *In the optimal mechanism*, *generically there is at most one type in* $\mathcal{N}_1$ *who obtains rent*.

Therefore, in general, at most only one of the two scenarios in Prooposition 9 can arise, further simplifying the search for the optimal mechanism.

7. Conclusion

In this paper, we study the mechanism design problem when the principal can condition the agent's transfers on the realization of *ex post* signals that are correlated with the agent's types. Previous research identifies conditions that guarantee full surplus extraction; our objective is to understand the nature of the optimal mechanism when the signals and payoff functions may be such that full surplus extraction is not possible.

Our first result shows that without any loss of generality, the optimal use of the signals involves using lotteries—one for each type—that have zero expected value under truth-telling. Hence the signals are used solely for incentive compatibility purposes, even when the CM–MR conditions fail.

The second insight is that in the optimal mechanism, the correlated signals reduce the agent's expected rent by allowing the principal to, in effect, "reallocate" intrinsic rents. In CM–MR, the principal can use lotteries to raise the payments of *every* type who report another type. In our case, this can still be done for types corresponding to $\mathcal{N}_1$; consequently in the optimal mechanism no type can obtain rent by reporting types in $\mathcal{N}_1$. For types corresponding to $\mathcal{N}_2$, when a lottery is constructed to raise the expected payments of *some* types when falsely reporting θ_i, $i \in \mathcal{N}_2$, the same lottery reduces the expected payments of some *other* types falsely reporting θ_i. Nevertheless, the principal can choose which types' payments to reduce and which types' to raise. In particular, and unlike the independent case, a type that has no intrinsic incentive to report θ_i still plays a major role; the principal can use such intrinsic disincentives to reduce the expected rents of the types that have intrinsic incentives to report θ_i.

Our third contribution is to develop a method of working with ELP to study the optimal mechanism. We consider in detail the case when the number of types is one more than the number of signals. Here, there is a unique worst-off type: a type who cannot obtain rent under the optimal mechanism irrespective of the allocation chosen. This greatly facilitates the explicit characterization of the optimal mechanism. With an arbitrary number of signals and types—in particular when $\mathcal{N}_2$ is not a singleton—which types obtain rent and which sets of incentive compatibility constraints bind depends on the allocation chosen. As a result, well known techniques of working only with local incentive compatibility conditions cannot be utilized—a problem faced in the multidimensional-type mechanism design literature as well. Instead, one is reduced to using the Kuhn–Tucker technique (with possibly a large number of Lagrangian multipliers) and to checking which sets of incentive compatibility constraints bind at the optimum. Nevertheless, our methodology still simplifies the problem of searching for the optimal mechanism: as we show in the case when $N = S + 2$, generically there can only be a limited number of types obtaining rent or having slack incentive compatibility constraints. Further, these types can be identified through certain expressions involving the priors and signal similarity parameters.

The model in the paper has a single agent; with more agents, the principal's problem is to extract the maximum possible surplus from all these agents. Our basic methodology still applies in such situations; however, the principal's optimization problem is more complicated, especially if the report of one agent can be used as a signal for the others (as in CM and the many applications in MR).

Finally, we follow CM–MR–RS to assume that only the transfers are contingent on the signal realizations. If allocations can be chosen *after* the signals are realized, the optimal mechanism may involve signal-contingent allocation when full surplus cannot be extracted. (If the CM–MR–RS conditions hold, on the other hand, the optimal allocation is the full information allocation and is independent of the signal realizations.) An interesting research topic is to study the nature of the optimal mechanism when both the allocation and the transfers can be made to depend on the signals.

Acknowledgments

We thank Joydeep Bhattacharya, Catherine Kling, Corinne Langinier, Harvey Lapan, Philippe Marcoul, Emre Ozdenoren, Ennio Stacchetti, an anonymous referee, and an associate editor for insightful and extremely helpful comments. The usual disclaimer applies.

Appendix A. Comparison with the RS condition

The Proposition in RS gives a necessary and sufficient condition for full surplus extraction. Since our approach to obtaining the condition in Corollary 2 is different from RS, we show the two conditions are equivalent.

Continue to define $\boldsymbol{Q}$ as the signal matrix, and to be consistent with the notations in RS, let $\boldsymbol{Q}_{i\cdot}$ be the ith row of $\boldsymbol{Q}$ (i.e., $\boldsymbol{Q}_{i\cdot} = \boldsymbol{q}_i$), let $\boldsymbol{U}$ be the $N \times N$ utility matrix, with $u_{ik} \equiv u(x_i^{FI}, \theta_k)$ being the element in the kth row and ith column. Let $\boldsymbol{U}_{\cdot i}$ be the ith column of $\boldsymbol{U}$, i.e. $\boldsymbol{U}_{\cdot i} = (u(x_i^{FI}, \theta_1), \ldots, u(x_i^{FI}, \theta_N))'$. Then the RS condition is

Proposition 11 (*RS*). *Full surplus can be extracted in problem* (P1) *if and only if for each* $i = 1, \ldots, N$, *there does not exist an* $(N+1)$*-element vector* $\boldsymbol{\rho} \geqslant 0$, *such that*[9]

$$\boldsymbol{\rho}\begin{bmatrix} \boldsymbol{Q} \\ -\boldsymbol{Q}_{i\cdot} \end{bmatrix} = 0 \quad \text{and} \quad \boldsymbol{\rho}\begin{bmatrix} \boldsymbol{U}_{\cdot i} \\ -u_{ii} \end{bmatrix} = 1. \tag{10}$$

First (in step 1) we transform the RS condition in (10) to a representation that uses our notations. Then we show in step 2 that it is equivalent to Corollary 2

Step 1: Since $\boldsymbol{Q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)'$ and $\boldsymbol{Q}_{i\cdot} = \boldsymbol{q}_i$, the first condition in (10) becomes

$$\boldsymbol{q}_i = \frac{1}{\rho_{N+1} - \rho_i} \sum_{j \neq i} \rho_j \boldsymbol{q}_j. \tag{11}$$

Because the $\boldsymbol{q}$'s are (row) vectors of conditional probabilities, post-multiplying both sides of (11) by a $N \times 1$ vector of one's, we know

$$\rho_{N+1} - \rho_i = \sum_{j \neq i} \rho_j > 0. \tag{12}$$

Thus, (11) implies that $\boldsymbol{q}_i$ is in the convex hull of the other signal vectors. Consequently, such $\boldsymbol{\rho} \geqslant 0$ *does not* exist for types θ_i, $i \in \mathcal{N}_1$, and *exists* for $i \in \mathcal{N}_2$. Hence, to check for full surplus extraction, we only need to check the second condition in (10), and only for type $\theta_{\tilde{n}}$ (i.e., only for $i = \tilde{n}$).

The second condition in (10) when $i = \tilde{n}$ is

$$\sum_{j \in \mathcal{N} \backslash \tilde{n}} \rho_j u(x_{\tilde{n}}^{FI}, \theta_j) + (\rho_{\tilde{n}} - \rho_{N+1}) u(x_{\tilde{n}}^{FI}, \theta_{\tilde{n}}) = 1,$$

which, using (12), can be rewritten as

$$\sum_{j \in \mathcal{N} \backslash \tilde{n}} \rho_j \left(u(x_{\tilde{n}}^{FI}, \theta_j) - u(x_{\tilde{n}}^{FI}, \theta_{\tilde{n}}) \right) = 1. \tag{13}$$

Using our notation, the set $\mathcal{N} \backslash \tilde{n}$ is $\mathcal{N}_1$ and $u(x_{\tilde{n}}^{FI}, \theta_j) - u(x_{\tilde{n}}^{FI}, \theta_{\tilde{n}}) = f_j(x_{\tilde{n}}^{FI})$. Further, $\boldsymbol{q}_{\tilde{n}} = \sum_{j \in \mathcal{N}_1} \lambda_j \boldsymbol{q}_j$, and the λ's are unique since $\{\boldsymbol{q}_j, j \in \mathcal{N}_1\}$ forms a basis for $\mathbb{R}^S$. Thus, from (11) and (12), we know $\lambda_j = \rho_j / \sum_{k \in \mathcal{N}_1} \rho_k$ for all $j \in \mathcal{N}_1$. Then (13) can be rewritten in our notation as

$$\left(\sum_{k \in \mathcal{N}_1} \rho_k \right) \left(\sum_{j \in \mathcal{N}_1} \lambda_j f_j(x_{\tilde{n}}^{FI}) \right) = 1. \tag{14}$$

Therefore, in the case of $N = S + 1$, the RS condition in (10) reduces to the condition in (14).

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[9] *In RS's setup, the principal is the buyer and the agent is the seller so the transfer goes from the principal to the agent, whereas in our setup the transfer is paid by the agent to the principal. Furthermore, in RS, the "utility" of the agent represents* costs *to the agent of producing the good,* $c_{ik} = C(x_i^{FI}, \theta_k)$. *Hence, in their proposition, the conditions in* (10) *are written as* $\boldsymbol{\rho}\begin{bmatrix} -\boldsymbol{Q} \\ \boldsymbol{Q}_{i\cdot} \end{bmatrix} = 0$ *and* $\boldsymbol{\rho}\begin{bmatrix} -\boldsymbol{C}_{\cdot i} \\ c_{ii} \end{bmatrix} = 1$.

Step 2: In Corollary 2, full surplus can be extracted if and only if $\sum_{j\in\mathcal{N}_1} \lambda_j f_j(x_{\tilde{n}}^{FI}) \leqslant 0$. However, if $\sum_{j\in\mathcal{N}_1} \lambda_j f_j(x_{\tilde{n}}^{FI}) \leqslant 0$, (14) has *no solution* for non-negative $\boldsymbol{\rho}$. On the other hand, if $\sum_{j\in\mathcal{N}_1} \lambda_j f_j(x_{\tilde{n}}^{FI}) > 0$, then there is a solution to (14). Thus, our condition in Corollary 2 is equivalent to the RS condition.

Appendix B. Proofs

Proof of Lemma 1. Since $i \in \mathcal{N}_1$ implies $\boldsymbol{q}_i \notin co\{\boldsymbol{q}_k, k \in \mathcal{N}\}$, this result is a direct application of Farkas' Lemma which implies that if a vector $\boldsymbol{q}_i$ does not belong to the cone generated by (or is not in the convex hull of) a set of other vectors, no matter how many the other vectors there are, there exists a hyperplane separating $\boldsymbol{q}_i$ from the cone. However, to relate to the earlier results in the literature, we provide a proof starting with the CM setup. Consider an arbitrary $\boldsymbol{q}_i$ for some $i \in \mathcal{N}_1$. Parallel to Theorem 2 of CM, Farkas' Lemma implies existence of a vector $\boldsymbol{y}_i \in \mathbb{R}^S$, such that $\boldsymbol{q}_i\boldsymbol{y}_i = 0$ and $\boldsymbol{q}_k\boldsymbol{y}_i > 0$ for all $k \in \mathcal{N}_1$. All we need to do then is to show that $\boldsymbol{q}_j\boldsymbol{y}_i > 0$ for $j \in \mathcal{N}_2$ as well. Since $\boldsymbol{q}_j \in co\{\boldsymbol{q}_k, k \in \mathcal{N}_1\}$ (see Remark 1), there exist scalars $\{\lambda_k(j), k \in \mathcal{N}_1\}$, $\lambda_k(j) \geqslant 0$, $\sum_{k\in\mathcal{N}_1} \lambda_k(j) = 1$, such that $\sum_{k\in\mathcal{N}_1} \lambda_k(j)\boldsymbol{q}_k = \boldsymbol{q}_j$. Hence we have $\boldsymbol{q}_j\boldsymbol{y}_i = \sum_{k\in\mathcal{N}_1} \lambda_k(j)\boldsymbol{q}_k\boldsymbol{y}_i > 0$. □

Proof of Lemma 2. We prove the equivalence of the two schedules by showing that they represent the same total contingent transfer, contingent on the realized signals. The total contingent transfer under the original schedule is $\hat{t}_i\mathbf{1}+\hat{\boldsymbol{y}}_i$, which can be rewritten as $(\hat{t}_i+\boldsymbol{q}_i\hat{\boldsymbol{y}}_i)\mathbf{1}+(\hat{\boldsymbol{y}}_i-\boldsymbol{q}_i\hat{\boldsymbol{y}}_i\mathbf{1}) = t_i\mathbf{1} + \boldsymbol{y}_i$, which is precisely the total contingent transfer under the new schedule.

Since the signal vectors are conditional probabilities, we know $\boldsymbol{q}_j\mathbf{1} = 1\ \forall j \in \mathcal{N}$. Thus, $\boldsymbol{q}_i\boldsymbol{y}_i = \boldsymbol{q}_i\hat{\boldsymbol{y}}_i - \boldsymbol{q}_i\boldsymbol{q}_i\hat{\boldsymbol{y}}_i\mathbf{1} = 0$, i.e., $\boldsymbol{y}_i$ is a PL. □

Proof of Proposition 1. We first note that the allocation is not changed for any type. Also, types whose lotteries are already PLs in the mechanism $\{\hat{t}_i, x_i, \hat{\boldsymbol{y}}_i, i \in \mathcal{N}\}$ have the same fixed transfers and lotteries in the mechanism $\{t_i, x_i, \boldsymbol{y}_i, i \in \mathcal{N}\}$. Now, consider the case where $j \in \mathcal{N}_1$. As shown in Lemma 1 and Corollary 1, there exist SLs for θ_j, and we choose $\boldsymbol{y}_j$ to be a SL with $\|\boldsymbol{y}_j\|$ sufficiently large. If no type in the original mechanism obtains rent from having incentive to report θ_j, then the principal's expected payoff in the new mechanism equals that in the original one. If however there is a type, say θ_k, in the original mechanism who obtains rent from having incentive to report θ_j, then similar to CM, t_k can be increased from $\hat{t}_k$ and the new mechanism is a strict improvement for the principal.

Next, for the case when $j \in \mathcal{N}_2$, lottery $\boldsymbol{y}_j$ can be chosen according to Lemma 2, so that the new lottery for θ_j is a PL and the total expected payment of every type remains unchanged upon reporting θ_j. Since the allocation is not changed for any type, and since the transfers of types, if any, whose lotteries were already PLs in the original mechanism are not changed, the new mechanism must satisfy participation and incentive compatibility constraints if the original mechanism did. It follows that the principal's expected payoff in the mechanism $\{t_i, x_i, \boldsymbol{y}_i, i \in \mathcal{N}\}$ remain unchanged when $j \in \mathcal{N}_2$ and is at least as high when $j \in \mathcal{N}_1$ as compared to the expected payoff in the mechanism $\{\hat{t}_i, x_i, \hat{\boldsymbol{y}}_i, i \in \mathcal{N}\}$. □

Proof of Proposition 2. Let $\varepsilon_i > 0$ be the size of $\theta_i' s$ rent, i.e.,

$$t_i = u(x_i, \theta_i) - \varepsilon_i, \tag{15}$$

or from (C-i) in (P2)

$$z_i = f_i(x_{\tilde{n}}) - \varepsilon_i. \tag{16}$$

Suppose $\lambda_j \neq 0$, and type θ_j receives no rent but has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$. That is, $u(x_j, \theta_j) - t_j = 0$, but $u(x_j, \theta_j) - t_j > f_j(x_{\tilde{n}}) - z_j$, implying $z_j > f_j(x_{\tilde{n}})$. Let $\varepsilon_j > 0$ be such that

$$z_j = f_j(x_{\tilde{n}}) + \varepsilon_j. \tag{17}$$

We will show that this schedule cannot be optimal as the principal can improve on it.

Let $\varepsilon = \min\left[\varepsilon_i, \frac{\lambda_j}{\lambda_i}\varepsilon_j\right]$. Since $\lambda_j > 0$, we know $\varepsilon > 0$. Consider a new set of transfers$\{t'_k, k \in \mathcal{N}_1\}$ satisfying $t'_i = t_i + \varepsilon$, and $t'_k = t_k$ for all $k \in \mathcal{N}_1 \backslash i$. Since μ_i, the probability that the agent is of type θ_i, is strictly positive, if the same allocation profile can be implemented by the new transfers $\{t'_k, k \in \mathcal{N}_1\}$ and a new ELP vector, say $\boldsymbol{z}'$, that satisfies $\boldsymbol{\lambda z}' = 0$, then the original schedule cannot be optimal. Thus, all we need is to find a $\boldsymbol{z}'$ that satisfies $\boldsymbol{\lambda z}' = 0$, and together with $\{t'_k, k \in \mathcal{N}_1\}$ satisfies (C-i) for all $i \in \mathcal{N}_1$ in (P2). We break up the discussion into two cases.

Case 1: $\varepsilon_i < \frac{\lambda_j}{\lambda_i}\varepsilon_j$. Define the new ELPs $\boldsymbol{z}'$ as: $z'_i = z_i + \varepsilon_i$, $z'_j = z_j - \frac{\lambda_i}{\lambda_j}\varepsilon_i$, and $z'_k = z_k$, for all $k \in \mathcal{N}_1 \backslash \{i, j\}$. Notice first that since $\boldsymbol{\lambda z} = 0$,

$$\begin{aligned}\boldsymbol{\lambda z}' &= \sum_{k \in \mathcal{N}_1 \backslash \{i,j\}} \lambda_k z'_k + \lambda_i z'_i + \lambda_j z'_j = \sum_{k \in \mathcal{N}_1 \backslash \{i,j\}} \lambda_k z_k + \lambda_i \left[z_i + \varepsilon_i\right] + \lambda_j \left[z_j - \frac{\lambda_i}{\lambda_j}\varepsilon_i\right] \\ &= \sum_{k \in \mathcal{N}_1} \lambda_k z_k = 0.\end{aligned}$$

Now consider constraint (C-i) in (P2). Substituting t_i and z_i from (15) and (16) into $t'_i = t_i + \varepsilon = t_i + \varepsilon_i$ and $z'_i = z_i + \varepsilon_i$, we get $t'_i = u(x_i, \theta_i)$ and $z'_i = f_i(x_{\tilde{n}})$, satisfying (C-$i$).

For constraint (C-j), substituting z_j from (17) into $z'_j = z_j - \frac{\lambda_i}{\lambda_j}\varepsilon_i$, we get $z'_j = f_j(x_{\tilde{n}}) + \varepsilon_j - \frac{\lambda_i}{\lambda_j}\varepsilon_i$, or

$$f_j(x_{\tilde{n}}) - z'_j = \frac{\lambda_i}{\lambda_j}\varepsilon_i - \varepsilon_j < 0. \tag{18}$$

Since θ_j obtains zero rent under the original schedule, $t_j = u(x_j, \theta_j)$. This, together with (18), implies that $t'_j = t_j$ and z'_j satisfy (C-j).

For $k \neq i, j$, since $t'_k = t_k$ and $z'_k = z_k$, the fact that (t_k, z_k) satisfies (C-k) in (P2) implies that (t'_k, z'_k) satisfies (C-k) as well.

Note that in the new mechanism, θ_i obtains no rent and θ_j still has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$.

Case 2: $\varepsilon_i > \frac{\lambda_j}{\lambda_i}\varepsilon_j$. Define the new EPL vector $\boldsymbol{z}'$ as $z'_i = z_i + \frac{\lambda_j}{\lambda_i}\varepsilon_j$, $z'_j = z_j - \varepsilon_j$, and $z'_k = z'_k$, for all $k \in \mathcal{N}_1 \backslash i, j$. Following steps similar to case 1, we can show $\boldsymbol{\lambda z}' = 0$, and the new mechanism with $\{t'_k, k \in \mathcal{N}_1\}, \boldsymbol{z}'$ satisfies all the combined constraints (C-l), $l \in \mathcal{N}_1$, in (P2). In the new mechanism, θ_j's incentive compatibility constraint binds and θ_i still obtains positive (but lower) rent.

As long as there exists some type θ_i who gets rent and some type θ_j whose incentive compatibility constraint is not binding, we can go on applying the steps in Cases 1 or 2 until either no type gets any rent or some type gets rent but no type has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$. □

Proof of Proposition 3. Suppose in a schedule with transfers $\{t_k, k \in \mathcal{N}_1\}$ and ELPs $\boldsymbol{z}$, rent is obtained by a type θ_j where $\frac{\mu_j}{\lambda_j} \neq \min\left\{\frac{\mu_k}{\lambda_k}, k \in \mathcal{N}_1\right\}$. We show that this schedule cannot be optimal as it can be improved upon.

Let $\varepsilon_j > 0$ be the rent of type θ_j, i.e., $\varepsilon_j = u(x_j, \theta_j) - t_j > 0$. Let θ_i, $i \in \mathcal{N}_1$, be such that $\frac{\mu_i}{\lambda_i} < \frac{\mu_j}{\lambda_j}$. Consider a new schedule with transfers $\{t'_k; k \in \mathcal{N}_1\}$ and ELPs $\boldsymbol{z}'$, where $t'_i = t_i - \frac{\lambda_j}{\lambda_i}\varepsilon_j$, $t'_j = t_j + \varepsilon_j$, $z'_i = z_i - \frac{\lambda_j}{\lambda_i}\varepsilon_j$, $z'_j = z_j + \varepsilon_j$, and $t'_k = t_k$ and $z'_k = z_k$ for $k \in \mathcal{N}_1 \backslash i, j$. Similar to the proof of Proposition 2, we can check that the new schedule satisfies all the combined constraints (C-l), $l \in \mathcal{N}_1$, in (P2). Further,

$$\boldsymbol{\lambda z}' = \sum_{k \in \mathcal{N}_1 \backslash i, j} \lambda_k z'_k + \lambda_i z'_i + \lambda_j z'_j = \sum_{k \in \mathcal{N}_1 \backslash i, j} \lambda_k z_k + \lambda_i \left[z_i - \frac{\lambda_j}{\lambda_i}\varepsilon_j\right] + \lambda_j \left[z_j + \varepsilon_j\right]$$
$$= \sum_{k \in \mathcal{N}_1} \lambda_k z_k = 0.$$

Thus $\{t'_k; k \in \mathcal{N}_1\}$ and $\boldsymbol{z}'$ implement the allocation profile $\{x_k, k \in \mathcal{N}\}$. Finally, the difference in the principal's payoff under the two mechanism is

$$\sum_{k \in \mathcal{N}_1} \mu_k t'_k - \sum_{k \in \mathcal{N}_1} \mu_k t_k = -\mu_i \frac{\lambda_j}{\lambda_i}\varepsilon_j + \mu_j \varepsilon_j = \lambda_j \varepsilon_j \left[\frac{\mu_j}{\lambda_j} - \frac{\mu_i}{\lambda_i}\right] > 0,$$

where the inequality follows since $\frac{\mu_j}{\lambda_j} > \frac{\mu_i}{\lambda_i}$ $\lambda_j > 0$, and $\varepsilon_j > 0$. Thus, the mechanism in which θ_j gets rent is not optimal. □

Proof of Proposition 4. Let $\mathcal{I} = \left\{i \in \mathcal{N}_1, \frac{\mu_i}{\lambda_i} = \min\left\{\frac{\mu_k}{\lambda_k}, k \in \mathcal{N}_1\right\}\right\}$. The total expected rent is zero if no types get any rent, and is equal to $\sum_{i \in \mathcal{I}} \mu_i \left[f_i(x_{\tilde{n}}) - z_i\right]$ when the agent gets rent in the optimal mechanism. Note that

$$\sum_{i \in \mathcal{I}} \mu_i \left[f_i(x_{\tilde{n}}) - z_i\right] = \sum_{i \in \mathcal{I}} \frac{\mu_i}{\lambda_i} \left[\lambda_i f_i(x_{\tilde{n}}) - \lambda_i z_i\right] = \eta \left(\sum_{i \in \mathcal{I}} \left[\lambda_i f_i(x_{\tilde{n}}) - \lambda_i z_i\right]\right). \tag{19}$$

Recall that $\boldsymbol{\lambda z} = \sum_{i \in \mathcal{I}} \lambda_i z_i + \sum_{j \in \mathcal{N}_1 \backslash \mathcal{I}} \lambda_j z_j = 0$. Since the incentive compatibility constraint is binding for every type (from Proposition 2) and since the rent is zero for all θ_j, $j \in \mathcal{N}_1 \backslash \mathcal{I}$, we know that $z_j = f_j(x_{\tilde{n}})$, for all $j \in \mathcal{N}_1 \backslash \mathcal{I}$. Hence, we have $\sum_{i \in \mathcal{I}} \lambda_i z_i = -\sum_{j \in \mathcal{N}_1 \backslash \mathcal{I}} \lambda_j z_j = -\sum_{j \in \mathcal{N}_1 \backslash \mathcal{I}} \lambda_j f_j(x_{\tilde{n}})$. Substituting this into (19), we obtain the expression in the Proposition.

To show that the total expected rent is a continuous function of $\boldsymbol{\lambda}$, notice that $\Psi_i(\boldsymbol{\lambda}) \equiv \frac{\mu_i}{\lambda_i} \sum_{k \in \mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}})$ is continuous in $\boldsymbol{\lambda}$ (with $\sum_{k \in \mathcal{N}_1} \lambda_k = 1$) for all $i \in \mathcal{N}_1$. Then $\Psi(\boldsymbol{\lambda}) \equiv \min_{i \in \mathcal{N}_1} \left[\Psi_i(\boldsymbol{\lambda})\right]$ is also continuous in $\boldsymbol{\lambda}$. Thus the total expected rent, $\max\{0, \Psi(\boldsymbol{\lambda})\}$ is a continuous function of $\boldsymbol{\lambda}$. □

Proof of Proposition 5. From Proposition 4, we know that if full rent cannot be extracted, then the total rent is $\sum_{k \in \mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}})$, which by definition must be strictly positive. Thus if $\sum_{k \in \mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}}) \leqslant 0$, full rent can be extracted.

Conversely, we now show that if full rent can be extracted then $\sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}}) \leqslant 0$. Suppose not, i.e., suppose $\sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}}) > 0$. We show this leads to a contradiction.

Since full rent can be extracted, i.e., θ_k gets no rent $\forall k \in \mathcal{N}_1$, we know $z_k \geqslant f_k(x_{\tilde{n}})$, $\forall k \in \mathcal{N}_1$. Since $\lambda_k \geqslant 0$, for all $k \in \mathcal{N}_1$, $\lambda_k z_k \geqslant \lambda_k f_k(x_{\tilde{n}})$, which implies $\sum_{k\in\mathcal{N}_1} \lambda_k z_k \geqslant \sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}})$. Thus $\sum_{k\in\mathcal{N}_1} \lambda_k f_k(x_{\tilde{n}}) > 0$ implies $\sum_{k\in\mathcal{N}_1} \lambda_k z_k > 0$, which contradicts the condition $\boldsymbol{\lambda z} = 0$. $\square$

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