Appendix C: Technical Results for $N = S + 2$

In this appendix, we present the technical results for the case of $N = S+2$, including the proofs of the propositions in Section 6 and some additional results to characterize the optimal mechanism. Note that for the most part, we use the Kuhn-Tucker technique to obtain our results. For example, results characterizing types that obtain rent or types that have slack incentive compatibility constraints in the optimal solution follow from the respective values of the Lagrangian multipliers. Similarly, the number of types who either obtain rent or have slack incentive compatibility constraints arises from the number of constraints on the values of the Lagrangian multipliers.

C.1 $\mathcal{N}_2$ is a singleton

We first show that the choice of the S basis vectors does not affect the search for the optimal mechanism; in particular, choosing different basis vectors does not affect the feasible set of the choice variables $\{z_i; i \in \mathcal{N}_1\}$ in (P3).

Proposition C1 *Suppose there are two sets of basis vectors in $\{\boldsymbol{q}_i, i \in \mathcal{N}_1\}$, the convex hull of each contains $\boldsymbol{q}_{\tilde{n}}$. That is, there exist $\mathcal{N}_b \subset \mathcal{N}_1$ and $\mathcal{N}_{b'} \subset \mathcal{N}_1$ such that both $\{\boldsymbol{q}_i, i \in \mathcal{N}_b\}$ and $\{\boldsymbol{q}_j, j \in \mathcal{N}_{b'}\}$ form bases for $\mathbb{R}^S$, and there exist $\boldsymbol{\lambda} = (\lambda_i, i \in \mathcal{N}_b) \geq 0$ and $\boldsymbol{\lambda}' = (\lambda'_j, j \in \mathcal{N}_{b'}) \geq 0$ such that $\boldsymbol{q}_{\tilde{n}} = \boldsymbol{\lambda} \boldsymbol{Q}_{\mathcal{N}_b}$ and $\boldsymbol{q}_{\tilde{n}} = \boldsymbol{\lambda}' \boldsymbol{Q}_{\mathcal{N}_{b'}}$. Then using the two sets of basis vectors in constructing the optimization problem (P3) results in the same optimal mechanism.*

Proof. The objective function in problem (P3) is independent of the choice of the basis vectors. We only need to show that using different sets of basis vectors results in the same restrictions on $\{z_i, i \in \mathcal{N}_1\}$ where $z_i = \boldsymbol{q}_i \boldsymbol{y}_{\tilde{n}}$, $i \in \mathcal{N}_1$.

As before, we continue to let $m = \mathcal{N}_1 \setminus \mathcal{N}_b$ and let $\{\gamma_i, i \in \mathcal{N}_b\}$ be the unique scalars such that

$$\sum_{i \in \mathcal{N}_b} \gamma_i \boldsymbol{q}_i = \boldsymbol{q}_m. \tag{20}$$

Similarly, let $m' = \mathcal{N}_1 \setminus \mathcal{N}_{b'}$ and let $\{\gamma_i', i \in \mathcal{N}_{b'}\}$ be the unique scalars such that

$$\sum_{i \in \mathcal{N}_{b'}} \gamma_i' \boldsymbol{q}_i = \boldsymbol{q}_{m'}. \tag{21}$$

Since $\mathcal{N}_1$ has exactly $S+1$ elements, we know $m' \in \mathcal{N}_b$, $m \in \mathcal{N}_{b'}$, and $\mathcal{N}_1 = m \cup \mathcal{N}_b = m' \cup \mathcal{N}_{b'}$.

We showed in formulating problem (P3) that (20), together with

$$\sum_{i \in \mathcal{N}_b} \lambda_i \boldsymbol{q}_i = \boldsymbol{q}_{\tilde{n}}, \tag{22}$$

leads to the following constraints on $\{z_i, i \in \mathcal{N}_1\}$ in (P3):

$$\sum_{i \in \mathcal{N}_b} \lambda_i z_i = 0, \tag{23}$$

$$\sum_{i \in \mathcal{N}_b} \gamma_i z_i = z_m. \tag{24}$$

Similarly, (21), together with

$$\sum_{i \in \mathcal{N}_{b'}} \lambda_i' \boldsymbol{q}_i = \boldsymbol{q}_{\tilde{n}}, \tag{25}$$

implies the following constraints:

$$\sum_{i \in \mathcal{N}_{b'}} \lambda_i' z_i = 0, \tag{26}$$

$$\sum_{i \in \mathcal{N}_{b'}} \gamma_i' z_i = z_{m'}. \tag{27}$$

To prove the Proposition, we need to show that (26) and (27) imply (23) and (24).

Step 1: (27) implies (24). Substituting the value of $\boldsymbol{q}_m$ from (20) into (21) and adjusting, we get

$$\sum_{i \in \mathcal{N}_{b'} \setminus m} \left(\gamma_i' + \gamma_m' \gamma_i\right) \boldsymbol{q}_i + \left(\gamma_m' \gamma_{m'} - 1\right) \boldsymbol{q}_{m'} = 0. \tag{28}$$

Since $(\mathcal{N}_{b'} \setminus m) \cup m' = \mathcal{N}_b$, the vectors $\{q_j, j \in (\mathcal{N}_{b'} \setminus m) \cup m'\}$ are linearly independent. Then (28) implies

$$\begin{aligned} \gamma_m' &= \frac{1}{\gamma_{m'}}, \\ \gamma_i' &= -\gamma_m' \gamma_i = -\frac{\gamma_i}{\gamma_{m'}}, \quad i \in \mathcal{N}_{b'} \setminus m. \end{aligned}$$

Substituting these into (27), we get

$$\sum_{i\in\mathcal{N}_{b'}\backslash m} -\frac{\gamma_i}{\gamma_{m'}} z_i + \frac{z_m}{\gamma_{m'}} = z_{m'}.$$

Multiplying both sides by $\gamma_{m'}$ and rearranging, we get (24).

Step 2: (26) implies (23). Substituting $\boldsymbol{q}_m$ from (20) into (25), we get

$$\sum_{i\in\mathcal{N}_{b'}\backslash m} \left(\lambda_i' + \lambda_m'\gamma_i\right) \boldsymbol{q}_i + \lambda_m'\gamma_{m'}\boldsymbol{q}_{m'} = \boldsymbol{q}_{\tilde{n}}.$$

Since $(\mathcal{N}_{b'}\backslash m) \cup m' = \mathcal{N}_b$, this equation and (22) imply that

$$\begin{aligned} \lambda_i &= \lambda_i' + \lambda_m'\gamma_i, \quad i \in \mathcal{N}_{b'}\backslash m \\ \lambda_{m'} &= \lambda_m'\gamma_{m'} \end{aligned}$$

Hence, we have,

$$\begin{aligned} \sum_{i\in\mathcal{N}_{b'}} \lambda_i' z_i &= \sum_{i\in\mathcal{N}_{b'}\backslash m} \left(\lambda_i - \lambda_m'\gamma_i\right) z_i + \lambda_m' z_m \\ &= \sum_{i\in\mathcal{N}_b} \lambda_i z_i - \lambda_{m'} z_{m'} - \sum_{i\in\mathcal{N}_{b'}\backslash m} \lambda_m'\gamma_i z_i + \lambda_m' z_m \\ &= \sum_{i\in\mathcal{N}_b} \lambda_i z_i - \lambda_m'\gamma_{m'} z_{m'} - \sum_{i\in\mathcal{N}_{b'}\backslash m} \lambda_m'\gamma_i z_i + \lambda_m' z_m \\ &= \sum_{i\in\mathcal{N}_b} \lambda_i z_i - \lambda_m' \left\{ z_m - \left[\sum_{i\in\mathcal{N}_{b'}\backslash m} \gamma_i z_i + \gamma_{m'} z_{m'} \right] \right\} \\ &= \sum_{i\in\mathcal{N}_b} \lambda_i z_i - \lambda_m' \left\{ z_m - \sum_{i\in\mathcal{N}_b} \gamma_i z_i \right\} \\ &= \sum_{i\in\mathcal{N}_b} \lambda_i z_i \end{aligned}$$

where we use the fact that $(\mathcal{N}_{b'}\backslash m) \cup m' = \mathcal{N}_b$ in the second and second to last equalities. ■

Before proving Proposition 7, we provide a lemma characterizing the type in $\mathcal{N}_b$ who obtains rent. Let the Lagrangian for problem (P3) be

$$\begin{aligned} L = \sum_{i\in\mathcal{N}_1} \mu_i \left[W(x_i) + u(x_i,\theta_i) - \max\{0, f_i(x_{\tilde{n}}) - z_i\}\right] \\ + \mu_{\tilde{n}} \left[W(x_{\tilde{n}}) + u(x_{\tilde{n}},\theta_{\tilde{n}})\right] - \alpha \sum_{i\in\mathcal{N}_1} \lambda_i z_i - \beta(z_m - \sum_{i\in\mathcal{N}_1} \gamma_i z_i), \end{aligned}$$

where α and $-\beta$ are the Lagrangian multipliers of constraints $\boldsymbol{\lambda z} = 0$ and $\boldsymbol{\gamma z} = z_m$ respectively.

Lemma C1 *Suppose allocation $x_{\tilde{n}}$ is given and in the optimal mechanism, θ_i, $i \in \mathcal{N}_b$, obtains rent. Then we must have*

$$\frac{\mu_i}{\lambda_i} + \beta\frac{\gamma_i}{\lambda_i} = \min\left\{\frac{\mu_j}{\lambda_j} + \beta\frac{\gamma_j}{\lambda_j}, j \in \mathcal{N}_b\right\}. \tag{29}$$

Proof. The Lagrangian is not differentiable everywhere; however it is piecewise linear in $\boldsymbol{z}$ and z_m with kinks appearing only where $z_i = f_i(x_{\tilde{n}})$ and $z_m = f_m(x_{\tilde{n}})$. Hence the necessary conditions for an optimal solution are given by:

$$\begin{cases} \alpha\lambda_i - \beta\gamma_i = \mu_i & \text{if } z_i < f_i(x_{\tilde{n}}) \\ \alpha\lambda_i - \beta\gamma_i = 0 & \text{if } z_i > f_i(x_{\tilde{n}}) \\ 0 \leq \alpha\lambda_i - \beta\gamma_i \leq \mu_i & \text{if } z_i = f_i(x_{\tilde{n}}) \end{cases} \quad i \in \mathcal{N}_b \tag{30}$$

and

$$\begin{cases} \beta = \mu_m & \text{if } z_m < f_m(x_{\tilde{n}}) \\ \beta = 0 & \text{if } z_m > f_m(x_{\tilde{n}}) \\ \beta \in [0, \mu_m] & \text{if } z_m = f_m(x_{\tilde{n}}) \end{cases} \tag{31}$$

If θ_i obtains rent, we know $z_i < f_i(x_{\tilde{n}})$, which from (30) implies

$$\frac{\mu_i}{\lambda_i} + \beta\frac{\gamma_i}{\lambda_i} = \alpha. \tag{32}$$

For any other type θ_j who does not obtain rent, (30) shows that there are two possibilities: $\alpha\lambda_j - \beta\gamma_j = 0$, implying

$$\frac{\mu_j}{\lambda_j} + \beta\frac{\gamma_j}{\lambda_j} = \alpha + \frac{\mu_j}{\lambda_j} > \alpha, \tag{33}$$

or $\alpha\lambda_j - \beta\gamma_j \leq \mu_j$, implying

$$\frac{\mu_j}{\lambda_j} + \beta\frac{\gamma_j}{\lambda_j} \geq \alpha. \tag{34}$$

(32) - (34) imply (29). ■

Proposition 7 *Suppose allocation $x_{\tilde{n}}$ is given and full rent cannot be extracted. In the optimal* *mechanism, generically there is at most one type in $\mathcal{N}_b$ who can obtain rent. In particular,*

(a) If θ_m has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$, i.e., if $z_m > f_m(x_{\tilde{n}})$, then type $\theta_i, i \in \mathcal{N}_b$, can obtain rent (i.e. $z_i < f_i(x_{\tilde{n}})$) only if

$$\frac{\mu_i}{\lambda_i} = \min\left\{\frac{\mu_k}{\lambda_k}, k \in \mathcal{N}_b\right\};$$

(b) If θ_m obtains rent, i.e., $z_m < f_m(x_{\tilde{n}})$, then θ_i, $i \in \mathcal{N}_b$, can obtain rent only if

$$\frac{\mu_i}{\lambda_i} + \mu_m\frac{\gamma_i}{\lambda_i} = \min\left\{\frac{\mu_k}{\lambda_k} + \mu_m\frac{\gamma_k}{\lambda_k}, k \in \mathcal{N}_b\right\}$$

Proof. The Proposition follows almost immediately from Lemma C1.

Case (a): $z_m > f_m(x_{\tilde{n}})$. From (31), we know $\beta = 0$. This case then immediately follows from Lemma C1. Further, if two types, say θ_i and θ_j, both obtain rents, we have $\mu_i/\lambda_i = \mu_j/\lambda_j$, a non-generic case. In other words, generically θ_i is the only type in $\mathcal{N}_b$ who obtains rent.

Case (b): $z_m < f_m(x_{\tilde{n}})$. From (31), we know $\beta = \mu_m$, and again this case then follows immediately from Lemma C1. Further, if two types, say θ_i and θ_j, both obtain rents, it must be that $\mu_i/\lambda_i + \mu_m\gamma_i/\lambda_i = \mu_j/\lambda_j + \mu_m\gamma_j/\lambda_j$, a non-generic case. ■

Note that there is a third case, of $z_m = f_m(x_{\tilde{n}})$, that Proposition 7 does not cover. The following proposition gives the value of β and shows which type obtains rent in this case.

Proposition C2 *Suppose allocation $x_{\tilde{n}}$ is given and full rent cannot be extracted. Suppose also that in the optimal mechanism, $z_m = f_m(x_{\tilde{n}})$. Then generically there are two possible situations.*

(a) There are two types, θ_i and θ_j, $i, j \in \mathcal{N}_b$, such that θ_i obtains rent and θ_j has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$: $z_i < f_i(x_{\tilde{n}})$ and $z_j > f_j(x_{\tilde{n}})$, while $z_k = f_k(x_{\tilde{n}})$ for all $k \neq i, j$, $k \in \mathcal{N}_b$. In this case,

$$\beta = \frac{\mu_i/\lambda_i}{\gamma_j/\lambda_j - \gamma_i/\lambda_i}. \quad (35)$$

(b) There are two types, θ_i and θ_j, $i, j \in \mathcal{N}_b$, who obtain rent, i.e., $z_i < f_i(x_{\tilde{n}})$ and $z_j < f_j(x_{\tilde{n}})$,

while $z_k = f_k(x_{\tilde{n}})$ *for all* $k \neq i, j$, $k \in \mathcal{N}_b$. *In this case,*

$$\beta = \frac{\mu_j/\lambda_j - \mu_i/\lambda_i}{\gamma_i/\lambda_i - \gamma_j/\lambda_j}. \tag{36}$$

Proof. From (30) and (31), the value of β is entirely determined by conditions in (30) when $z_m = f_m(x_{\tilde{n}})$. For each type in $\mathcal{N}_b$ that obtains rent or has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$, there is one equation in (30) (the first or the second equation) that imposes a condition on β and α, the two unknowns. Since there are only two unknowns, generically for there to be a solution, there can at most be two such equations. That is, generically only cases (a) and (b) of the Proposition are possible.

Case (a): From (30), the two conditions determining α and β are $\alpha\lambda_i - \mu_i - \beta\gamma_i = 0$ and $\alpha\lambda_j - \beta\gamma_j = 0$. Solving for α and β, we get (35).

Case (b): From (30), the two conditions determining α and β are $\alpha\lambda_i - \mu_i - \beta\gamma_i = 0$ and $\alpha\lambda_j - \mu_j - \beta\gamma_j = 0$. Solving for α and β, we get (36).

Further, in both cases we can verify that $\beta \in [0, \mu_m]$. We sketch the argument for $\beta \leq \mu_m$ in case (a); the other case can be shown similarly. So, suppose in case (a), the value of β in (35) is strictly greater than μ_m. We will show a contradiction.

From (35), $\beta > \mu_m$ implies $\frac{\mu_i}{\lambda_i} + \mu_m \frac{\gamma_i}{\lambda_i} > \mu_m \frac{\gamma_j}{\lambda_j}$. Suppose the principal raises z_i by a small amount dz_i, reduces z_j by a corresponding small amount dz_j, and keeps all the other $z_k, k \in \mathcal{N}_b \backslash i, j$ fixed, so as to continue to satisfy $\boldsymbol{\lambda z} = 0$. Since $z_j > f_j(x_{\tilde{n}})$, a small reduction in z_j does not affect $\theta_j' s$ (zero) rent, and the impact of this change on the principal's expected payoff is equal to $\mu_i dz_i + \mu_m dz_m$. From the two constraints, $\lambda_i dz_i + \lambda_j dz_j = 0$ and $\gamma_i dz_i + \gamma_j dz_j = dz_m$, we can solve for dz_m as a function of dz_i, and substituting it into $\mu_i dz_i + \mu_m dz_m$, we get the change in the principal's expected payoff as $\left(\frac{\mu_i}{\lambda_i} + \mu_m \frac{\gamma_i}{\lambda_i} - \mu_m \frac{\gamma_j}{\lambda_j}\right)\left(\frac{dz_i}{\lambda_i}\right) > 0$. Thus if $\beta > \mu_m$ the mechanism cannot be optimal. ■

We next prove Proposition 8.

Proposition 8 *Generically, in the optimal mechanism,*

(a) There cannot be more than two types who obtain rent.

(b) If two types obtain rent, then there cannot be a type whose incentive compatibility constraint of reporting $\theta_{\tilde{n}}$ *is slack.*

(c) If full rent cannot be extracted, there can be at most one type who has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$.

Proof. From (30) and (31), there is one equation or restriction on the values of α and β corresponding to each type in $\mathcal{N}_1 = \mathcal{N}_b \bigcup m$ who either obtains rent or has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$. Since there are two unknowns, α and β, generically there cannot be three or more such equations. The Proposition follows since whenever any of the three conditions (a),(b) or (c) are violated we have more than three equations determining the value of the two variables α and β. ■

The type in Proposition 8(c), i.e., the one who has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}}$, can either be θ_m or any other type from $\mathcal{N}_b$. In the latter case, this type can be identified as follows.

Proposition C3 *Suppose* $z_m \leq f_m(x_{\tilde{n}})$ *and* $\beta > 0$. *Then type* θ_i, $i \in \mathcal{N}_b$, *satisfies* $z_i > f_i(x_{\tilde{n}})$ *only if*

$$\frac{\gamma_i}{\lambda_i} = \max\left\{\frac{\gamma_j}{\lambda_j}, j \in \mathcal{N}_b\right\}. \tag{37}$$

Proof. Since $\beta > 0$ and $z_i > f_i(x_{\tilde{n}})$, (30) implies that $\alpha = \beta\frac{\gamma_i}{\lambda_i}$ while $\alpha \geq \beta\frac{\gamma_j}{\lambda_j}$ for $j \in \mathcal{N} \setminus i$. Then (37) follows. ■

C.2 $\mathcal{N}_2$ contains two elements

We first show that both $\theta_{\tilde{n}_1}$ and $\theta_{\tilde{n}_2}$ cannot obtain rents. In particular, if $\theta_{\tilde{n}_1}$ obtains rent, $\theta_{\tilde{n}_2}$ must have a slack incentive compatibility constraint.

Proposition C4 *If $\theta_{\tilde{n}_1}$ obtains rent, i.e., if $z_{\tilde{n}_1} < f_{\tilde{n}_1}(x_{\tilde{n}_2})$, $\theta_{\tilde{n}_2}$ cannot obtain any rent. Further, $\theta_{\tilde{n}_2}$'s incentive compatibility constraint of reporting $\theta_{\tilde{n}_1}$ must be slack. That is, $z_{\tilde{n}_2} > f_{\tilde{n}_2}(x_{\tilde{n}_1})$.*

Proof. The incentive compatibility constraints in (P1) between $\theta_{\tilde{n}_1}$ and $\theta_{\tilde{n}_2}$ are, in terms of the z's, $u(x_{\tilde{n}_1}, \theta_{\tilde{n}_1}) - t_{\tilde{n}_1} \geq u(x_{\tilde{n}_2}, \theta_{\tilde{n}_1}) - t_{\tilde{n}_2} - z_{\tilde{n}_1}$, and $u(x_{\tilde{n}_2}, \theta_{\tilde{n}_2}) - t_{\tilde{n}_2} \geq u(x_{\tilde{n}_1}, \theta_{\tilde{n}_2}) - t_{\tilde{n}_1} - z_{\tilde{n}_2}$. Adding these two inequalities and adjusting, we get $[u(x_{\tilde{n}_2}, \theta_{\tilde{n}_1}) - u(x_{\tilde{n}_2}, \theta_{\tilde{n}_2}) - z_{\tilde{n}_1}] + [u(x_{\tilde{n}_1}, \theta_{\tilde{n}_2}) - u(x_{\tilde{n}_1}, \theta_{\tilde{n}_1}) - z_{\tilde{n}_2}] \leq 0$, which is $[f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}] + [f_{\tilde{n}_2}(x_{\tilde{n}_1}) - z_{\tilde{n}_2}] \leq 0$. Therefore, if $z_{\tilde{n}_1} < f_{\tilde{n}_1}(x_{\tilde{n}_2})$, it must be that $z_{\tilde{n}_2} > f_{\tilde{n}_2}(x_{\tilde{n}_1})$, and thus $\theta_{\tilde{n}_2}$ does not obtain rent. ∎

Similar to Section C.1, we form the Lagrangian of the optimization problem in (P4):

$$\begin{aligned} L = & \sum_{i \in \mathcal{N}_1} \mu_i \left[W(x_i) + u(x_i, \theta_i) - \max\{0,\ f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}),\ f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2)\} \right] \\ & + \mu_{\tilde{n}_1} \left[W(x_{\tilde{n}_1}) + u(x_{\tilde{n}_1}, \theta_{\tilde{n}_1}) - (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}) \right] + \mu_{\tilde{n}_2} \left[W(x_{\tilde{n}_2}) + u(x_{\tilde{n}_2}, \theta_{\tilde{n}_2}) \right] \\ & - \alpha(\tilde{n}_1) \sum_{i \in \mathcal{N}_1} \lambda_i(\tilde{n}_1) z_i(\tilde{n}_1) - \beta(\tilde{n}_1)(z_{\tilde{n}_2} - \sum_{i \in \mathcal{N}_1} \lambda_i(\tilde{n}_2) z_i(\tilde{n}_1)) \\ & - \alpha(\tilde{n}_2) \sum_{i \in \mathcal{N}_1} \lambda_i(\tilde{n}_2) z_i(\tilde{n}_2) - \beta(\tilde{n}_2)(z_{\tilde{n}_1} - \sum_{i \in \mathcal{N}_1} \lambda_i(\tilde{n}_1) z_i(\tilde{n}_2)) \end{aligned}$$

where $\alpha(\tilde{n}_i)$ and $-\beta(\tilde{n}_i)$ are the Lagrangian multipliers of constraints $\boldsymbol{\lambda}(\tilde{n}_i)\boldsymbol{z}(\tilde{n}_i) = 0$ and $\boldsymbol{\lambda}(\tilde{n}_j)\boldsymbol{z}(\tilde{n}_i) = z_{\tilde{n}_j}$ respectively, $i, j = 1, 2,\ j \neq i$.

Proposition 9 *Suppose allocations $x_{\tilde{n}_1}$ and $x_{\tilde{n}_2}$ are given. In the optimal mechanism,*

(a) If θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_1}$, then it must be that θ_i has the lowest $\mu/\lambda(\tilde{n}_1)$:

$$i = \arg\min_{j \in \mathcal{N}_1} \left\{ \frac{\mu_j}{\lambda_j(\tilde{n}_1)} \right\}.$$

(b) If θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_2}$, then it must be that

$$i = \arg\min_{j \in \mathcal{N}_1} \left\{ \frac{\mu_j}{\lambda_j(\tilde{n}_2)} + \left(\mu_{\tilde{n}_1} + \sum_{k \in \mathcal{N}_1(\tilde{n}_1)} \mu_k \right) \frac{\lambda_j(\tilde{n}_1)}{\lambda_j(\tilde{n}_2)} \right\},$$

where $\mathcal{N}_1(\tilde{n}_1) \subseteq \mathcal{N}_1$ is the index set of the types who obtain rent from having incentive to report $\theta_{\tilde{n}_1}$.

Proof. Again, the Lagrangian is piecewise linear in all the z_i's. Similar to the proof of Lemma C1, a set of conditions must be satisfied for there to be an optimal solution. The difference is that here we have two sets of z_i's, i.e., $z_i(\tilde{n}_1)$'s and $z_i(\tilde{n}_2)$'s, for the types in $\mathcal{N}_1$, and we are only exploring the case when $\theta_{\tilde{n}_1}$ obtains rent while $\theta_{\tilde{n}_2}$ has a slack incentive compatibility constraint. The necessary conditions are

$$\begin{cases} \alpha(\tilde{n}_1)\lambda_i(\tilde{n}_1) - \beta(\tilde{n}_1)\lambda_i(\tilde{n}_2) = \mu_i, & \text{if } (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}) > \max\{0, f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2)\}, \\ \alpha(\tilde{n}_1)\lambda_i(\tilde{n}_1) - \beta(\tilde{n}_1)\lambda_i(\tilde{n}_2) = 0, & \text{if } (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}) < \max\{0, f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2)\}, \\ \alpha(\tilde{n}_1)\lambda_i(\tilde{n}_1) - \beta(\tilde{n}_1)\lambda_i(\tilde{n}_2) \in [0, \mu_i], & \text{if } (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1}) = \max\{0, f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2)\}, \end{cases} \quad i \in \mathcal{N}_1, \tag{38}$$

$$\begin{cases} \alpha(\tilde{n}_2)\lambda_i(\tilde{n}_2) - \beta(\tilde{n}_2)\lambda_i(\tilde{n}_1) = \mu_i, & \text{if } f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2) > \max\{0, (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1})\}, \\ \alpha(\tilde{n}_2)\lambda_i(\tilde{n}_2) - \beta(\tilde{n}_2)\lambda_i(\tilde{n}_1) = 0, & \text{if } f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2) < \max\{0, (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1})\}, \\ \alpha(\tilde{n}_2)\lambda_i(\tilde{n}_2) - \beta(\tilde{n}_2)\lambda_i(\tilde{n}_1) \in [0, \mu_i], & \text{if } f_i(x_{\tilde{n}_2}) - z_i(\tilde{n}_2) = \max\{0, (f_i(x_{\tilde{n}_1}) - z_i(\tilde{n}_1)) + (f_{\tilde{n}_1}(x_{\tilde{n}_2}) - z_{\tilde{n}_1})\}, \end{cases} \quad i \in \mathcal{N}_1, \tag{39}$$

$$\begin{cases} \beta(\tilde{n}_2) & = \mu_{\tilde{n}_1} + \sum_{i \in \mathcal{N}_1(\tilde{n}_1)} \mu_i \\ \beta(\tilde{n}_1) & = 0, \end{cases} \tag{40}$$

where (38) and (39) are derived from $\partial L / \partial z_i(\tilde{n}_1)$ and $\partial L / \partial z_i(\tilde{n}_2)$, and (40) is derived from $\partial L / \partial z_{\tilde{n}_1}$ and $\partial L / \partial z_{\tilde{n}_2}$. Although (38) and (39) are rather messy, they are parallel to (30) of the last section. Equation (40) is parallel to (31), but is simpler since here $\theta_{\tilde{n}_1}$ obtains rent and $\theta_{\tilde{n}_2}$ has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_1}$.

Repeating the same procedure as in proving Lemma C1, part (a) of the Proposition is obtained by substituting $\beta(\tilde{n}_1) = 0$ into (38), and part (b) is obtained by substituting the value of $\beta(\tilde{n}_2)$ in (40) into (39). ■

Proposition 10 *Generically in the optimal mechanism, for $k = 1, 2$, (a) there is at most one type in $\mathcal{N}_1$ who can obtain rent from having incentive to report $\theta_{\tilde{n}_k}$;*

(b) if a type obtains rent from having incentive to report $\theta_{\tilde{n}_k}$, there cannot be a type (with a nonzero $\lambda(\tilde{n}_k)$) whose incentive compatibility constraint of reporting $\theta_{\tilde{n}_k}$ is slack.

Proof. When $k = 1$, substituting $\beta(\tilde{n}_1) = 0$ into (38), we know that if two types, say θ_i and θ_j, $i, j \in \mathcal{N}_1$, obtain rent from having incentive to report $\theta_{\tilde{n}_1}$, it must be that $\alpha(\tilde{n}_1) = \mu_i / \lambda_i(\tilde{n}_1) = \mu_j / \lambda_j(\tilde{n}_1)$, a non-generic case. Similarly, if θ_i obtains rent, we have $\alpha(\tilde{n}_1) = \mu_i / \lambda_i(\tilde{n}_1) \neq 0$. If θ_j has a slack incentive compatibility constraint and $\lambda_j(\tilde{n}_1) \neq 0$, we know $\alpha(\tilde{n}_1) = 0$, which is a contradiction.

The proof is exactly the same when $k = 2$. ■

Essentially, the Proposition is based on the number of constraints that will be imposed on the values of the α's and β's, similar to Section C.1. Since we have assumed that $\theta_{\tilde{n}_1}$ obtains rent and $\theta_{\tilde{n}_2}$ has a slack incentive compatibility constraint, the values of the β's are fixed. That leaves only the values of the α's to be determined, limiting the number of restrictions that can be imposed.

The following proposition extends Proposition 10.

Proposition C5 *Generically, in the optimal mechanism,*

(a) there cannot be two or more types in $\mathcal{N}_1$ *that have slack incentive compatibility constraints of reporting* $\theta_{\tilde{n}_2}$.

(b) If a type θ_i, $i \in \mathcal{N}_1$, *has a slack incentive compatibility constraints of reporting* $\theta_{\tilde{n}_2}$, *it must be that*

$$\frac{\lambda_i(\tilde{n}_1)}{\lambda_i(\tilde{n}_2)} = \max\left\{\frac{\lambda_j(\tilde{n}_1)}{\lambda_j(\tilde{n}_2)}, j \in \mathcal{N}_1\right\}.$$

Proof. Part (a): From (39) and the value of $\beta(\tilde{n}_2)$ in (40), if there are two types, say θ_i and θ_j, $i, j \in \mathcal{N}_1$, who have slack incentive compatibility constraints of reporting $\theta_{\tilde{n}_2}$, it must be that

$$\alpha(\tilde{n}_2)/\beta(\tilde{n}_2) = \lambda_i(\tilde{n}_1)/\lambda_i(\tilde{n}_2) = \lambda_j(\tilde{n}_1)/\lambda_j(\tilde{n}_2), \tag{41}$$

which is non-generic.

Part (b): From (39), if a type θ_j, $j \neq i$, $j \in \mathcal{N}_1$, does not have a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_2}$, it must be that $\alpha(\tilde{n}_2)\lambda_j(\tilde{n}_2) - \beta(\tilde{n}_2)\lambda_j(\tilde{n}_1) \geq 0$, or $\frac{\lambda_j(\tilde{n}_1)}{\lambda_j(\tilde{n}_2)} \leq \alpha(\tilde{n}_2)/\beta(\tilde{n}_2)$. This and (41) imply part (b) of the Proposition. ■

If a type θ_i, $i \in \mathcal{N}_1$, obtains rent from having incentive to report $\theta_{\tilde{n}_1}$, it must have a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_2}$. Proposition C5 then indicates that there cannot be another type in $\mathcal{N}_1$ who has a slack incentive compatibility constraint of reporting $\theta_{\tilde{n}_2}$. This observation, together with Proposition10(b) when $k = 2$, implies that

Corollary C1 *If there is a type in* $\mathcal{N}_1$ *who obtains rent in the optimal mechanism, generically there is no other type in* $\mathcal{N}_1$ *who has a slack incentive compatibility constraint of reporting* $\theta_{\tilde{n}_2}$.