INCOMPLETE INFORMATION AGGREGATION GAMES

GORDON C. RAUSSER, LEO K. SIMON, AND JINHUA ZHAO

ABSTRACT. This paper concerns a class of collective decision-making problems under incomplete information. Members of a group receive private signals and the group must take a collective decision—e.g., make a group purchase—based on the aggregate of members' private information. Since members have diverse preferences over the outcome of this decision, each has an incentive to manipulate the decision-making process by mis-reporting his private information. Whenever there are natural bounds on the set of admissible reports—e.g., if one's signal is one's endowment, one cannot report a negative signal—an asymmetry arises between those who would over-report, and those who would under-report, their true information. To model the incentives to misreport, and the associated asymmetry, we introduce a new kind of incomplete information game called an *aggregation game*. In such a game, each player is characterized by two parameters: the first—the player's *type*—is privately known, the second is publicly observed. Players simultaneously report their types, resulting in an *outcome*, which is a function of the aggregate of these reports. Each player's payoff depends on his observable characteristic, the aggregate of players' type realizations and the game's outcome. Every aggregation game has a pure-strategy equilibrium. We characterize these equilibria, and study their comparative statics properties, under a variety of restrictions.

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The authors are, respectively, Robert Gordon Sproul Distinguished Professor, Adjunct Professor and Associate Professor. Rausser and Simon are at the Department of Agricultural and Resource Economics, University of California at Berkeley. Zhao is at the Department of Economics, Iowa State University. The authors are grateful to seminar participants at the University of Washington, at the Economic Theory Seminar Series at Iowa State University, and at the Graduate Seminar, Department of Agricultural and Resource Economics, University of California at Berkeley. Susan Athey and Chris Shannon provided very helpful advice relating to the application of Athey's methodology to our problem. Larry Karp provided useful comments on an earlier draft.

Address Correspondence to: Leo Simon, Department of Agricultural Economics, University of California at Berkeley, Berkeley, CA 94709-3301. Email: simon@are.berkeley.edu.

1. INTRODUCTION

We consider a class of games that are naturally characterized as *aggregation games*. In an aggregation game, a finite collection of players with diverse characteristics simultaneously make reports to a central authority who then makes a decision that affects all of them. The authority's decision rule is fixed, commonly known, and based solely on the aggregate value of players' reports. Each player has two characteristics: the first (his *unobservable characteristic*) is his private information; the second (his *observable characteristic*) commonly known.[1] A player's unobservable characteristic is identified with his *type*, which is continuously distributed; the distribution of types is common knowledge. The defining property of an aggregation game is that each player's payoff depends—in addition to his observable characteristic and the central authority's decision—on the *aggregate* of all players' unobservable characteristics, but *not* on either the player's individual report or on his type. A player's *strategy* in an aggregation game is to report his unobservable characteristic. The link between reports and payoffs is that individual reports affect the aggregate of all reports, which in turn determines the central authority's response, and this response is payoff-relevant.

One interpretation of the model is that players' payoffs depend on the unknown mean of some distribution, from which each player receives an i.i.d. signal. The best available estimator of this mean is the average of players' signals. If the central authority can solicit information about players' signals, but has no independent information about the population mean, it is natural to model the authority's response to the state of the world as a function defined on the space of mean signals. Players' observable characteristic could be any payoff-relevant variable. In this context, the arguments in each player's payoff would be: (a) the mean signal; (b) the central player's response; (c) the player's individual observable characteristic.

The economic issue we address here arises in a variety of contexts. Market research surveys provide a familiar example, but in this case, information is aggregated from a large number of sources, so that the issue of strategic manipulation is secondary. Our work is more pertinent to contexts where an individual who makes a report has significant scope, as well as incentive, to influence the central outcome. For example, spot natural gas price indices, on which forward

[1] Throughout this paper, when we say that a player is "larger/smaller" we refer only to the magnitude of his observable characteristic. Indeed, players may be "large" in some conventional sense (e.g., market share) but be "small" for our purposes. Moreover, "player heterogeneity" refers to heterogeneity in their observable characteristics.

contract prices are based, are compiled by companies such as Platts from reports by relatively few contributors, who also participate in forward markets (Moody, 2003). Recently, the data from which these indices are constructed has come under intense scrutiny, as evidence accumulates that contributors have misreported their private information in order to manipulate these indices, and, thereby, the forward prices that determine the value of forward transactions. The Livestock Mandatory Reporting Act of 1999 provides another example. This Act, which requires large meat packers to report to the USDA price/quantity data relating to transactions involving cattle, hogs and lambs was designed to provide timely, accurate, pooled market information in an environment where the details of an increasing fraction of transactions is not publicly disclosed (Azzam, 2003). Once again, since the entities that must reveal their information have both scope and incentives to manipulate the published indices, the difficulty of ensuring truthful reporting is a major concern.

To elucidate the nature of strategic interaction in aggregation games, we briefly discuss an application. Suppose a group of doctors is considering how much malpractice insurance to buy for their medical partnership. Each doctor receives a private signal about the partnership's vulnerability to a malpractice suit and is asked to report this signal. The level of insurance is then computed using a pre-determined formula, which is monotone increasing in the average of the doctors' reports. Doctors have different net worths, and these levels are common knowledge; the deeper the doctor's pockets, the greater is his exposure in the event of a liability claim, and hence the greater is his willingness to pay for insurance. Expressed in terms of an aggregation game, a doctor's type is his private evaluation of the partnership's liability exposure; his observable characteristic is his net worth; the central authority's decision is the level of insurance. A doctor's expected payoff is a function of his net worth, the level of insurance, and the best estimate of the practice's risk, which is the aggregate of the doctors' signals. It should be emphasized that the doctor's private signal enters his payoff function only through its effect on the *aggregate* signal.

Since doctors with higher net worths (*high doctors*) prefer higher levels of insurance coverage, and the level of coverage depends on the aggregate report, high doctors have incentives to over-report their private signals, while low doctors may wish to under-report. Further, relative to the case in which all parties reveal the truth, high doctors' incentives to over-report will be exacerbated by the knowledge that low doctors are simultaneously under-reporting. Note that a doctor's *reported* signal enters his payoff function only through its effect on the pre-determined formula for computing insurance coverage, which is a function of the *aggregate* reported signal. If strategy spaces were unbounded, the differences between doctors' incentives would result in an endless "tug-of-war," with increased under-reporting by some leading to increased over-reporting by others, and Nash equilibria would fail to exist. When strategies are bounded in at least one direction, however, a pure strategy Nash equilibrium exists. In this context, as in the other applications we consider, there is a natural candidate for the lower bound on strategies: since "liability exposure" is necessarily nonnegative, it is natural to assume that doctors receive, and then report, nonnegative signals.

It will be apparent from the above example that bounds on the space of unobservable characteristics, and hence on players' strategy spaces, play a pivotal role in determining the equilibrium properties of aggregation games. Indeed, as the example suggests, player-types will fare very differently in equilibrium, depending on whether their equilibrium reports are constrained to lie on the boundary. In many applications, it is natural to assume, as we do in this paper, that the only relevant bound is nonnegativity. In others—for example, when players must report a probability estimate or a proper fraction—a model with two-sided bounds would be more appropriate. In some applications, the bounds that players face are "softer" than the absolute bounds that our players face: for example, while index compilers will sometimes reject data points that are outliers, data providers cannot predict with certainty the boundary between outlying and admissible reports.

This paper is related to the large literature on information aggregation in groups, in either voting or committee decision making (Feddersen and Pesendorfer (1996), Feddersen and Pesendorfer (1997), Austen-Smith and Banks (1996), Ottaviani and Sorensen (2001), Banerjee and Somanathan (2001), Krishna and Morgan (2001), and Austen-Smith and Feddersen (2002)). A major finding of this literature is that strategic reporting by agents with private information can prevent complete information transmission, depending on the institution through which the information is transmitted and aggregated. For example, Austen-Smith and Banks (1996) found that under majority voting, "sincere voting" by every player cannot be a Nash Equilibrium. We contribute to this literature by studying agents' incentives to strategically manipulate their private signals, and by characterizing the resulting distortion in aggregate information.

Our model differs from the literature in several respects. First, following the seminal work of Crawford and Sobel (1982), the literature has focused primarily on the problems facing agents who are required to choose a signal from a discrete set. In our model, the signal space is continuous, restricted only by a lower bound. Second, the structure of our space of announcements limits how extreme certain types of players can be when they strategically manipulate their signals. Banerjee and Somanathan (2001) showed that extremists are more likely to communicate misleading information, and in certain cases, the presence of too many extremists can block communication altogether. By contrast, the most extreme (i.e., the largest) player in our model in an *ex ante* sense controls the game: he always obtains, in expectation, the outcome that he most prefers. Third, the "central agency" in our model is nonstrategic in that it responds to players' signals as if they were truthful rather than trying to design mechanisms that achieve truthful revelation. In the language of Dessein (2002), the institution we study involves "delegation" rather than "communication." (While delegating behavior in this sense—i.e., the nonstrategic processing of signals transmitted by agents—seems naive relative to the the kind of communicating behavior modeled in Crawford and Sobel (1982), Dessein (2002) and Ottaviani and Squintani (2002) have identified conditions under which delegation will be more efficient than communication.) Finally, we emphasize the role of heterogeneity in agents' observable characteristics, analyzing how heterogeneous agents fare in equilibrium and how varying degrees of heterogeneity affect the aggregate report. Our treatment of agent heterogeneity—it has an observable and an unobservable component—extends the treatments that have appeared so far in the literature. For example, in Gilligan and Krehbiel (1989), agent heterogeneity is observable; in Banerjee and Somanathan (2001), it is unobservable; in Krishna and Morgan (2001), a single expert has an observable bias and an unobservable signal.

The paper is organized as follows. In §2 we introduce our model and prove that every aggregation game has a pure strategy equilibrium in which players' equilibrium strategies are strictly monotone in their types. §3 demonstrates that incentives to misreport do not arise when players have *ex ante* identical characteristics. Following a wide-spread practice in the information aggregation literature (e.g., Gilligan and Krehbiel (1989), Austen-Smith (1993), and Krishna and Morgan (2001)), we make further progress by restricting attention to two-player games. In this setting, a weak condition ensures that truthful revelation is the unique equilibrium. §4 contains results on equilibrium misreporting when players' observable characteristics diverge. In general, larger players over-report and smaller players under-report. In two-player games, an increase in heterogeneity results in increased over-(under-)reporting by the larger (smaller) player. This and an earlier result imply a corollary with far-reaching consequences: the larger player is never constrained by the lower bound on announcements. §5 introduces a restricted specification of our model that admits sharp comparative statics results. While the restrictions we impose limit the applicability of our model, an offsetting benefit is that in the stark context of §5-§7, the inner workings of our model can be revealed in a relatively transparent way. The main restriction is that the first argument of players' utilities is related to the aggregate signal and the outcome by an affine transformation. §6 contains comparative statics results for a two-player version of this specification. We show that in an *ex ante* sense, the larger player essentially controls the game while *ex post*, high types of the smaller player have total control. In §7, we further restrict the specification in §5 to facilitate analysis of the *n*-player game. We show in particular that in the unique equilibrium for this restricted game, each player but the largest plays the same strategy as he plays in the two-person game against the largest player alone. Propositions marked with asterisks are proved in the appendix; proofs for the remaining propositions are omitted as they follow immediately from arguments in the text.

2. The Model

In this section we formalize the notion of an aggregation game and show that every aggregation game has a pure-strategy equilibrium in which each player's strategy is monotone in his type. An aggregation game is an incomplete information simultaneous-move game between n players, indexed by $r = 1, \ldots, n$.

Player characteristics: Player r's *unobservable characteristic* is θ_r, which is his private information. (Following the standard terminology, we will sometimes refer to θ_r as r's *type*.) We assume that the θ_r's are identically, independently and continuously distributed on the closed interval $\Theta \subset \mathbb{R}_+$, where $\underline{\theta}$ and $\overline{\theta}$ are its left and right boundary points. (The restriction that $\underline{\theta} \geq 0$ involves no loss of generality.) Let $\eta(\cdot)$ denote the density of players' types. Let $\Theta = \Theta^n$ denote the space of *type profiles*, with generic element θ. Similarly, let $\Theta_{-r} = \Theta^{n-1}$. For $\theta_{-r} \in \Theta_{-r}$, let $\eta_{-r}(\theta_{-r}) = \prod_{j \neq r} \eta(\theta_j)$. Given two functions $v, w : \Theta \to \mathbb{R}$, we will write $v \gneqq w$ (resp. $v \lneqq w$) if $v \geq w$ (resp. $v \leq w$) and $\int_\Theta (v(\vartheta) - w(\vartheta)) d\eta(\vartheta) > 0$ (resp. < 0). The following technical requirement is required to ensure that pure-strategy equilibria exist:

Assumption A1: The density, $\eta(\cdot)$, of players' types is bounded.

Player r's *observable characteristic* will be denoted by k_r. We normalize the k_r's to lie in the unit interval. We will refer to the vector $k = (k_r)_{r=1}^n$ as the *observable characteristic profile* and to the pair (θ_r, k_r) as r's *characteristics*.

The payoff function: The *payoff function* is a map $u : \mathbb{R}_+ \times \Theta \times [0,1] \to \mathbb{R}$. The first argument of u can be interpreted as the decision taken by a governing body or central authority, in response to information provided by the players: $u(\tau, \theta, k)$ is the payoff to a player when the central authority's decision is τ, the unobservable characteristic profile is θ and the player's observable characteristic is k. We assume:

Assumption A2: u is bounded and thrice continuously differentiable. For each k, θ, $u(\cdot, \theta, k)$ is strictly concave in τ.

The essence of an aggregation game is that a player's unobservable characteristic affects the payoff function only through its effect on the sum, denoted $\Sigma\theta$, of all player's unobservable characteristics:

Assumption A3: for each τ, k and θ, $\theta' \in \Theta$, if $\Sigma\theta - \Sigma\theta' = 0$, then $u(\tau, \theta, k) = u(\tau, \theta', k)$.

We assume that players with higher unobservable and/or observable characteristics derive higher marginal benefits from an increase in the central authority's decision:[2]

Assumption A4: For all (τ, θ, k), $\frac{\partial^2 u(\tau,\theta,k)}{\partial\tau\partial\theta_r} > 0$.

Assumption A5: For all (τ, θ, k), $\frac{\partial^2 u(\tau,\theta,k)}{\partial\tau\partial k} > 0$.

In the medical insurance application described on page 3, $u(\tau, \theta, k)$ represents the benefit to a doctor with net worth k, beliefs about the likelihood of a malpractice suit representedy by θ, and malpractice insurance coverage level at τ. Assumptions A4 and A5 imply that doctors with higher private signals about the likelihood of a suit or with higher net worths will derive higher marginal benefits from an increase in the partnership's insurance coverage.

Our final assumption on u is that no player of any type has an insatiable appetite for τ.

Assumption A6: There exists $\bar{\tau} \in \mathbb{R}_+$ such that $\frac{du(\bar{\tau},(\bar{\theta},\ldots,\bar{\theta}),1)}{d\tau} = 0$.

[2] Assumption A4 is a strict version of the "single crossing property of incremental returns (SCP-IR)" (Milgrom and Shannon, 1994) in $(\tau;\theta)$ when the payoff function is differentiable (Athey, 2001, Definition 1).

Because u is strictly concave, assumptions A4-A6 imply that *any* player of *any* type would view any $\tau > \bar{\tau}$ as excessive. We shall refer to $\bar{\tau}$ as the *satiation level* of the central agency's activity.

As a benchmark notion of social efficiency, we adopt the utilitarian perspective, defining the *social welfare function*, $\check{w}$, to be the mean of players' individual utility functions. That is, we define $\check{w}(\tau,\theta,k) = \sum_r u(\tau,\theta,k_r)/n$.

Pure strategies: A *pure strategy* for the r'th player is a function $s_r : \Theta \to A \subset \mathbb{R}_+$ where A is a closed interval representing the set of admissible announcements, and $s_r(\theta_r)$ is the announcement of player r when his type is θ_r. (We will henceforth reserve the symbol s to denote *functions* from types to A and use the symbol a_r to denote a particular value of $s_r(\theta_r)$.) In §3 and below, we shall focus particular attention on a class of strategies that we call *unit affine*, i.e., strategies of the form $s_r(\cdot) = \iota(\cdot) + \lambda$, where $\lambda \in \mathbb{R}$ and $\iota(\cdot)$ denotes the identity map on Θ. We assume that $\Theta \subset A$, so that truthful revelation—i.e., the strategy $s_r = \iota$— is admissible. Let $\underline{a}$ denote the left boundary of A. We assume that A is unbounded above.[3] In most of the applications we consider, it is natural to equate A with $\mathbb{R}_+$. We will demonstrate later that the existence of the satiation level $\bar{\tau}$ guarantees that our specification is equivalent to an otherwise identical one with a compact action space. The strategy vector $s = (s_1,\ldots,s_n)$ is called a *pure strategy profile*. Thus a strategy profile is a mapping from Θ to $A = A^n$.

A pure strategy s_r is said to be *monotone* if there exists $\underset{\sim}{\theta}_r \geq \underline{\theta}$ such that s_r equals $\underline{a}$ on $[\underline{\theta},\underset{\sim}{\theta}_r)$, and is strictly increasing on $(\underset{\sim}{\theta}_r,\bar{\theta}]$.[4] Given a monotone strategy profile, s, define the *threshhold type vector* for s, $\underset{\sim}{\theta}(s)$, by, for each r,

$$\underset{\sim}{\theta}_r(s) = \begin{cases} \underline{\theta} & \text{if } s_r(\underline{\theta}) > \underline{a} \\ \sup\{\theta \in \Theta : s_r(\theta) = \underline{a}\} & \text{if } s_r(\underline{\theta}) = \underline{a} \end{cases}. \tag{1}$$

[3]Since we have declared announcements below $\underline{a} \leq \underline{\theta}$ to be implausible, and hence inadmissible, one might object that we should take the same view of announcements above $\bar{\theta}$. In many applications—particularly when the natural choice for Θ would be the unit interval—this objection would be unassailable. We adopt our asymmetric specification because, while fascinating, the analysis of "bi-bounded" games would be much more difficult. In particular, the kind of determinate comparative statics results we obtain in sections 6 and 7 would not be available. Moreover, nonnegativity constraints play such an ubiquitous role in economics that their implications warrant special attention.

[4]Note that if $s_r(\cdot) > \underline{a}$ on Θ then the interval $[\underline{\theta},\underset{\sim}{\theta}_r(s))$ will be empty.

If a player's realized type is below his threshhold type, the lower bound $\underline{a}$ on actions will be a binding constraint.

The outcome function: Strategy profiles are mapped into *outcomes*. The *outcome function*, $t : A \times [0,1]^n \to \mathbb{R}_+$, can be interpreted as a mapping from announcements by players and the observable characteristics profile to actions by the central authority. A key restriction is that t depends only on the *sum* of players' announcements:

Assumption A7: for $a, a' \in A$, $\sum_{i=1}^{n} \left[a_r - a'_r\right] = 0$ implies $t\left(a_1, ..., a_n, k\right) = t\left(a'_1, ..., a'_n, k\right)$.

Since t depends only on aggregate reports, we will sometimes write it as a mapping from $\mathbb{R}_+ \times [0,1]^n$ to $\mathbb{R}_+$ and replace the notation "$t(s,k)$" by "$t(\Sigma s, k)$." We say that an outcome function, $t(\cdot, k)$, is *complete information socially efficient (CISE)* if it maximizes the social welfare function (p. 6) when all players truthfully reveal their types, i.e., if for each $\theta \in \Theta$, $t(\theta, k) = \text{argmax}\ \check{w}(\cdot, \theta, k)$. We assume:

Assumption A8: $t(\cdot, k)$ is a CISE outcome function.

We will refer to the *outcome* implemented by a CISE outcome function as a CISE outcome.[5] The *social value function* is now defined as the mapping from player-type profiles to $\mathbb{R}$, i.e., $\check{W}(\theta, k) = \check{w}(t(\theta, k), \theta, k)$. Recalling the definition of $\bar{\tau}$ (p. 6), we assume:

Assumption A9: t is strictly increasing and thrice continuously differentiable. For all k, there exists $\bar{a} \in A$ such that $t(\bar{a}, k) = \bar{\tau}$.

In any equilibrium, no *individual* player will ever announce more than $\bar{a}$ because if one did, then with probability one, the outcome would exceed the satiation level, $\bar{\tau}$. Because of this restriction, our game with an unbounded strategy space is equivalent to an alternative specification in which players are required to choose actions within the compact space $[0, \bar{a}]$.[6]

In the analysis that follows, it will be helpful to ensure that $u(t(\cdot, k), \theta, k)$ is strictly concave in Σs. Since u is not monotone in Σs, this property is not completely straightforward. To see this, let f

[5]Assumption A9 below ensures that argmax $\check{w}(\cdot, \theta, k)$ is nonempty, even though A is unbounded. Assumption A2 ensures that argmax $\check{w}(\cdot, \theta, k)$ is unique. Hence the concept of a CISE outcome is well-defined.

[6]The advantage of our specification over requiring A to be compact in the first place is that we don't have to specify Kuhn Tucker conditions for the upper end of the strategy space.

and g map $\mathbb{R}$ to $\mathbb{R}$, with f concave and consider $m(x) = f(g(x))$. Unless f' can be signed, the only straightforward way to ensure that $m'' = f''(g')^2 + f'g'' \leq 0$ is to require that $g'' = 0$. We will, nonetheless, impose:

Assumption A10: For all k and θ, $u(t(\cdot,k),\theta,k)$ is strictly concave in Σs.

While the combination of assumptions A2 and A10 is restrictive in general, sufficient conditions for the pair can readily be identified when, as in the various applications considered in this paper, u is a net benefit function. For example, a natural specification in our context might be that $u(t,\theta,k) = b(t,k) - c(t,\Sigma\theta)$, where b is a concave benefit function and c is a convex cost function; if b and c are, respectively, homogeneous of degree $\alpha_b < 1$ and $\alpha_c > 1$ in t, and t is homogeneous of degree $\alpha_t \in (\frac{1}{\alpha_c}, \frac{1}{\alpha_b})$ in Σs, then b, $-c$ and hence u will all be strictly concave in Σs.

Player objective functions: Following Athey (2001), we now synthesize the various components defined above into an *objective function* for each player. The objective function for player r is a mapping from player r's own action and type into his payoff, treating other players' strategies as parameters. We suppress reference to r's observable characteristic and the outcome function. Formally, given a profile, s_{-r}, of strategies for players other than r, we define player r's objective function as $U_r : A \times \Theta \to \mathbb{R}_+$, where

$$U_r(a,\theta;s_{-r}) \quad = \quad \int_{\Theta_{-r}} u\left(t\left(a,s_{-r}(\vartheta_{-r}),k\right),(\theta,\vartheta_{-r}),k_r\right) d\eta(\vartheta_{-r}) \tag{2}$$

$U_r(a,\theta;s_{-r})$ is player r's conditional expected payoff from playing the action a, given that his realized type is θ. When confusion can be avoided, we will write U_r' rather than $\frac{\partial U_r}{\partial a}$. Observe that since $U_r' = \int_{\Theta_{-r}} \frac{\partial u}{\partial t}\frac{dt}{da} d\eta(\vartheta_{-r})$, assumption A10 implies that for all r, all θ and all s_{-r}, $U_r(\cdot,\theta;s_{-r})$ is strictly concave in a. Moreover, since t is strictly increasing (A9), it follows from A4 that[7]

$$\text{for all } r\text{, all } a\text{, all } \theta \text{ and all all } s_{-r}, \quad \frac{\partial^2 U_r(a,\theta;s_{-r})}{\partial a \partial \theta} \quad > \quad 0.^8 \tag{3}$$

[7]Inequality (3) states that U satisfies SCP-IR in $(a;\theta)$ (cf. p. 6). In our context, this property implies Athey's sufficiency condition, SCC, for existence of a pure-strategy equilibrium, i.e., "the single crossing condition for games of incomplete information" (Athey, 2001, Definition 3). This condition requires that U satisfies SCP-IR only if other players play non-increasing strategies. In our model, U satisfies SCP-IR regardless of other players' choices.

Equilibrium: A *monotone pure strategy Nash equilibrium* (MPSNE) for an aggregation game is a monotone strategy profile s satisfying for all r, all θ and all $a \in A$, $U_r(s_r(\theta), \theta; s_{-r}) \geq U_r(a, \theta; s_{-r})$. If s is a MPSNE then, necessarily,

$$U_r'(s_r(\theta), \theta; s_{-r}) \leq 0, \text{ with equality holding whenever } s_r(\theta) > \underset{\sim}{a} \tag{4}$$

Proposition 1 (Existence of a monotone MPSNE)*: *Every aggregation game satisfying assumptions A1-A10 has a monotone pure-strategy Nash equilibrium, s, with the property that for each r, s_r is continuously differentiable on* $(\underset{\sim}{\theta}_r(s), \overline{\theta}]$.

We will observe below that certain player-types "do particularly well" in equilibrium. We define an action $s_r(\theta)$ to be the *best conceivable* action for player-type (r, θ) if for all s'_{-r} and all a, $U_r(s_r(\theta), \theta; s_{-r}) \geq U_r(a, \theta; s'_{-r})$. A player-type for whom such an action is available will attain the same payoff as he would obtain if he has full control over the strategies played by all other players! We will say that a PSNE is *efficient* if the action chosen by every type of every player is a best-conceivable action. This is, clearly, the strongest conceivable notion of efficiency.

3. Aggregation games when players are *ex ante* identical.

In this section we explore the properties of MPSNE's when players have the same observable characteristic, $\bar{k}$, and are hence *ex ante* identical. This analysis provides a useful benchmark when we consider games in which players' observable characteristics are heterogeneous. For two person games, we show that *all* equilibria are efficient.[9] We also show that in n-player games, there are equilibria in which players' strategies are unit affine (p 7); moreover, in two-player games,

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$$\begin{aligned}\frac{\partial^2 U_r(a,\theta;s_{-r})}{\partial a \partial\theta} &= \frac{\partial^2}{\partial a \partial \theta}\left\{\int_{\Theta_{-r}} u\left(t\left(a, s_{-r}(\vartheta_{-r}), k\right), \left(\theta, \vartheta_{-r}\right), k_r\right) d\eta(\vartheta_{-r})\right\} \\ &= \int_{\Theta_{-r}} \left\{\frac{\partial^2}{\partial\theta\partial t} u\left(t\left(a, s_{-r}(\vartheta_{-r}), k\right), \left(\theta, \vartheta_{-r}\right), k_r\right) \frac{dt}{d\Sigma s}\right\} d\eta(\vartheta_{-r}) \quad > \quad 0\end{aligned}$$

The inequality holds because both $\frac{\partial^2 u}{\partial\theta\partial t} \geq 0$ and $\frac{dt}{d\Sigma s} > 0$.

9 It remains an open question whether this result can be extended to multi-player games.

equilibrium strategies are *necessarily* unit affine. The class of efficient equilibria include truthful revelation equilibria.

We will establish that when players have the same observable characteristic, a profile satisfying the following properties is an efficient equilibrium: each player's strategy is a unit affine function of his type and the intercepts of all players' strategies sum to zero. We shall refer to such profiles as *zero-sum unit affine (ZSUA)*. Specifically, let $\Lambda = \{\lambda \in \mathbb{R}^n : \sum_{r=1}^n \lambda_r = 0 \text{ and } \lambda_r \geq (\underline{a} - \underline{\theta}), \forall r\}$. A profile is ZSUA if for some $\lambda \in \Lambda$, $s_r = \theta_r + \lambda_r$, for each r. (Clearly, for any vector λ with $\lambda_r < (\underline{a} - \underline{\theta})$, $s_r = \theta_r + \lambda_r$ would not be admissible for types in some neighborhood of $\underline{\theta}$.)

Proposition 2 (ZSUA profiles are efficient equilibria): *Consider an aggregation game satisfying assumptions A1-A10 in which $k_r = \bar{k}$, for all r. A sufficient condition for a strategy profile to be an efficient equilibrium is that it is ZSUA.*

A special case of a ZSUA equilibrium is when all agents truthfully reveal their types, i.e., when $\lambda = 0$. The proposition thus highlights the intuitive fact that in an aggregation game, incentives for strategic behavior arise only when there are differences between agents' *observable* characteristics.

The proof of Prop. 2 is immediate. Consider $\lambda = (\lambda_i, \lambda_{-i}) \in \Lambda$. Necessarily, $\lambda_i = -\sum_{r \neq i} \lambda_r$. In the ZSUA strategy profile corresponding to λ, player r plays the strategy $s_r(\theta) = \theta + \lambda_r$. Consequently

$$EU_i(s_i(\theta), \theta; s_{-i}) \quad = \quad \int_\Theta u(t(\sum_r s_r(\theta_r), k), \Sigma\theta, \bar{k}) \eta(\theta) d\theta \quad = \quad \int_\Theta u(t(\Sigma\theta, k), \Sigma\theta, \bar{k}) \eta(\theta) d\theta$$

Since players' observable characteristics are all identical, $\check{w}(t, \theta, k) = u(t, \theta, \bar{k})$. By assumption, the outcome function is CISE, so that for every $\theta \in \Theta$, $t = \operatorname{argmax} u(\cdot, \theta, \bar{k})$, and hence $u(t(\sum_r s_r(\theta_r), k), \Sigma\theta, \bar{k}) = \check{W}(\theta, k)$.

When there are only two players with identical observable characteristics, we can go much further. In this case, Prop. 2 and Prop. 3 below establish that a profile is an equilibrium if and only if it is ZSUA, i.e., *all* equilibria are efficient![10] (We conjecture that this result can be extended to general games.)

Proposition 3 (Efficient equilibria are ZSUA)*: *Consider a two player aggregation game satisfying assumptions A1-A10 with $k_i = k_j$. A necessary condition for a strategy profile to be a PSNE is that it is ZSUA.*

[10]While we have not formally analyzed the implications of restricting the space of player announcements to Θ, it will be immediately apparent from the argument below that if we did do so, then truth-telling would be the *unique* pure-strategy equilibrium for any aggregation game with two players with identical observable characteristics.

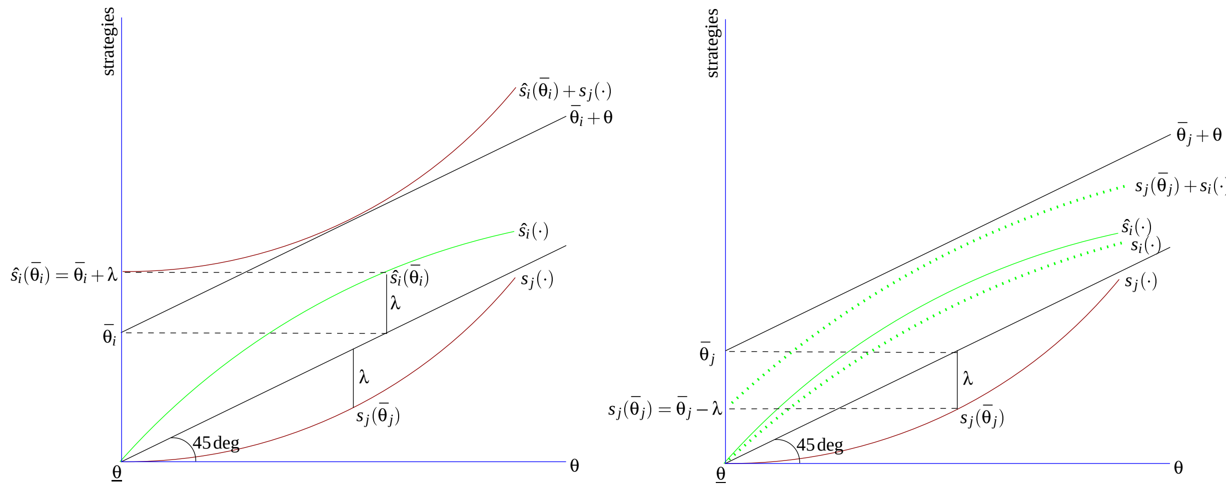

FIGURE 1. Intuition for Prop. 3

Figure 1 provides some intuition for this result. Consider a strategy, such as s_j in the left panel of the figure, that is not unit affine. Recalling from p. 7 that $\iota(\cdot)$ denotes the identity map, the minimum value of $(s_j(\cdot)-\iota(\cdot))$ is $-\lambda$, which is achieved at $\bar{\theta}_j$. A necessary condition for s_i to be a best response to s_j is that the maximum value of $(s_i(\cdot)-\iota(\cdot))$ is strictly less than λ. To see why this must be so, consider a strategy such as $\hat{s}_i$ satisfying, for some $\bar{\theta}_i$, $(\hat{s}_i(\bar{\theta}_i)-\bar{\theta}_i)\geq\lambda$. Given any such strategy for i, the aggregate strategy $\hat{s}_i(\bar{\theta}_i)+s_j(\cdot)$—i.e., the highest curve in the left panel—must lie above the line $\bar{\theta}_i+\iota(\cdot)$ with probability one. That is, conditional on player i being of type $\bar{\theta}_i$, the sum of players' announced types exceeds the sum of their actual types with probability one. Since by assumption, $t(\cdot)$ is CISE, the outcome it implements when $(s_j,\hat{s}_i)$ is played and i's type is $\bar{\theta}_i$ must be sub-optimal with probability one. We thus conclude that $\bar{\theta}_i+\lambda$ is not a best response for player i of type $\bar{\theta}_i$ against the strategy $s_j(\cdot)$, or more generally, a necessary condition for s_i to be optimal against a strategy s_j that is not unit affine is that $(s_i(\cdot)-\iota(\cdot))<\max_{\vartheta\in\Theta}(\vartheta-s_j(\vartheta))$. Now consider any strategy for i that satisfies this necessary condition—such as the dashed curve $s_i(\cdot)$ in the right panel—and observe that for all θ_i, the aggregate strategy $s_j(\bar{\theta}_j)+s_i(\cdot)$ is everywhere *below* the line $\bar{\theta}_j+\iota(\cdot)$. Thus for player j of type $\bar{\theta}_j$, the action $s_j(\bar{\theta}_j)$ cannot be a best response against *any* strategy that could possibly be a best response against s_j. This establishes that players' strategies must be unit affine.

4. AGGREGATION GAMES WITH HETEROGENEOUS PLAYERS

Prop. 1 established the existence of an equilibrium in which players' announcements are monotone with respect to their types. We show below that in any such equilibrium, players' strategies will also be monotone with respect to their *observable* characteristics. That is, if i's observable characteristic exceeds j's, but both are of the same type, i's announcement will strictly exceed j's except for types of i below i's threshhold type. In our medical partnership example, if two doctors with different levels of net worth receive the same private estimate of liability risk, the one with the deeper pockets will report a higher estimate of that risk than the other.

Proposition 4 (Monotonicity w.r.t. observable characteristics)*: *Let s be a MPSNE strategy profile for an aggregation game satisfying assumptions A1-A10. If $k_i > k_j$ and if $\underset{\sim}{\theta}_i(s) < \overline{\theta}$, then (a) $\underset{\sim}{\theta}_i(s) < \underset{\sim}{\theta}_j(s)$ and (b) $s_i(\cdot) > s_j(\cdot)$ on $(\underset{\sim}{\theta}_i(s), \overline{\theta}]$.*

Part (a) implies that for any player whose observable characteristic is lower than some other player's, sufficiently low types of the former will always be constrained by the lower bound of the action space.

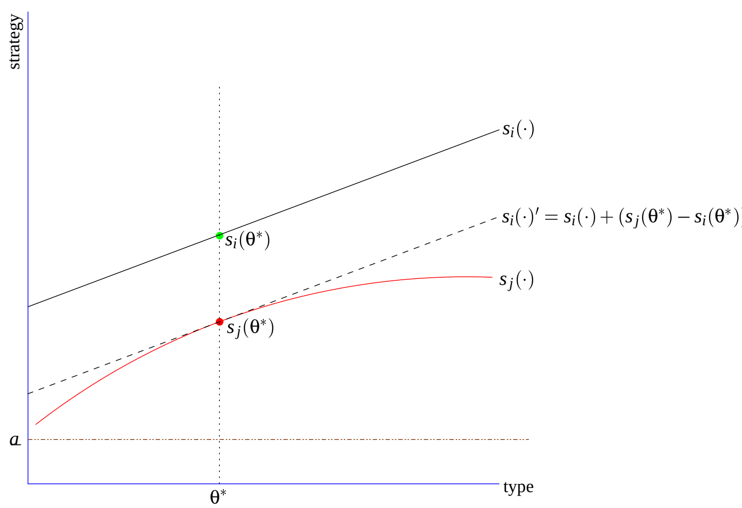

FIGURE 2. Intuition for Prop. 4

The proof of Prop. 4 involves two steps. First, if $k_j < k_i$ then player j cannot be at an interior optimum when his type is $\theta^* = \text{argmax}(s_j(\cdot) - s_i(\cdot))$. To see this, consider the arbitrary strategy pair $s = (s_i, s_j)$ drawn in Figure 2, in which s_j attains an optimum at θ^*, and $s_j(\theta^*) > \underset{\sim}{a}$. Assume that s_j is a best response to s_i; we will argue that s_i cannot be a best response to s_j. Because t depends only on Σs, if i were to play the strategy $s_i{}' = s_i + (s_j(\theta^*) - s_j(\theta^*))$ rather than s_i, then

j's optimal announcement would be $s_i(\theta^*)$ rather than $s_j(\theta^*)$. But if $s_i(\theta^*)$ is optimal for j against $s_i{}'$, then $s_i(\theta^*)$ cannot also be optimal for i against s_j: because $s_j \leq s_i{}'$, $t(s_i(\theta^*), s_j(\cdot), s_{-ij}(\cdot), k) \leq t(s_i{}'(\cdot), s_i(\theta^*), s_{-ij}(\cdot), k)$, yet when both i and j are type θ^*, i's optimal choice of t strictly exceeds j's. Conclude that $s_j(\theta^*) = \underset{\sim}{a} \leq s_i(\cdot)$ so that, by definition of θ^*, $s_j(\cdot) - s_i(\cdot) \leq 0$. It is now straightforward to show that $s_i(\cdot) > s_j(\cdot)$ whenever j's type exceeds his threshhold type.

In two-player games, we can compare the equilibrium strategies for games with different profiles of observable characteristics. If player i's observable characteristic increases and j's does not, then in the sense made precise in Prop. 5-2P, i's equilibrium strategy also increases, while j's decreases.

Proposition 5-2P (Heterogeneity: two players)*: *Consider two aggregation games satisfying assumptions A1-A10 with observable characteristic profiles, k and $\hat{k}$ such that $\hat{k}_i > k_i$ and $\hat{k}_j \leq k_j$. Let s and $\hat{s}$ be MPSNE's for the two games. Then $\underset{\sim}{\theta}_i(\hat{s}) < \underset{\sim}{\theta}_i(s)$ and $\hat{s}_i(\theta) > s_i(\theta)$ for all $\theta > \underset{\sim}{\theta}_i(\hat{s})$, while $\underset{\sim}{\theta}_j(\hat{s}) > \underset{\sim}{\theta}_j(s)$ and $\hat{s}_j(\theta) < s_j(\theta)$ for all $\theta > \underset{\sim}{\theta}_j(s)$.*

The intuition for this result is straightforward. As i's observable characteristic increases, his preferred outcome increases also, for each of his possible types. Consequently, *holding j's strategy constant*, i's best response must strictly increase with k_i. Similarly, holding i's strategy constant, j's best response must weakly decrease with k_j. Now suppose that s is an equilibrium profile for the game with k and consider a profile s' for the game with $\hat{k}$ such that s'_i is a best response to s'_j, but s'_i is *not* strictly greater than s_j. In this case there must be some type of j whose announcement has increased. Indeed, the largest value of $(s'_j - s_j)$ must strictly exceed the largest value of $(s_i - s'_i)$. But by a parallel argument, a necessary condition for s'_j to be a best reply to s_i is that the largest value of $(s'_j - s_j)$ *does not* strictly exceed the largest value of $(s_i - s'_i)$. Since these two requirements are mutually incompatible, s'_j cannot be a best reply to s'_i, so that the equilibrium profile $\hat{s} \neq s'$.

Props. 2, 3 and 5-2P have an immediate corollary. As we have noted, Prop. 4 implied that in any heterogeneous two-player game, there will be some types of the player with the smaller observable characteristic whose optimal action is a corner solution. We now have the converse result: the player with the larger observable characteristic will *never* be constrained by the lower bound on admissible actions.

Proposition 6-2P (The larger player is never constrained): *Let s be an MPSNE for a two-person game satisfying assumptions A1-A10. If $k_i > k_j$, then $\underset{\sim}{\theta}_i(s) = \underline{\theta}$.*

The proof of Prop. 6-2P is immediate. Props. 2 and 3 established that when $k_i = k_j$, truthtelling is an equilibrium. Prop. 5-2P now implies that when i's type increases relative to to j's, i's strategy must overstate the truth, i.e., in any equilibrium s, $s_i(\cdot) > \iota(\cdot)$. Since by assumption, $\underline{\theta} \geq \underline{a}$, $s_i(\cdot) > \underline{a}$. From (1), therefore, $\underset{\sim}{\theta}_i(s) = \underline{\theta}$.

A question naturally arises concerning the *net* effect of changes in k on the expected value of the equilibrium outcome. Intuitively, we would expect that this value would increase with heterogeneity, because of the asymmetric structure of the strategy space: while there is no restriction on the rate at which strategies can increase, there are restrictions on the rate at which they can decrease. In order to obtain this intuitive result, however, we need to impose additional conditions on u and t. A set of sufficient conditions is identified in the following section.

5. Aggregation games with quasi-linear payoffs

In this and the following two sections, we analyze in detail a class of aggregation games in which there is an affine relationship between players' types and the outcome function. In this section we introduce the specification. In section 6 we analyze in detail a two-person variant. In section 7 we consider a restricted version of the n-person game, which bears a striking relationship to the two-person variant.

Throughout sections 5-7, we assume that player i has the largest observable characteristic. Without loss of generality we assume that $\underline{a} = \underline{\theta} = 0$ and $\sum_{r=1}^{n} k_r = 1$.

A player with observable characteristic $k \in [0,1]$ has the quasi-linear payoff function

$$u(\tau, \theta, k) \quad = \quad \mu - \tau - f(\phi(\theta, \tau), k) \tag{5}$$

where $\phi(\cdot,\cdot)$ is an affine function which increases with the sum of players' types and decreases with $\tau \geq 0$. In the context of our malpractice insurance story (p. 2), the function u can be interpreted as follows: μ is the partnership's net income excluding insurance costs; τ is the level of expenditure on malpractice insurance; $\phi(\theta, \tau)$ is the partnership's best estimate of its net liability exposure, after deducting the level of insurance coverage obtainable by spending τ, when the vector of doctors' private signals is θ; f is the partnership's disutility—from the perspective of a doctor with net worth k—of a net liability exposure of $\phi(\theta,\tau)$.

Since $\phi(\cdot,\cdot)$ is affine, there exist scalars ϕ_0, ϕ_1 and ϕ_2 such that for all θ and all τ, $\phi(\theta,\tau) = \phi_0 + \phi_1\Sigma\theta - \phi_2\tau$. We will refer to $\phi(\theta,\tau)$ as the *net impact* of the aggregate signal $\Sigma\theta$, after factoring in the central response level τ.

We assume throughout this section that f satisfies assumptions A11-A14 below. It is straightforward to check that under assumptions A11-A13, the specification in this section satisfies assumptions A1-A10 of the model presented in section 2. The additional assumption A14 resolves some comparative statics indeterminacies.

Assumption A11: $\phi_1, \phi_2 > 0$.

Assumption A12: $f > 0$ is twice continuously differentiable; $f_1 > 0$; $f_{11} > 0$; $f_{12} > 0$.

Note that under assumptions A11 and A12, u is strictly concave in both θ_r and τ. The role of our next assumption is to ensure that for each type vector and each player, there is a unique, positive level of τ that is globally optimal for that player-type:

Assumption A13: $\phi_2 f_1(0,1) < 1 < \phi_2 f_1(\phi(0,0),0)$.[11]

Assumptions A11-A13 together imply[12]

$$\text{for all } k \in [0,1], \text{ and all } \theta \in \Theta, \quad \text{there exists } \tau > 0 \text{ such that } \frac{du(\tau,\theta,k)}{d\tau} = 0. \tag{6}$$

[11] If $f(\phi,k) = (\alpha_1 + k)\phi^{1+\gamma}$ and $\phi(\theta,\tau) = \alpha_2 + \Sigma\theta/n - \tau$, with $\alpha \geq 1$ and $\gamma > 0$, then ϕ and $f(\phi(\cdot,\cdot),\cdot)$ satisfy assumptions A11-A13. To see this, observe that f_1, f_{11} and f_{12} are all positive, $\phi_2 = 1 = n\phi_1$ and

$$\phi_2 f_1(0,1) = (\alpha_1+1)(1+\gamma)(0)^\gamma = 0 < 1 < \alpha_1(1+\gamma)(\alpha_2)^\gamma = \phi_2 f_1(\phi(0,0),0)$$

[12] To verify (6), note that for all $\theta \in \Theta$ and all $k \in [0,1]$,

$$\frac{du(0,\theta,k)}{d\tau} = \phi_2 f_1(\phi(\theta,0),k) - 1 \geq \phi_2 f_1(\phi(0,0),0) - 1 > 0$$

while if τ is sufficiently large that $\phi(1,\tau) \leq 0$, then for all $\theta \in \Theta$ and all $k \in [0,1]$,

$$\frac{du(\tau,\theta,k)}{d\tau} \leq \phi_2 f_1(\phi(1,\tau),k) - 1 \leq \phi_2 f_1(0,1) - 1 < 0$$

Property (6) now follows from the intermediate value theorem.

Assumption A14, which signs some of the third derivatives of f, will enable us to obtain determinate comparative statics results. The first condition implies that a mean-preserving spread in net impact, ψ, will induce players to prefer (weakly) higher central agency response levels; the second implies that a player's degree of risk aversion is nonincreasing in his observable characteristic.

Assumption A14: For all $k \in [0,1]$ and all $\psi \in \mathbb{R}$, $f_{111}(\psi,k) \geq 0 \geq f_{112}(\psi,k)$

Because it provides a useful benchmark for our incomplete information game, we begin by analyzing its complete information counterpart. Throughout this section, a "ˇ" over a symbol indicates that it relates to this version, while the same symbol without this identifier relates to the incomplete information version. For each r, we define player r's *complete information personally optimal (CIPO) net impact level* to be net impact ϕ that maximizes the right hand side of (5). Note that this level, which we shall denote by $g_r(k)$, is independent of $\Sigma\theta$: since f_1 is strictly monotone, it is uniquely defined by the first order condition for (5), i.e., $\phi_2 f_1(g_r(k), k_r) = 1$. Property (6) implies that $g_r(k) > 0$. To identify outcome functions that implement such a CIPO net impact level, consider, for $\gamma \in \mathbb{R}$, the affine outcome function $\check{t}_\gamma$,

$$\check{t}_\gamma(\theta) \quad = \quad \frac{\phi_0 + \phi_1 \Sigma\theta - \gamma}{\phi_2}. \tag{7}$$

The associated net impact level is independent of $\Sigma\theta$:

$$\phi(\theta, \check{t}_\gamma(\theta)) \quad = \quad \phi_0 + \phi_1\Sigma\theta - \phi_2\left(\frac{\phi_0 + \phi_1\Sigma\theta - \gamma}{\phi_2}\right) \quad = \quad \gamma, \quad \text{for all } \theta \in \Theta. \tag{8}$$

Indeed, it is easy to check that the net impact $\phi(\cdot, t(\cdot))$ will be independent of $\Sigma\theta$ if *and only if* $t = \check{t}_\gamma$, for some γ. It follows that the unique, complete information outcome function that implements r's CIPO impact level with certainty is $\check{t}_{g_r(k)}$, since

$$\text{for all } \theta \in \Theta,\ \check{t}_{g_r(k)}(\theta) \text{ maximizes } u(\cdot, \theta, k_r).^{13} \tag{9}$$

We shall refer to this function as player r's *CIPO outcome* and will observe below that certain player-types succeed in implementing this outcome in equilibrium with probability one. Since

[13] Checking second order conditions, u is concave in ϕ, (from (5) and assumption A12) and $\frac{d\phi(\theta, \check{t}_\gamma(\theta))}{d\gamma} = 1$.

$\frac{dg_r(k)}{dk_r} = -\frac{f_{12}(g_r(k),k_r)}{f_{11}(g_r(k),k_r)} < 0$, players with higher k-levels prefer lower net impacts:

$$g_i(k) \ < \ g_j(k) \quad \text{iff} \quad k_i \ > \ k_j \tag{10}$$

Given a profile k, the "utilitarian" social welfare function (see p. 6) is defined as

$$\breve{w}(\tau,\theta,k) \quad = \quad \mu - \tau - \frac{1}{n}\sum_{r=1}^{n} f(\phi(\theta,\tau),k_r)$$

$\breve{w}(\cdot,\theta,k)$ is maximized w.r.t. τ at the net impact level $\breve{g}(k)$, defined by the condition that $\phi_2 \sum_{r=1}^{n} f_1(\breve{g}(k),k_r) = n$. We shall refer to $\breve{g}(k)$ as the *CISE net impact level*. Note that the CIPO net impact for the player with the largest k-level is necessarily smaller than the CISE net impact

$$\text{for all } k\text{, if } k_i \geq k_r,\ \forall r \text{ then } g_i(k) < \breve{g}(k), \text{ with strict inequality unless all players' } k\text{'s are equal.} \tag{11}$$

For each k, the *CISE outcome function* is the uniquely defined function, $\breve{t}_{\breve{g}(k)}$, which implement $\breve{g}(k)$, yielding player r a payoff of $u(\breve{t}_{\breve{g}(k)}(\theta),\theta,k_r) = \mu - \breve{t}_{\breve{g}(k)}(\theta) - f(\breve{g}(k),k_r)$. For each r and k, we define player r's *CISE expected utility*, $E\breve{u}_r = \int_\Theta u(\breve{t}_{\breve{g}(k)}(\vartheta),\vartheta,k_r)d\eta(\vartheta)$, to be player r's expected payoff when players' types are publicly known and the CISE net impact is implemented.

We now return to the incomplete information game. Its outcome function, t, delivers the outcome that would be socially optimal if players truthfully revealed their types (i.e., it is a CISE outcome function, c.f. assumption A8). That is, for each $\theta \in \Theta$:

$$t(s(\theta),k) \quad = \quad \breve{t}_{\breve{g}(k)}(s(\theta)) \quad = \quad \frac{\phi_0 + \phi_1 \sum_{r=1}^{n} s_r(\theta_r) - \breve{g}(k)}{\phi_2} \tag{12}$$

$$\text{so that} \quad \phi(\theta, t(s(\theta),k)) \quad = \quad \phi_1 \left(\textstyle\sum_{r=1}^{n}(\theta_r - s_r(\theta_r))\right) + \breve{g}(k) \tag{13}$$

Recall from page 9 that player r's *objective function* is defined as:

$$U_r(a_r,\theta_r,k;s_{-r}) \quad = \quad \int_\Theta u(\breve{t}_{\breve{g}(k)}(a_r,s_{-r}(\vartheta_{-r})),(\theta_r,\vartheta_{-r}),k_r)d\eta(\vartheta_{-r})$$

while the expected outcome conditional on r-type θ_r playing a_r is

$$Et(a_r,\theta_r;k,s_{-r}) \quad = \quad \int_\Theta \breve{t}_{\breve{g}(k)}(a_r,s_{-r}(\vartheta_{-r}))d\eta(\vartheta_{-r})$$

From (5), $\frac{du(\cdot,\theta,k)}{d\tau} = \phi_2 f_1(\phi(\theta,\cdot),k) - 1$, while from (12), $\frac{d\check{t}_{\breve{g}(k)}(s(\theta))}{da_r} = \frac{\phi_1}{\phi_2} > 0$. Hence

$$U_r'(a,\theta_r,k;s_{-r}) \;=\; \frac{\partial U_r}{\partial \tau}\frac{\partial \check{t}_{\breve{g}(k)}}{\partial a} \;=\; \frac{\phi_1}{\phi_2}\left(\phi_2 \int_{\underline{\theta}}^{\overline{\theta}} f_1(\phi(\theta,\check{t}_{\breve{g}(k)}(a_r,s_{-r}(\vartheta_{-r}))),k)d\eta(\vartheta_{-r}) - 1\right)$$

$$\text{which, from (13)} \;=\phi_1 \left\{ \int_{\underline{\theta}}^{\overline{\theta}} f_1\Big(\big[\breve{g}(k)+\phi_1\big(\theta_r - a + \Sigma(\vartheta_{-r} - s_{-r}(\vartheta_{-r}))\big)\big],k\Big)d\eta(\vartheta_{-r}) - 1/\phi_2 \right\} \tag{14}$$

$$\text{Moreover, } U_r''(a,\theta_r,k;s_{-r}) \;=\; -\,\phi_1\phi_2 \int_{\underline{\theta}}^{\overline{\theta}} f_{11}(\phi(\theta,\check{t}_{\breve{g}(k)}(a_r,s_{-r}(\vartheta_{-r}))),k)d\eta(\vartheta_{-r}) \;<\; 0, \tag{15}$$

verifying that U_r is strictly concave in player r's action.

6. QUASI-LINEAR AGGREGATION GAMES WITH TWO PLAYERS

In this section we analyze a two-person version of the game specified in §5. We obtain this version by requiring that two of the n players, labeled i and j, play strategically, while the other players truthfully report their private information. Formally, for each $r \neq i,j$, we require that player r plays the identity strategy $\iota(\cdot)$. An alternative interpretation is that the signals received by all but i and j are publicly observable. (With this formulation, we can consider the games between different pairs drawn from our original set of n players, while maintaining a common outcome function throughout. We shall exploit this common factor in §7 below.) After establishing that each two-person game has a unique pure-strategy equilibrium, we consider the impact of increasing the degree of heterogeneity between the two players' observable characteristics, focusing on: (a) players' tendency to misreport their signals; (b) the expected level of the outcome; and (c) players' ex ante expected equilibrium payoffs. In section 7, we impose an additional restriction on f and show how the unique equilibrium of the n-person game can be constructed by melding together the equilibria analyzed in this section.

For each observable characteristic profile, k, our game can be solved recursively. We first assume that player i plays the unit affine strategy s_i^{λ}, defined by, for some $\lambda \geq 0$, $s_i^{\lambda}(\cdot) = \iota(\cdot) + \lambda$. Given this strategy for i, we construct the optimal response for j, $s_j^*(\cdot,\lambda)$, which is piecewise affine in ϑ_j. Finally, we prove that there exists $\lambda^* > 0$ such that $s_i^{\lambda^*}$ is an optimal response to $s_j^*(\cdot,\lambda^*)$.

If there were no restrictions on his strategy space, j could exactly counter-balance s_i^λ and implement his CIPO net impact with probability one. The unit affine strategy that would accomplish this is $s'_j(\cdot,\lambda)$, defined by $s'_j(\cdot,\lambda) = \iota(\cdot) - \lambda + (\breve{g}(k) - g_i(k))/\phi_1$. From (12) and the definition of $\breve{t}_{\breve{g}(k)}$, the outcome generated by $\left(s_i^\lambda(\cdot), s'_j(\cdot,\lambda), \boldsymbol{\iota}_{-i,j}(\cdot)\right)$ would be, for all $\theta \in \Theta$

$$\begin{aligned}
t(\theta; k, \left(s_i^\lambda(\cdot), s'_j(\cdot,\lambda), \boldsymbol{\iota}_{-i,j}(\cdot)\right)) &= \breve{t}_{\breve{g}(k)}\left(s_i^\lambda(\theta_i), s'_j(\theta_j,\lambda), \theta_{-i,j}\right) \\
&= \frac{\phi_0 + \phi_1\left(\Sigma\theta + \lambda - \lambda + (\breve{g}(k) - g_i(k))/\phi_1\right) - \breve{g}(k)}{\phi_2} \\
&= \frac{\phi_0 + \phi_1\Sigma\theta - g_j(k)}{\phi_2} \quad = \quad t_{g_j(k)}(\theta)
\end{aligned}$$

so that if it were an admissible strategy, $s'_j(\cdot,\lambda)$ would (from (9)) be a best response to i's strategy. But in fact, $s'_j(\theta_j,\lambda)$ is an inadmissible action, for all θ_j's below j's threshhold type, since

$$s'_j(\theta_j,\lambda) \;<\; 0 = \underset{\sim}{a}, \qquad \text{for all } \theta_j \;<\; \underset{\sim}{\theta}(\lambda,k) \;=\; \max(0, \lambda + (g_j(k) - \breve{g}(k))/\phi_1). \tag{16}$$

We thus conclude that against s_i^λ, the optimal admissible strategy for j is

$$s^*_j(\theta_j,\lambda) \quad = \quad \begin{cases} 0 & \text{if} \quad \theta_j < \underset{\sim}{\theta}(\lambda,k) \\ s'_j(\theta_j,\lambda) & \text{if} \quad \theta_j \geq \underset{\sim}{\theta}(\lambda,k) \end{cases} \tag{17}$$

Note that $s^*_j(\cdot,\lambda)$ is piecewise affine in θ_j and does not depend on the distribution of θ_i. To summarize what we have established so far, if players select the strategy pair $\left(s_i^\lambda(\cdot), s^*_j(\cdot,\lambda), \boldsymbol{\iota}_{-i,j}(\cdot)\right)$, for some $\lambda \geq 0$, then from (12), the outcome resulting from a type vector θ will be

$$t(\theta; k, \left(s_i^\lambda(\cdot), s^*_j(\cdot,\lambda), \boldsymbol{\iota}_{-i,j}(\cdot)\right)) \;=\; \begin{cases} (\phi_2)^{-1}\left(\phi_0 + \phi_1(\Sigma\theta_{-j} \;+\; \lambda) - \breve{g}(k)\right) & \text{if} \quad \theta_j < \underset{\sim}{\theta}(\lambda,k) \\ (\phi_2)^{-1}\left(\phi_0 + \phi_1\Sigma\theta - g_j(k)\right) & \text{if} \quad \theta_j \geq \underset{\sim}{\theta}(\lambda,k) \end{cases} \tag{18}$$

while, from (8), the net impact of $\Sigma\theta$ will be $\phi\left(\theta, \breve{t}_{\breve{g}(k)}\left(s_i^\lambda(\theta_i), s^*_j(\theta_j,\lambda), \theta_{-i,j}\right)\right) \triangleq \psi(\theta_j|k,\lambda)$,[14] where

$$\psi(\theta_j|k,\lambda) \;=\; \begin{cases} \phi_1(\theta_j \;-\; \lambda) + \breve{g}(k) & \text{if} \quad \theta_j < \underset{\sim}{\theta}(\lambda,k) \\ g_j(k) & \text{if} \quad \theta_j \geq \underset{\sim}{\theta}(\lambda,k) \end{cases} \tag{19}$$

Observe that this expression is independent of θ_i. We can now establish that every two-person game has a unique PSNE. An important property of this equilibrium is that the optimal action for low types of player j is a corner solution.

Proposition 7 (Existence of a unique Equilibrium)*: *Let f satisfy assumptions A11-A13 and assume that all players other than i and j truthfully reveal their types. For all k, there exists $\lambda^* \geq 0$ such that $s^* = \left(s_i^{\lambda^*}, s_j^*(\cdot,\lambda^*)\right)$ is the unique PSNE for the game defined by f and k. Moreover, $\underset{\sim}{\underline{\theta}}(\lambda^*,k) > \underline{\theta}$. For all $\theta \geq \underset{\sim}{\underline{\theta}}(\lambda^*,k)$, $s_j^*(\theta,\lambda^*)$ is a best conceivable action for (j,θ).*

Figure 3 provides the intuition for our proof of existence. The horizontal broken lines in the

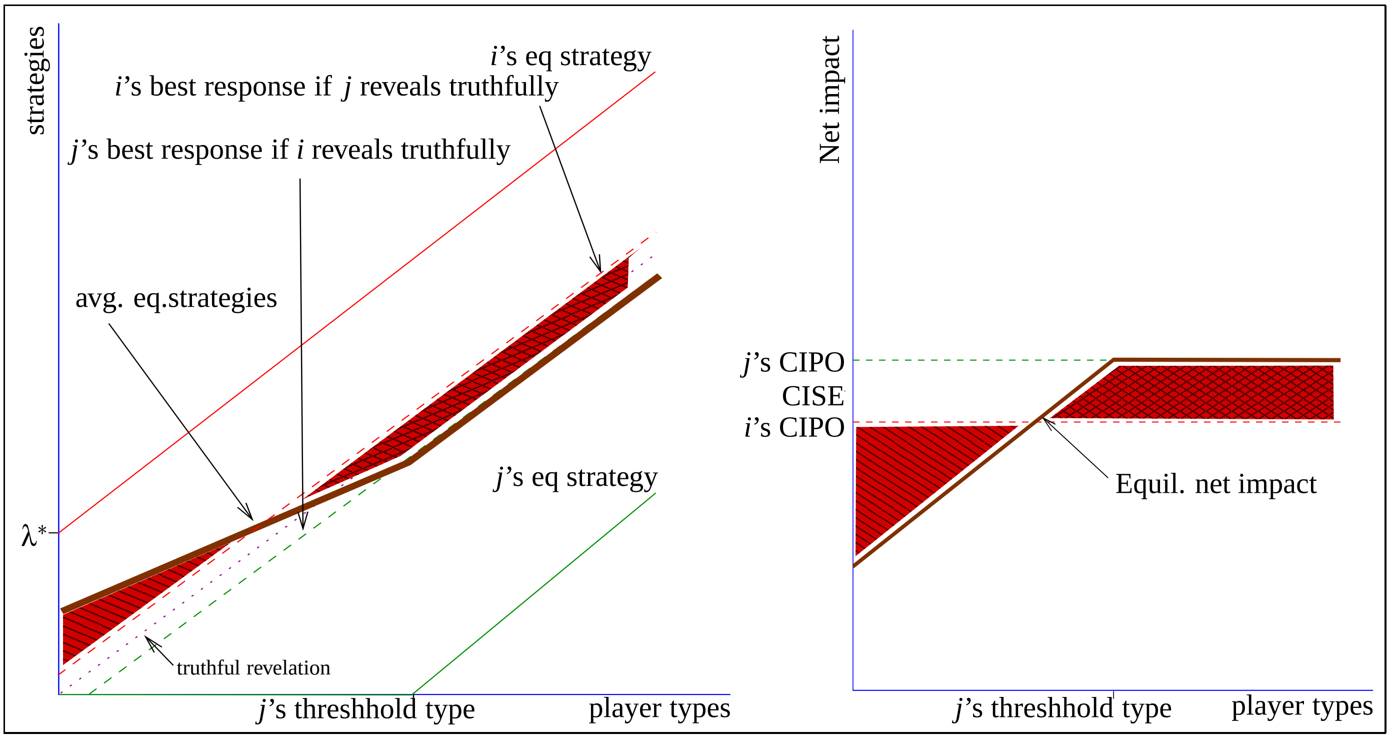

FIGURE 3. Intuition for the existence part of Prop. 7

right panel represent the CISE net impact and the CIPO net impacts for i and j, all of which are independent of player types. The solid line represents the net impact that results from the equilibrium profile s^*: it is a function of θ_j but is independent of θ_i. The equilibrium strategies themselves are depicted by the thin solid lines in the left panel; the thick solid line is the average of these strategies. The dotted line would be the average strategy if both players truthfully revealed their types; the two dashed lines are the strategies that would be optimal for each player if the

[14]We use the symbol "$\triangleq$" to mean "defined equal to".

other truthfully revealed his type: in this case (except for the lowest types of j), r's strategy would implement r's CIPO net impact. For the case in which $f(\cdot,k_i)$ is a quadratic function, the defining characteristic of our equilibrium strategies is that (in either panel) the area of the striped shaded region above i's dashed line equals the area of the cross-hatched region below it. This condition can be satisfied only if i over-reports and j under-reports: if i did *not* over-report, the striped region would be eliminated. On the other hand, if i over-reports by so much that even the highest types of j are pushed to the corner, then the cross-hatched region would be eliminated. The intermediate value theorem now implies the existence of an optimal level of over-reporting for i. To understand this, note that j can *costlessly* under-report to counteract i's over-report, unless constrained by the lower bound. On the other hand, i's over-reporting comes at a cost: the average equilibrium outcome is too low when j's type is high and too high when j's type is low (compared with i's CIPO net impact level). The optimal report for i is defined by the condition that these two effects (or the two shaded areas in the Figure) exactly offset each other.

The figure also illustrates another important point: because i's strategy is to announce his true type plus some intercept term that is independent of his type, j can *exactly* offset i's over-reporting provided he is not pushed to the corner. Thus to the right of the kink, the equilibrium strategies implement j's CIPO net impact. That is, high types of j play best conceivable actions (p. 10). On the other hand, because j's strategy is kinked, no type of player i has a best conceivable action.

To prove uniqueness, it suffices to show that a necessary condition for an equilibrium is that i's strategy is unit affine. Figure 4 extends the idea used to prove Prop. 2, to motivate this condition. Suppose i plays a strategy such as $\hat{s}_i$ which is not unit affine and let $\underline{\lambda}$ denote the largest λ such that $\iota(\cdot)+\lambda$ lies below $\hat{s}_i$. We will show that $\underline{\lambda}$ cannot be less than the equilibrium level, λ^*, of i's intercept. An analogous argument shows that $\underline{\lambda}$ cannot strictly exceed λ^*. Clearly, j's best response, $\hat{s}_j$, to $\hat{s}_i$ must lie below j's best response to $s_i^{\underline{\lambda}}$. But we have established that the level, $\underline{\lambda}$, of over-reporting is insufficient to offset j's under-reporting if j plays $s_j^*(\cdot,\underline{\lambda})$. Hence it must be even more inadequate when j under-reports even more by playing $\hat{s}_j$. Hence $\hat{s}_i(\underline{\vartheta})$ cannot be a best response for player-type $(i,\underline{\vartheta})$ against $\hat{s}_j$.

In the remainder of this section, we will consider the comparative statics effects of an increase in i's observable characteristic, while j's either remains constant or decreases. Formally, we will fix two observable characteristic profiles $\tilde{k}$ and $\hat{k}$ such that $\hat{k}_i > \tilde{k}_i$ and $\hat{k}_j \leq \tilde{k}_j$. Letting $k(\alpha) =$

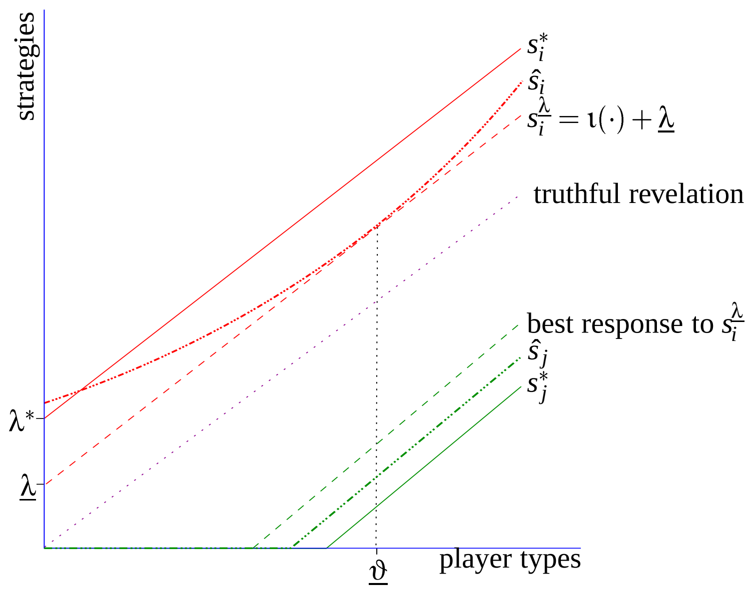

FIGURE 4. Intuition for the uniqueness part of Prop. 7

$\alpha\hat{k} + (1-\alpha)\tilde{k}$, $\breve{g}(\alpha) = \breve{g}(k(\alpha))$ and $g_i(\alpha) = g_i(k(\alpha))$, we can now study the comparative statics effects of increasing α from zero to one. We will assume that as α increases, i's CIPO net impact declines at a faster rate than the CISE net impact (which may in fact increase):[15]

$$\breve{g}'(\alpha) \quad > \quad g_i'(\alpha) \tag{20}$$

Prop. 7 confirms that our model can be reduced to one equation in one unknown, $\lambda(\alpha)$. We can therefore streamline notation by defining, for all $\alpha \in [0,1]$, $s^\alpha = \left(s_i^{\lambda(\alpha)}, s_j^*(\cdot, \lambda(\alpha))\right)$, as well as:

$$\begin{aligned}
\underset{\sim}{\theta}(k) &= \underset{\sim}{\theta}(\lambda(\alpha),k); \quad \nu(\alpha) = \int_{\underline{\theta}}^{\underset{\sim}{\theta}(k)} d\eta(\vartheta_j)\\
\psi_\alpha(\theta_j) &= \psi(\vartheta_j|k,\lambda(\alpha)) = \begin{cases} \phi_1(\theta_j - \lambda(\alpha)) + \breve{g}(\alpha) & \text{if} \quad \theta_j < \underset{\sim}{\theta}(k) \\ g_j(\alpha) & \text{if} \quad \theta_j \geq \underset{\sim}{\theta}(k) \end{cases}
\end{aligned} \tag{21}$$

$$\frac{d\psi_\alpha(\theta_j)}{d\alpha} \triangleq \psi_\alpha'(\theta_j) = \begin{cases} \breve{g}'(\alpha) - \phi_1\lambda'(\alpha) & \text{if} \quad \theta_j < \underset{\sim}{\theta}(k) \\ g_j'(\alpha) & \text{if} \quad \theta_j \geq \underset{\sim}{\theta}(k) \end{cases} \tag{$19^{\alpha'}$}$$

[15] There are a number of ways to ensure that (20) will be satisfied. Since $\breve{g}'(\alpha) = -\frac{f_{12}(\breve{g}(k),k_i)\frac{\partial k_i(\alpha)}{\partial\alpha} + \sum_{r\neq i} f_{12}(\breve{g}(k),k_r(\alpha))\frac{\partial k_r(\alpha)(\alpha)}{\partial\alpha}}{\sum_r f_{11}(\breve{g}(k),k_r(\alpha))}$, the simplest is to require that $\hat{k}_r(\alpha) \leq \tilde{k}_r(\alpha)$, for all $r \neq i$.

$$t_\alpha = t(\cdot;\alpha,s^\alpha); \quad Et(\alpha) = \int_\Theta t_\alpha(\vartheta)d\eta(\vartheta) = (\phi_2)^{-1}\big(\phi_0 + n\phi_1 E\theta - E\psi_\alpha\big) \tag{22}$$

$$\check{t}_\alpha = \check{t}_{\breve{g}(\alpha)}; \quad E\check{t}(\alpha) = \int_\Theta \check{t}_\alpha(\vartheta)d\eta(\vartheta) = (\phi_2)^{-1}\big(\phi_0 + n\phi_1 E\theta - \breve{g}(\alpha)\big) \tag{$\check{22}$}$$

$$Eu_r(\alpha) = \int_\Theta u(t_\alpha(\vartheta),\vartheta,k_r(\alpha)(\alpha))d\eta(\vartheta) \tag{23}$$

We will refer to t_α as the *equilibrium outcome* and to ψ_α as the *equilibrium net impact* when the observable characteristics profile is $k(\alpha)$. From (22), there is a very close relationship between these two concepts. Indeed, (18) and (19) imply that $\frac{dt_\alpha(\theta)}{d\alpha} = -\frac{d\psi_\alpha(\theta_j)}{d\alpha}/\phi_2$.

<u>Comparative statics of heterogeneity</u>: We now investigate the effects of increasing α and thus heightening the tension between i's and j's strategic objectives. We focus on three sets of dependent variables: (a) players' *strategies*; (b) the *expected equilibrium outcome* (relative to the CISE outcome); and (c) the difference between players' *expected equilibrium utilities*, again relative to their CISE expected utilities. Our first result follows immediately from Prop. 5-2P.

Proposition 8 (Comparative statics of λ): *If f satisfies A11-A13 then $\lambda'(\cdot) > 0$ on $[0,1]$.*

We next consider the effect of increasing heterogeneity on the expected equilibrium outcome, $Et(\alpha)$, relative its effect on the expected CISE outcome, $E\check{t}(\alpha)$. It will be apparent from (22) and ($\check{22}$) that an essentially equivalent issue is the relationship between the expected equilibrium net impact and the CISE net impact. As α increases, i's CIPO net impact level increases while j's does not increase. Even though j-types above j's threshhold type implement j's CIPO impact with probability one, the *ex ante* expected equilibrium impact is weakly below i's CIPO impact! To see this, recall that the first order condition defining $\lambda(\alpha)$ is, for all α:

$$U_i'(\lambda(\alpha),\alpha) = \phi_1\left\{\int_{\underline{\theta}}^{\overline{\theta}} f_1\big(\psi_\alpha(\vartheta_j),\, k_i(\alpha)\big)d\eta(\vartheta_j) \;-\; \frac{1}{\phi_2}\right\} \equiv 0 \tag{24}$$

Since $f_1(\cdot,\, k_i(\alpha))$ is convex (A14) a necessary condition for (24) is that

$$\int_{\underline{\theta}}^{\overline{\theta}} \psi_\alpha(\vartheta_j)d\eta(\vartheta_j) \leq g_i(\alpha), \tag{25}$$

since if this inequality were reversed, then by Jensen's inequality

$$\int_{\underline{\theta}}^{\overline{\theta}} f_1\big(\psi_\alpha(\vartheta_j),\, k_i(\alpha)\big)d\eta(\vartheta_j) \quad \geq \quad f_1\big(\int_{\underline{\theta}}^{\overline{\theta}} \psi_\alpha(\vartheta_j)d\eta(\vartheta_j),\, k_i(\alpha)\big) \quad > \quad f_1\big(g_i(\alpha),\, k_i(\alpha)\big) \quad = \quad \frac{1}{\phi_2}$$

violating (24). It now follows immediately from (22), ($\check{22}$), (25) and (11) that[16]

$$Et(\alpha) - E\check{t}(\alpha) \quad = \quad (\phi_2)^{-1}(\check{g}(\alpha) - E\psi_\alpha) \quad \geq \quad (\phi_2)^{-1}(\check{g}(\alpha) - g_i(\alpha)) \quad > \quad 0 \quad (26)$$

$$\text{while} \quad Et'(\alpha) - E\check{t}'(\alpha) \quad = \quad (\phi_2)^{-1}\big(\check{g}'(\alpha) - E\psi'_\alpha\big) \quad (27)$$

Unless a further assumption is imposed on f, the sign of (27) is indeterminate.[17] We cannot, therefore, conclude that $Et(\alpha) - E\check{t}(\alpha)$ increases monotonically with α. However, since $\check{g}'(\alpha) - g'_i(\alpha)$ is positive by assumption and since the the gap between $Et(\alpha)$ and $E\check{t}(\alpha)$ weakly exceeds the gap between $\check{g}(\alpha)$ and $g_i(\alpha)$, $Et(\alpha) - E\check{t}(\alpha)$ is bounded below by a monotone increasing function. In other words, there may exist $\alpha' > \alpha$ such that $(Et(\cdot) - E\check{t}(\cdot))$ declines on $[\alpha, \alpha')$, but it never declines by enough to offset all of the gains made on $[0, \alpha)$. Formally, the result below follows immediately from (26) and (20), with $\zeta(\cdot) = (\phi_2)^{-1}\big(\check{g}(\cdot) - g_i(\cdot)\big)$:

Proposition 9-2P (Comparative statics of $Et - E\check{t}$): *If f satisfies A11-A14 and* (20) *holds, then there exists a positive, strictly increasing function ζ such that $(Et(\cdot) - E\check{t}(\cdot)) \geq \zeta(\cdot)$ on* $[0,1]$.

We now turn to the impact on i's and j's utility of an increase in heterogeneity. Since the disutility to i of a given net impact level increases with heterogeneity, i becomes more willing, while j becomes less willing, to incur costs in order to lower this impact. Consequently, even if i could implement in equilibrium his CIPO net impact, $g_i(\alpha)$, with probability one, i's equilibrium payoff would still decline with heterogeneity. It is, therefore, more instructive to focus on the *payoff gap*, $\Delta Eu_r(\alpha) = Eu_r(\alpha) - E\check{u}_r(\alpha)$, between the expected values of player r's equilibrium and CISE expected utilities. Integrating (5) and multiplying both sides by ϕ_2, we obtain

$$\phi_2 \Delta Eu_r(\alpha) = \phi_2\left\{\Big(E\check{t}(\alpha) - Et(\alpha)\Big) - \Big(\int_{\underline{\theta}}^{\overline{\theta}} f\big(\psi_\alpha(\vartheta_j),\, k_r(\alpha)\big)d\eta(\vartheta_j) - f(\check{g}(\alpha),\, k_r(\alpha))\Big)\right\}$$

$$\text{which, from (27)} = \Big(\phi_2 f(\check{g}(\alpha),\, k_r(\alpha)) - \check{g}(\alpha)\Big) - \Big(\phi_2 \int_{\underline{\theta}}^{\overline{\theta}} f\big(\psi_\alpha(\vartheta_j),\, k_r(\alpha)\big)d\eta(\vartheta_j) - E\psi_\alpha\Big) \quad (28)$$

[16] Note that if f_{11} is a constant, then $E\psi_\alpha = g_i(\alpha)$. This fact will be used in the proof of Prop. 13 below.

[17] It is straightforward to check that if we assume in addition to assumptions A11-A14 that f_{11} is concave and f_{12} is convex, then (27) (as well as the expression $\Delta Eu_i(\alpha) - \Delta Eu_j(\alpha)$, defined below) will be positive.

Our first observation is that the sign of i's payoff gap is indeterminate. To see this, set $r = i$ in expression (28) and suppose that f is quadratic in net impact, i.e., f_{11} is constant. In this case, $E\psi_\alpha = g_i(\alpha)$. Now the expression $F_r(x) = (\phi_2 f(x,\, k_r(\alpha)) - x)$ is minimized w.r.t. x at $g_i(\alpha)$. The term inside the first parentheses in (28) exceeds the minimum value of $F_i(\cdot)$ because the CISE net impact level $\breve{g}(\alpha)$ is superoptimal from i's perspective; the term inside the second achieves *in expectation* the minimum value of F_i but has a positive variance and F_i is convex. Hence the sign of $\Delta Eu_i(\alpha)$ will be negative or positive, depending on the extent of i's risk aversion (i.e., the convexity of F_i) relative to the difference between i's CIPO and the CISE net impact levels. Now reinstate the assumption that $f_{111} \geq 0$ and consider $\Delta Eu_j(\alpha)$. The sign of this expression is determinate if and only if $g_j(\alpha) \leq \breve{g}(\alpha)$. In this case, from (25) and (11), $E\psi_\alpha \leq g_i(\alpha)$ is further from $g_j(\alpha)$ than $\breve{g}(\alpha)$, so that the first parenthesized term in (28) is dominated in expectation by the second. Moreover, once again the second has a positive variance and $F_j(\cdot)$ is convex. Hence, whenever $g_j(\alpha) < \breve{g}(\alpha)$, j's payoff gap will be unambiguously negative. Summarizing and formalizing the above, we have:

Proposition 10 (Signs of players' payoff gaps): *There exists* $\bar{\delta} > \underline{\delta} > 0$ *such that* $\Delta Eu_i(\alpha) \begin{cases} > 0 & \text{if } \sup\ f_{11}(\cdot, k_i) < \underline{\delta} \\ < 0 & \text{if } \inf\ f_{11}(\cdot, k_i) > \bar{\delta} \end{cases}$. *A sufficient condition for* $\Delta Eu_j(\alpha) < 0$ *is that* $g_j(\alpha) < \breve{g}(\alpha)$.

While players' individual payoff gaps cannot be signed, the difference between them can be. Our last result establishes that the difference between i's and j's payoff gaps is bounded below by a function ζ that is positive and monotone increasing in heterogeneity.[18] The intuition for this result is as follows: the difference $(g_i(\alpha) - \breve{g}(\alpha))$ between the equilibrium and CISE net impacts is negative. Because $f_{12} > 0$, this difference benefits i more than j, so that the bounding function ζ is positive. As k_i increases with α, $g_i(\alpha)$ declines; $\breve{g}(\alpha)$ may also decline, but by assumption (20) it declines by no more than $g_i(\alpha)$. To ensure that ζ increases with α, it is sufficient that that the net effect of these declines benefits i more than j. For this, it is in turn sufficient that the difference between the benefits of the decline in $g_i(\alpha)$ and in $\breve{g}(\alpha)$ is no less for i than for j. Since $\breve{g}(\alpha) > g_i(\alpha)$, this in turn is ensured by our assumption (A14) that $f(\cdot, k_i(\alpha))$ is not more convex than $f(\cdot, k_j(\alpha))$.

[18] footnote 17

Proposition 11-2P (Comparative statics of $\Delta Eu_i(\alpha) - \Delta Eu_j(\alpha)$)*: *If f satisfies A11-A14 and* (20) *holds, there is a positive, strictly increasing function ζ such that $(\Delta Eu_i(\cdot) - \Delta Eu_j(\cdot)) \geq \zeta(\cdot)$.*

7. Quasi-linear games with n players.

The model as specified in § 5 yields indeterminate comparative statics results when there are more than two players. However, if we assume in addition that f_1 is quadratic in net impact, i.e.,

Assumption A15: For all $k \in [0,1]$, $f_{11}(\cdot,k)$ is a constant

then, because we can now interchange the integral and the "f" operators, the multi-player model becomes simple to analyze. In particular, letting $\Sigma Es_{-r} = \sum_{j\neq r} \int_{\underline{\theta}}^{\overline{\theta}} s_j(\vartheta)d\eta(\vartheta)$, we have, from (14)

$$U_r'(a,\theta_r;s_{-r}) \quad = \quad \phi_1 \left\{ f_1\Big(\left[\breve{g}(k) + \phi_1(\theta_r - a + \Sigma Es_{-r} - (n-1)E\theta)\right], k_r(\alpha)\Big) - 1/\phi_2 \right\} \tag{29}$$

Property (29) has a number of immediate and significant consequences. First,

$$\text{Given A11-A15, } U_r'(a,\theta_r;s_{-r}) \underset{=}{\overset{<}{}} U_r'(a,\theta_r;s_{-r}') \text{ if and only if } (\Sigma Es_{-r} - \Sigma Es_{-r}') \underset{=}{\overset{>}{}} 0 \tag{30}$$

Second, given ΣEs_{-r}, the first order condition $U_r'(a,\theta_r,k;s_{-r}) = 0$ implicitly defines a unique value of $\theta_r - a$, which in turn implies that every PSNE is piecewise affine, i.e.,

$$\text{Given A11-A15, if } s \text{ is a PSNE, } \exists \boldsymbol{\lambda} \in \mathbb{R}^n \text{ such that for each } r, s_r(\cdot) = \max(\underline{a}, \iota(\cdot) + \lambda_r) \tag{31}$$

The third consequence of (29) is slightly less immediate. For each r, the expected equilibrium outcome conditional on r's type exceeding his threshhold type is r's expected CIPO outcome.[19]

$$\text{Given A11-A15, if } s \text{ is a PSNE then, for each } r, Et\left(\cdot;k,s|\theta_r \geq \underset{\sim}{\theta}_r(s)\right) = E\check{t}_{g_r(k)}. \tag{32}$$

Property (31) implies the n-person analog of Prop. 6-2P, which has extensive consequences.

[19]To verify (32), note that for each r and each $\theta_r \geq \underset{\sim}{\theta}_r(s)$

$$U_r'(s_r(\theta_r),\theta_r,k;s_{-r}) \underbrace{=}_{\text{expr (4)}} 0 \underbrace{=}_{\text{expr (14)}} f_1\Big(\left[\breve{g}(k) + \phi_1(\Sigma Es - nE\theta)\right], k\Big) - 1/\phi_2 \underbrace{=}_{f_{11}>0} f_1(g_i(k),k) - 1/\phi_2$$

Proposition 6-nP (The largest player is never constrained)*: *Let s be an MPSNE for an n-person game with f satisfying A11-A15, and $k_i > k_r(\alpha)$, for $r \neq i$. Then $\underset{\sim}{\theta}_i(s) = \underline{\theta}$.*

Prop. 6-nP and property (32) then implies that the (unconditional) expected outcome of any PSNE coincides with the expectation of i's CIPO outcome.

Proposition 12 (PSNE's implement *i*'s CIPO expected outcome): *Let s be an MPSNE for an n-person game with f satisfying A11-A15, and $k_i > k_r(\alpha)$, for $r \neq i$. Then $Et(\cdot;k,s) = E\check{t}_{g_i(k)}$.*

We can now characterize the unique equilibrium of the n-person game. It has a somewhat surprising property: for each player r other than i (who has the largest observable characteristic), r's equilibrium strategy in the unique equilibrium for the n-person game coincides with r's strategy in the unique equilibrium for the two-person game between r and i. That is, if we start out with a game in which only i and one other player act strategically, while all other players passively reveal their private information, and then successively introduce strategic play by other players, then player i makes all of the adjustments, and the original participants in the game make none. Formally, for each $j \neq i$, define the profile s^j as follows: for $r \neq j, i$, $s^j_r = \iota(\cdot)$ while for i and j, s^j_i and s^j_j form an equilibrium for the induced game between these two players. Now define the strategy profile $\hat{s}$ by the condition that for each $j \neq i$, $\hat{s}_j = s^j_j$, while $\hat{s}_i$ is a best reply to $\hat{s}_{-i}$.

Proposition 13 (Unique equilibrium for the *n*-person Game)*: *Let s be an MPSNE for an n-person game with f satisfying A11-A13 and A15, and $k_i > k_r(\alpha)$, for $r \neq i$. Then $s = \hat{s}$.*

Prop. 13 follows almost immediately from Prop. 12. To see that the combination of two-player equilibria is an equilibrium for the n-player game, note that regardless of how many players are playing strategically, there is a unique scalar $\bar{\omega}$ such that the term in square brackets in (29) equals i's CIPO net impact. Whether i is playing against j alone, or against all other players, to construct a best response he must adjust the intercept of his unit affine strategy so that the expected sum of all players' strategies, ΣEs, equals $\bar{\omega}$. Therefore, because i has compensated exactly for the impact of all players other than himself and j, the number ΣEs_{-j} must be, in equilibrium, invariant to whether j is playing against i alone, or against all these other players. Now consider whether a

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That is, $E\phi\left(\cdot, t(\cdot;k,s)\middle|\theta_r \geq \underset{\sim}{\theta}_r(s)\right) = g_i(k)$. From (7), therefore

$$Et\left(\cdot;k,s|\theta_r \geq \underset{\sim}{\theta}_r(s)\right) \quad = \quad \frac{\phi_0 + n\phi_1 E\theta - g_i(k)}{\phi_2} \quad = \quad E\check{t}_{g_r(k)}, \text{ verifying (32).}$$

profile $\hat{s}$ for the n-player game can be an equilibrium if for some $j \neq i$, $\hat{s}_j$ differs from s^j_j. Once again, i must adjust so that $\Sigma E\hat{s} = \bar{\omega}$. Choose the unit affine strategy $\tilde{s}_i$ such that $E\tilde{s}_i = \Sigma E\hat{s}_{-j}$. If $\hat{s}$ is to be an equilibrium, then the profile $(\hat{s}_j, \tilde{s}_i)$ must be an equilibrium for the game between i and j. But since this latter game has a unique equilibrium, this possibility cannot arise.

The comparative statics properties of our n-player game can now be obtained as easy corollaries to preceding results. The first is almost the exact counterpart to Prop. 5-2P, and follows immediately from that result, together with Props. 6-nP, 12 and 13.

Proposition 5-nP (Heterogeneity: n players): *Suppose that f satisfies A11-A13 and A15. Consider two observable characteristic profiles, k and $\hat{k}$ such that $\hat{k}_i > k_i$ and $\hat{k}_r \leq k_r$, for all $r \neq i$. Then $\hat{s}_i(\cdot) > s_i(\cdot)$, while for all $r \neq i$, $\underset{\sim}{\theta}_r(\hat{s}) > \underset{\sim}{\theta}_r(s)$ and $\hat{s}_r(\theta) < s_r(\theta)$ for all $\theta > \underset{\sim}{\theta}_r(s)$.*

Now let $\tilde{k} = (1/n, ..., 1/n)$ and let $\hat{k}$ be an observable characteristics profile such that $\hat{k}_i > \hat{k}_r$, for all $r \neq i$. For $\alpha \in [0,1]$, define $k(\alpha) = \alpha\hat{k} + (1-\alpha)\tilde{k}$. As α increases, players become increasingly heterogeneous in an *ex ante* sense. Note that in this context, a sufficient condition for inequality (20) is that $f_{122} \leq 0$.[20] Our last two results, relating to the comparative statics effects of increasing heterogeneity, are strong, n-player versions of Prop. 9-2P and Prop. 11-2P. Thanks to Prop. 12 we can in both cases replace the phrase "bounded below by an increasing function of α" in the 2-player result with, simply "increases with α". Defining $Et(\alpha)$ and $E\breve{t}(\alpha)$ as on p. 24, our first result is an immediate implication of Prop. 12.

Proposition 9-nP (Comparative statics of $Et - E\breve{t}$): *Suppose that f satisfies A11-A13 and A15. If inequality (20) is satisfied, then $(Et'(\cdot) - E\breve{t}'(\cdot)) > 0$ on $[0,1]$.*

Recalling the definition of player r's *payoff gap*, $\Delta Eu_r(\alpha)$ (p. 25), our second result is that larger players have larger payoff gaps, and this difference increases with heterogeneity.

[20] For all r, $\frac{\partial k_r(\alpha)}{\partial\alpha}(k_r(\alpha) - 1/n) \geq 0$, while, since $f_{122} \leq 0$, $f_{12}(\breve{g}(\alpha), k_r(\alpha)) - f_{12}(\breve{g}(\alpha), 1/n)(k_r(\alpha) - 1/n) \leq 0$, so that

$$\breve{g}'(k(\alpha)) = -\frac{f_{12}(\breve{g}(\alpha), k_i(\alpha))\frac{\partial k_i(\alpha)}{\partial\alpha} + \sum_{r\neq i} f_{12}(\breve{g}(\alpha), k_r(\alpha))\frac{\partial k_r(\alpha)}{\partial\alpha}}{\sum_r f_{11}(\breve{g}(\alpha), k_r(\alpha))} \geq -\frac{f_{12}(\breve{g}(\alpha), k_i(\alpha))\frac{\partial k_i(\alpha)}{\partial\alpha} + f_{12}(\breve{g}(\alpha), 1/n)\sum_{r\neq i}\frac{\partial k_r(\alpha)}{\partial\alpha}}{\sum_r f_{11}(\breve{g}(\alpha), k_r(\alpha))}$$

$$\geq -\frac{f_{12}(\breve{g}(\alpha), 1/n)\frac{\partial k_i(\alpha)}{\partial\alpha} + f_{12}(\breve{g}(\alpha), 1/n)\sum_{r\neq i}\frac{\partial k_r(\alpha)}{\partial\alpha}}{\sum_r f_{11}(\breve{g}(\alpha), k_r(\alpha))} = 0 > g_i'(k(\alpha))$$

Proposition 11-nP (Comparative statics of $\Delta Eu_r(\alpha) - \Delta Eu_j(\alpha)$): *If f satisfies A11-A15 and* (20) *holds, then for each r, j with $\hat{k}_r > \hat{k}_j$, $(\Delta Eu_r(\cdot) - \Delta Eu_j(\cdot))$ and $(\Delta Eu'_r(\cdot) - \Delta Eu'_j(\cdot))$ are both positive on* $[0,1]$.

While our analysis in this section mirrors in many respects our analysis in §6, there is one important respect in which it is different. We observed in §6 that high types of the smaller player play best conceivable actions in equilibrium. In general, this property has no counter-part in the *n*-player game. The reason is that best conceivable actions are available only when all other players play unit affine strategies. When there are more than two players (and a unique largest player), the strategies of all but the largest player are piecewise, but not unit affine. Consequently, like the larger player in §6, smaller players in this section cannot perfectly counter-balance the aggregate impact of other players' misreporting.

8. Conclusion

This paper introduces a new framework for studying the problem of information transmission, when players' individual signals must be aggregated in order to make decisions that affect all of them. Rather than seeking the most general formulation, we have imposed strong restrictions on the model, to reveal the inner workings of aggregation games in a relatively transparent way.

Prop. 1 established that every aggregation game has a pure strategy equilibrium in which players' equilibrium strategies are strictly monotone in their types. Props. 2 and 4 showed that truthful revelation is an equilibrium if and only if all players are identical *ex ante*. When they diverge *ex ante*, larger players over-report and smaller players under-report their types; from Prop. 5-2P, this mis-reporting is exacerbated as the divergence increases, at least in two-player games. For the remainder of the paper, the restrictions we impose ensure that every game we consider has a unique equilibrium. Prop. 3 proved that for two-player games with *ex ante* identical players truthtelling is the unique equilibrium provided that the lower bound on players' reports coincides with the lower bound on their types. This, together with Prop. 5-2P, implies that in two-person games, all types of a player have interior solutions to their optimization problems if and only if that player is the largest player. Our remaining results apply to a restricted specification of the model: the main restriction is that the first argument of players' utilities is related to the aggregate signal and the outcome by an affine transformation. Prop. 7 shows that in the unique equilibrium for the two-player version of this specfication, players play "unit affine" strategies. Props. 8-11-2P provide a number of results, showing that in an *ex ante* sense, the larger player essentially controls the game while *ex post*, high types of the smaller player have total control. To conclude our analysis, we add a further restriction to payoffs and return to n-player games. Our main result is that in the unique equilibrium for this specification, each player but the largest plays the same strategy as he plays in the two-person game against the largest player alone.

Our analysis might fruitfully be extended in a number of directions. One route is to consider situations where players face an upper, as well as a lower bound on strategies. Another is to relax our assumption that the central authority passively aggregates information transmitted from the periphery, and to then address some of the issues raised in the literature (see p. 3) concerning the relative merits of delegation versus communication.

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Appendix: Proofs

Proof of Proposition 1: To prove the proposition we apply Theorems 1 and 2 of Athey (2001). The first of these theorems is used to establish existence for finite-action aggregation games. The second implies existence for general aggregation games. To apply Athey's first theorem, we define a *finite action aggregation game* to be one in which players are restricted to choose actions from a finite subset of A. In all other respects, finite action aggregation games are identical to (infinite action) aggregation games. We now check that u satisfies Athey's Assumption A1. Clearly, our types have joint density w.r.t. Lebesgue measure which is bounded and atomless. Moreover, the integrability condition in Athey's A1 is trivially satisfied since u is bounded. Moreover, inequality (3) implies that the SCC holds. Therefore, every finite action aggregation game has a MPSNE in which player r's equilibrium strategy s_r is nondecreasing.

Athey's Theorem 2 requires that the action space A is a compact interval. In our model, however, A is unbounded above. To apply her theorem, therefore, we will establish any equilibrium for an aggregation game satisfying A1-A15 will also be an equilibrium for the restricted aggregation game that is identical to the original, except that players are required to choose actions in the compact set $[\underline{a},\bar{a}]$, where $\bar{a}$ was identified in assumption A9. By Athey's Theorem 2, the restricted game has an MPSNE, call it s^*. To show that s^* will also be an equilibrium for the original, unrestricted game, it suffices to show that for all r, all θ and all $a > \bar{a}$, $\frac{\partial U_r(a,\theta;s^*_{-r})}{\partial a} < 0$. To establish this, note that $s^*_{-r} \geq \underline{s}_{-r} \geq 0$, so that since t is strictly increasing, $a > \bar{a}$ implies

$$U_r'(a,\theta;s^*_{-r}) \quad < \quad U_r'(\bar{a},\theta;s^*_{-r}) \quad \leq \quad U_r'(\bar{a},\theta;\underline{s}_{-r}) \quad \leq \quad U_r'(\bar{a},\theta;0) \quad \leq \quad 0$$

Finally, to establish that s_r is strictly increasing and continuously differentiable on $(\underset{\sim}{\theta}_r(s),\bar{\theta}]$, note that $U_r'(s_r(\cdot),\cdot;s_{-r}) = 0$ on $(\underset{\sim}{\theta}_r(s),\bar{\theta}]$. From (3), assumption A10 and the implicit function theorem, we have, for all $\theta \in (\underset{\sim}{\theta}_r(s),\bar{\theta}]$, $\frac{ds_r(\theta)}{d\theta} = -\frac{\partial^2 U_r(s_r(\theta),\theta;s_{-r})}{\partial a \partial \theta} \Big/ \frac{\partial^2 U_r(s_r(\theta),\theta;s_{-r})}{\partial a^2} > 0$. ■

Proof of Proposition 3: We first assume that s is admissible and unit affine but not ZSUA. i.e., that there exists $\lambda \in \mathbb{R}^2$ such that $s_r = \theta_r + \lambda_r$, with $\sum_{r=i}^{j} \lambda_r \neq 0$. Admissibility implies that $\lambda_r \geq \underline{a} - \underline{\theta}$. Fix θ_j arbitrarily:

$$U_j(s_j(\theta_j),\theta_j;s_i) \quad = \quad \int_\Theta u(t(\vartheta_i + \theta_j + \sum_{r=i}^{j} \lambda_r), \vartheta_i + \theta_j, \bar{k}) d\eta(\vartheta_i)$$

which, since t is CISE

$$< \quad \int_\Theta u(t(\vartheta_i + \theta_j), \vartheta_i + \theta_j, \bar{k}) d\eta(\vartheta_i) \quad = \quad U_j(\theta_j - \lambda_i, \theta_j; s_i)$$

That is, $s_j(\cdot)$ is not a best response against s_i so that s is not an equilibrium profile. Now assume that s is not unit affine. Without loss of generality, assume that there exists $\lambda > 0$ such that $s_i(\theta) \geq \theta - \lambda$, with strict inequality holding for some $\bar{\theta}_i$. Since strategies are continuous (Prop. 1), $s_i(\theta) \gneqq \theta - \lambda$. We now show that if s_j is a best response to s_i, then $(s_j(\cdot) - \iota(\cdot)) < \lambda$. Consider s_j such that $s_j(\bar{\theta}_j) \geq \bar{\theta}_j + \lambda$, for some $\bar{\theta}_j$, so that $\bar{\theta}_j + \lambda + s_i(\cdot) \gneqq \bar{\theta}_j + \iota(\cdot)$. Assumption A9 then implies

$$t(s_j(\bar{\theta}_j) + s_i(\cdot)) \quad \geq \quad t(\bar{\theta}_j + \lambda + s_i(\cdot)) \quad \gneqq \quad t(\bar{\theta}_j + \iota(\cdot))$$

Since U_j is concave in t and, for all θ_i, $u(\cdot, \bar{\theta}_j + \theta_i, \bar{k})$ is maximized at $t(\bar{\theta}_j + \theta_i) = \bar{\theta}_j + \theta_i$, we have

$$U_j'(s_j(\bar{\theta}_j),\bar{\theta}_j;s_i) \; = \; \int_\Theta \frac{du}{da}(t(s_j(\bar{\theta}_j) + s_i(\vartheta_i)), \bar{\theta}_j + \vartheta_i, \bar{k}) d\eta(\vartheta_i) \; \leq \; \int_\Theta \frac{du}{da}(t(\bar{\theta}_j + \lambda + s_i(\vartheta_i)), \bar{\theta}_j + \vartheta_i, \bar{k}) d\eta(\vartheta_i)$$

$$< \int_\Theta \frac{du}{da}(t(\overline{\theta}_j+\vartheta_i),\overline{\theta}_j+\vartheta_i,\overline{k})d\eta(\vartheta_i) \;=\; 0$$

This establishes that if s_j is a best response to s_i, then $(s_j(\cdot)-\iota(\cdot))<\lambda$. But in this case, $s_i(\overline{\theta}_i)+s_j(\cdot)<\overline{\theta}_i+\iota(\cdot)$, implying that $t(s_i(\overline{\theta}_i)+s_j(\cdot)) \;<\; t(\overline{\theta}_i+\iota(\cdot))$, so that

$$U_i'(s_i(\overline{\theta}_i),\overline{\theta}_i;s_j) \;> 0$$

Therefore, $s_i(\cdot)$ is not a best response for i against $s_j(\cdot)$ at $\overline{\theta}_i$. ■

Proof of Proposition 4: Let s be an MPSNE and for each r. Assume that $k_j<k_i$ and pick $\theta^*\in\Theta^*=\mathrm{argmax}(s_j-s_i)$. Let $\gamma=s_j(\theta^*)-s_i(\theta^*)$. We have

$$\begin{aligned} U_j'(s_j(\theta^*),\theta^*;s_{-j}) \quad &= \quad U_j'(s_i(\theta^*)+\gamma,\theta^*;s_{-j}) \quad = \quad U_j'(s_i(\theta^*),\theta^*;s_{-j}+\gamma) \\ &< \quad U_i'(s_i(\theta^*),\theta^*;s_{-j}+\gamma) \quad \le \quad U_i'(s_i(\theta^*),\theta^*;s_{-i}) \quad = \quad 0 \end{aligned}$$

The strict inequality follows from assumption A5. The weak inequality holds because $s_j(\cdot)-\gamma\le s_i(\cdot)$, so that $s_{-i}-\gamma\le s_{-j}$, and from assumption A10. Since, $U_j'(s_j(\theta^*),\theta^*;s_{-j})<0$ it follows from (4) that $s_j(\theta^*)=\underline{a}$, so that $\Theta^*\subset[\underline{\theta},\underset{\sim}{\theta}_j(s)]$. By definition of θ^*, $0\ge(s_j(\theta^*)-s_i(\theta^*))\ge(s_j(\cdot)-s_i(\cdot))$. It follows that $s_j(\theta)>\underline{a}$ implies $s_i(\theta)>\underline{a}$, which in turn implies that $[\underline{\theta},\underset{\sim}{\theta}_i(s)]\subset[\underline{\theta},\underset{\sim}{\theta}_j(s)]$. Moreover, since $s_i(\cdot)$ is strictly increasing on $(\underset{\sim}{\theta}_i(s),\overline{\theta}]$, $\Theta^*\subset[\underline{\theta},\underset{\sim}{\theta}_i(s)]$.

Now consider $\underset{\sim}{\theta}_i(s)$. If $s_i(\underset{\sim}{\theta}_i(s))>\underline{a}$ then $[\underline{\theta},\underset{\sim}{\theta}_i(s)]=\{\underline{\theta}\}=\Theta^*$. If $s_i(\underset{\sim}{\theta}_i(s))=\underline{a}$ then $s_i(\underset{\sim}{\theta}_i(s))=s_j(\underset{\sim}{\theta}_i(s))=0$ and $\underset{\sim}{\theta}_i(s)\in\Theta^*$. In either case, therefore, $U_j'(s_j(\underset{\sim}{\theta}_i),\underset{\sim}{\theta}_i(s);s_{-j})<0$. By continuity, $U_j'(s_j(\cdot),\cdot;s_{-j})<0$ on a neighborhood of $\underset{\sim}{\theta}_i(s)$. Hence $\underset{\sim}{\theta}_j(s)>\underset{\sim}{\theta}_i(s)$, proving part (a) of the proposition. Now note that for $\theta>\underset{\sim}{\theta}_j(s)$:

$$0 \quad = \quad U_j'(s_j(\theta),\theta;s_{-j}) \quad \le \quad U_j'(s_j(\theta),\theta;s_{-i}) \quad < \quad U_i'(s_j(\theta),\theta;s_{-i}).$$

The weak inequality follow from A10, since $s_i\ge s_j$ implying $s_{-j}\ge s_{-i}$; the strict inequality is implied by A5. It now follows that $s_i(\theta)>s_j(\theta)$. ■

Proof of Proposition 5-2P: Let s be an equilibrium for the game with k and consider s' such that in the game with $\hat{k}$, $s_i{}'$ is a best reply to $s_j{}'$ but $\gamma_i=\max_{\vartheta\in[\underset{\sim}{\theta}_i(s),\overline{\theta}]}(s_i(\vartheta)-s'(\vartheta))\ge 0$ so that $s_i{}'+\gamma_i\ge s_i$. Pick $\vartheta_i\in\mathrm{argmax}_{\vartheta\in[\underset{\sim}{\theta}_i(s),\overline{\theta}]}(s_i(\vartheta)-s'(\vartheta))$. Let $\hat{U}_r$ be defined as in (2), with $\hat{k}_r$ replacing k_r. We will show that $s_j{}'-\gamma_i>s_j$. If this were not the case then

$$0 \quad \underbrace{\ge}_{\text{expr (4)}} \quad \hat{U}_i'(s_i'(\vartheta_i),\vartheta_i;s_j{}') \quad \underbrace{>}_{\hat{k}_i>k_i,\text{A5}} \quad U_i'(s_i'(\vartheta_i),\vartheta_i;s_j{}') \quad \underbrace{=}_{\text{defn of }\vartheta_i,\gamma_i} \quad U_i'(s_i(\vartheta_i)-\gamma_i,\vartheta_i;s_j{}')$$

$$\underbrace{=}_{\text{A7}} \quad U_i'(s_i(\vartheta_i),\vartheta_i;s_j{}'-\gamma_i) \quad \underbrace{\geq}_{s_j{}'-\gamma_i\leq s_j,\text{A10}} \quad U_i'(s_i(\vartheta_i),\vartheta_i;s_j)$$

contradicting the fact that since $\vartheta_i \in [\underset{\sim}{\theta}_i(s),\overline{\theta}]$, we have $U_i'(s_i(\vartheta_i),\vartheta_i;s_j) = 0$. Conclude that $\gamma_j = \max(s_j{}' - s_j) > \gamma_i$ and pick $\vartheta_j \in \operatorname{argmax}(s_j{}' - s_j)$. Note that $s_j{}'(\vartheta_j) > s_j(\vartheta_j) \geq a$, so that a necessary condition for $s_j{}'(\vartheta_j)$ to be a best reply to s_i is that $\hat{U}_j'(s_j'(\vartheta_j),\vartheta_j;s_i{}') = 0$. However:

$$\begin{aligned} 0 \quad \underbrace{\geq}_{\text{expr (4)}} \quad & U_j'(s_j(\vartheta_j),\vartheta_j;s_i) \quad \underbrace{=}_{\text{defn of }\vartheta_j,\gamma_j} \quad U_j'(s_j'(\vartheta_j)-\gamma_j,\vartheta_j;s_i) \quad \underbrace{>}_{\gamma_j>\gamma_i,\ \text{A10}} \quad U_j'(s_j'(\vartheta_j)-\gamma_i,\vartheta_j;s_i) \\ \underbrace{=}_{\text{A7}} \quad & U_j'(s_j'(\vartheta_j),\vartheta_j;s_i-\gamma_i) \quad \underbrace{\geq}_{s_i-\gamma_i\leq s_i{}',\text{A10}} \quad U_j'(s_j'(\vartheta_j),\vartheta_j;s_i{}') \quad \underbrace{\geq}_{\hat{k}_j\leq k_j,\text{A5}} \quad \hat{U}_j'(s_j'(\vartheta_j),\vartheta_j;s_i{}') \end{aligned}$$

Conclude that $s_j{}'(\vartheta_j)$ cannot be best reply to s_i, and hence that a necessary condition for $\hat{s}$ to be an equilibrium for the game with $\hat{k}$ is that $\hat{s}_i(\cdot) > s_i(\cdot)$ on the interval $[\underset{\sim}{\theta}_i(s),\overline{\theta}]$. In particular, $\hat{s}_i(\underset{\sim}{\theta}_i(s)) > a$, so that $\underset{\sim}{\theta}_i(\hat{s})) < \underset{\sim}{\theta}_i(s)$. But in this case, for all ϑ_j,

$$0 \quad \underbrace{\geq}_{\text{expr (4)}} \quad U_j'(s_j(\vartheta_j),\vartheta_j;s_i) \quad \underbrace{>}_{\hat{s}_i\gneqq s_i,\text{A10}} \quad U_j'(s_j(\vartheta_j),\vartheta_j;\hat{s}_i) \quad \underbrace{\geq}_{\hat{k}_j\leq k_j,\text{A5}} \quad \hat{U}_j'(s_j(\vartheta_j),\vartheta_j;\hat{s}_i)$$

Conclude that $\underset{\sim}{\theta}_j(\hat{s}) > \underset{\sim}{\theta}_j(s)$ and that $\vartheta_j \geq \underset{\sim}{\theta}_j(s)$ implies $\hat{s}_j(\vartheta_j) < s_j(\vartheta_j)$. ■

Proof of Proposition 7: To reduce notation, we will suppress reference throughout this proof to the passive strategies (i.e., $\boldsymbol{\iota}_{-i,j}(\cdot)$) played by the players other than i and j.

Existence: Since we have established that $s_j^*(\cdot,\lambda^*)$ is a best response to $s_i^{\lambda^*}$, we need only prove that $s_i^{\lambda^*}$ is a best response to $s_j^*(\cdot,\lambda^*)$. From (19), i's payoff from playing $s_i^{\lambda}(\theta_i)+a_i$ against $s_j^*(\cdot,\lambda)$ when his type is θ_i is given by:

$$\begin{aligned} U_i(s_i^{\lambda}(\theta_i)+a_i,\theta_i,k;s_j^*(\cdot,\lambda)) &= \mu - \int_{\underline{\theta}}^{\overline{\theta}} \check{t}_{\breve{g}(k)}\big(s_i^{\lambda}(\theta_i)+a_i, s_j^*(\vartheta_j,\lambda),\theta_{-i,j}\big)d\eta(\vartheta_j) \\ &\quad - \int_{\underline{\theta}}^{\overline{\theta}} f\big(\psi(\theta_j|k,\lambda)-\phi_1 a_i,\ k\big)d\eta(\vartheta_j) \\ \text{so that } U_i'(s_i^{\lambda}(\theta_i)+a_i,\theta_i,k;s_j^*(\cdot,\lambda))\Big|_{a_i=0} &= \phi_1\left\{\int_{\underline{\theta}}^{\overline{\theta}} f_1\big(\psi(\theta_j|k,\lambda),\ k\big)d\eta(\vartheta_j) - \frac{1}{\phi_2}\right\} \quad (33) \end{aligned}$$

Since the right hand side of (33) is independent of θ_i, we will henceforth abbreviate its left-hand side to $U_i'(\lambda,\alpha)$. Since U_i is concave in i's actions, a sufficient condition for $s_i^{\lambda^*}$ to be a best reply to $s_j^*(\cdot,\lambda^*)$ is that $U_i'(\lambda^*,\alpha)=0$. To establish that such a λ^* exists, we will identify $\underline{\lambda}$ and $\overline{\lambda}$ such that

$U_i'(\underline{\lambda},\alpha) > 0 > U_i'(\overline{\lambda},\alpha)$. Existence of λ^* will then follow from the intermediate value theorem. Moreover, since $f_{11} > 0$ and $\frac{d\psi(\theta_j|\alpha,\cdot)}{d\lambda} \le 0$, with strict inequality when $\theta_j < \underset{\sim}{\theta}(\lambda,k)$, it follows that $U_i'(\cdot,\alpha)$ is strictly decreasing in λ, implying that λ^* is uniquely defined.

Set $\underline{\lambda} = (\breve{g}(k) - g_i(k))/\phi_1$. Substituting into (33), we have

$$U_i'(\underline{\lambda},\alpha) \quad = \quad \phi_1 \left\{ \int_{\underline{\theta}}^{\underset{\sim}{\theta}(\underline{\lambda},k)} f_1\big(\psi(\theta_j|k,\lambda),\, k\big) d\eta(\vartheta_j) \quad + \quad \int_{\underset{\sim}{\theta}(\underline{\lambda},k)}^{\overline{\theta}} f_1\big(g_j(k),\, k\big) d\eta(\vartheta_j) \quad - \quad \frac{1}{\phi_2} \right\}$$

which, since $\psi(\theta_j|k,\lambda) = \phi_1\vartheta_j + g_i(k)$ on $\big[0, \underset{\sim}{\theta}(\underline{\lambda},k)\big)$, $f_{11} > 0$ and $\underset{\sim}{\theta}(\underline{\lambda},k) > 0$

$$> \quad \phi_1 \left\{ \int_{\underline{\theta}}^{\underset{\sim}{\theta}(\underline{\lambda},k)} \underbrace{f_1\big(g_i(k),\, k\big) d\eta(\vartheta_j)}_{=1/\phi_2} \quad + \quad \int_{\underset{\sim}{\theta}(\underline{\lambda},k)}^{\overline{\theta}} \underbrace{f_1\big(g_j(k),\, k\big)}_{>1/\phi_2} d\eta(\vartheta_j) \; - \; \frac{1}{\phi_2} \quad \right\} \;\ge\; 0$$

Now set $\overline{\lambda} = \overline{\theta} + (\breve{g}(k) - g_i(k))/\phi_1$ so that $\underset{\sim}{\theta}(\overline{\lambda},k) = \overline{\theta} + \big(g_j(k) - g_i(k)\big)/\phi_1$. Since $g_j(k) > g_i(k)$, $\underset{\sim}{\theta}(\overline{\lambda},k) > \overline{\theta}$, so that $s_j^*(\cdot,\overline{\lambda})$ is identically zero on $[\underline{\theta},\overline{\theta}]$. Hence:

$$\begin{aligned} U_i'(\overline{\lambda},\alpha) \quad &= \quad \phi_1 \left\{ \int_{\underline{\theta}}^{\overline{\theta}} f_1\big(\psi(\theta_j|k,\lambda),\, k\big) d\eta(\vartheta_j) \; - \; \frac{1}{\phi_2} \right\} \\ &< \quad \phi_1 \left\{ f_1\big(\psi(\overline{\theta}|\alpha,\overline{\lambda}),\, k\big) \int_{\underline{\theta}}^{\overline{\theta}} d\eta(\vartheta_j) \; - \; \frac{1}{\phi_2} \right\} \quad = \quad 0, \quad \text{since } \psi(\overline{\theta}|\alpha,\overline{\lambda}) = g_i(k). \end{aligned}$$

We have, therefore, identified $\underline{\lambda}$ and $\overline{\lambda}$ such that $U_i'(\underline{\lambda},\alpha) > 0 > U_i'(\overline{\lambda},\alpha)$, implying the existence of λ^* such that $U_i'(\lambda^*,\alpha) = 0$. Moreover, since $\lambda^* > \underline{\lambda} = (\breve{g}(k) - g_i(k))/\phi_1 > (\breve{g}(k) - g_j(k))/\phi_1$, it follows from (16) that $\underset{\sim}{\theta}(\lambda^*,k) > \underline{\theta}$. Indeed, when $k \gg 0.5$, so that $g_j(k) \gg g_i(k)$, it is possible that $\underset{\sim}{\theta}(\lambda^*,k) > \overline{\theta}$, so that $s_j(\cdot,\lambda^*)$ is identically zero.

Uniqueness: Since λ^* is unique, we have established that s^* is the unique equilibrium such that s_i is unit affine. We now consider an arbitrary strategy, $\hat{s}_i(\cdot) = \iota(\cdot) + \lambda(\cdot)$, let $\hat{s}_j$ be a best response to $\hat{s}_i$ and show that if $\lambda(\cdot)$ is non-constant, then $\hat{s} = \big(\hat{s}_i, \hat{s}_j\big)$ cannot be a PSNE. Prop. 6-2P implies that we can restrict our attention to the case in which $\lambda(\cdot) > 0$. Let $E\lambda = \int_\Theta \lambda(\vartheta) d\eta(\vartheta)$. Since $\hat{s}_i$ is not unit affine, there exists $\Theta^-, \Theta^+ \subset \Theta$, with $\Theta^- \cup \Theta^+ \ne \emptyset$, such that $\hat{s}_i(\cdot) < s_i^{E\lambda}(\cdot)$ on Θ^- and $\hat{s}_i(\cdot) > s_i^{E\lambda}(\cdot)$ on Θ^+. We will assume that both Θ^- and Θ^+ have positive measure, leaving to the reader the task of checking that no equilibrium can exist when at least one of these sets has measure zero. Since the argument is a straightforward extension of the one used to prove Prop. 2, we shall be rather terse.

Pick $\underline{\vartheta} \in \operatorname{argmin}_{\vartheta\in\Theta}\lambda(\cdot)$, $\underline{\lambda} = \lambda(\underline{\vartheta})$, $\overline{\vartheta} \in \operatorname{argmax}_{\vartheta\in\Theta}\lambda(\cdot)$ and $\overline{\lambda} = \lambda(\overline{\vartheta})$. Since both Θ^- and Θ^+ have positive measure, we have $s_i^{\bar{\lambda}} \gneqq \hat{s}_i \gneqq s_i^{\underline{\lambda}}$, where, as on p. 19, $s_i^{\lambda}(\cdot) = \iota(\cdot) + \lambda$. Assume first that $\underline{\lambda} \leq \lambda^*$. From (12), $t(\cdot; k, s)$ is strictly increasing in s_i, so that $t(\theta; k, (\hat{s}_i, s_j^*(\cdot, \underline{\lambda}))) \gneqq t(\theta; k, (s_i^{\underline{\lambda}}, s_j^*(\cdot, \underline{\lambda})))$. Consequently, for all $\theta_j \geq \underset{\sim}{\theta}(\underline{\lambda}, k)$, $t\left((\cdot, \theta_j);, k, (\hat{s}_i, s_j^*(\theta_j, \underline{\lambda}))\right) \gneqq t_{g_j(k)}$, so that j's best response, $\hat{s}_j(\theta_j)$, to $\hat{s}_i$ must be strictly less than $s_j^*(\theta_j, \underline{\lambda})$. Therefore,

$$U_i'(\hat{s}_i(\underline{\vartheta}) + a_i, \underline{\vartheta}, k; \hat{s}_j)\big|_{a_i=0} \underbrace{>}_{\hat{s}_j \lneqq s_j^*(\cdot, \underline{\lambda}), U_j''<0} U_i'(\hat{s}_i(\underline{\vartheta}) + a_i, \underline{\vartheta}, k; s_j^*(\cdot, \underline{\lambda}))\big|_{a_i=0} \underbrace{\geq}_{\hat{s}_i(\underline{\vartheta}) \triangleq s_i^{\underline{\lambda}}(\underline{\vartheta}) \leq s_i^{\lambda^*}(\underline{\vartheta})} 0$$

We have established, therefore, that if $\underline{\lambda} \leq \lambda^*$, then $\hat{s}$ cannot be a PSNE. Next, assume that $\bar{\lambda} > \underline{\lambda} > \lambda^*$, so that $\hat{s}_i \lneqq s_i^{\bar{\lambda}}$. Reversing the above argument, it follows that for all $\theta_j \geq \underset{\sim}{\theta}(\bar{\lambda}, k)$, j's best response, $\hat{s}_j(\theta_j)$, to $\hat{s}_i$ must strictly exceed $s_j^*(\theta_j, \bar{\lambda})$. On the other hand, $\hat{s}_i(\bar{\vartheta}) = s_i^{\bar{\lambda}}(\bar{\vartheta})$ and by assumption $\bar{\lambda} > \lambda^*$, so that

$$U_i'(\hat{s}_i(\bar{\vartheta}) + a_i, \bar{\vartheta}, k; \hat{s}_j)\big|_{a_i=0} \quad < \quad U_i'(\hat{s}_i(\bar{\vartheta}) + a_i, \bar{\vartheta}, k; s_j^*(\cdot, \bar{\lambda}))\big|_{a_i=0} \quad < \quad 0$$

We have now established that if $\underline{\lambda} > \lambda^*$, then $\hat{s}$ cannot be a PSNE. ■

Proof of Proposition 11-2P: From (28), we have:

$$\begin{aligned}
& \Delta E u_i(\alpha) - \Delta E u_j(\alpha) \\
= & \Big(f(\breve{g}(\alpha), k_i(\alpha)) - f(\breve{g}(\alpha), k_j(\alpha))\Big) - \int_{\underline{\theta}}^{\bar{\theta}} \Big(f\big(\psi_\alpha(\vartheta_j), k_i(\alpha)\big) - f\big(\psi_\alpha(\vartheta_j), k_j(\alpha)\big)\Big) d\eta(\vartheta_j) \\
= & \int_{k_j}^{k_i} \left\{ f_2(\breve{g}(\alpha), k') - \int_{\underline{\theta}}^{\bar{\theta}} f_2\big(\psi_\alpha(\vartheta_j), k'\big) d\eta(\vartheta_j) \right\} dk'
\end{aligned}$$

From (11) and (25), $\breve{g}(\bar{\alpha}) > g_i(\alpha) \geq \int_{\underline{\theta}}^{\bar{\theta}} \psi_{\bar{\alpha}}(\vartheta_j) d\eta(\vartheta_j)$. Moreover since $f_{111} \geq 0$, $f_2\big(\int_{\underline{\theta}}^{\bar{\theta}} \psi_\alpha(\vartheta_j) d\eta(\vartheta_j), k'\big) \geq \int_{\underline{\theta}}^{\bar{\theta}} f_2\big(\psi_\alpha(\vartheta_j), k'\big) d\eta(\vartheta_j)$. Therefore

$$\Delta E u_i(\alpha) - \Delta E u_j(\alpha) \geq \quad \zeta(\alpha) \quad \triangleq \quad \int_{k_j}^{k_i} \left\{ f_2(\breve{g}(\alpha), k') - f_2(g_i(\alpha), k') \right\} dk' \quad > \quad 0$$

Since $f_{112} \leq 0$, $f_{21}(g_i(\alpha), k') \geq f_{21}(\breve{g}(\alpha), k') \geq 0$ while from (20) $\breve{g}'(\alpha) > g_i'(\alpha)$. Hence

$$\begin{aligned}
\zeta'(\alpha) = & \sum_{r=i}^{j} \left(f_2(\breve{g}(\alpha), k_r(\alpha)) - f_2(g_i(\alpha), k_r(\alpha)) \right) \\
& + \int_{k_j}^{k_i} \left\{ f_{21}(\breve{g}(\alpha), k') \breve{g}'(\alpha) - f_{21}(g_i(\alpha), k') g_i'(\alpha) \right\} dk' \quad > \quad 0
\end{aligned}$$ ■

Proof of Proposition 6-nP: We will establish that given A11-A15, a necessary condition for s to be an equilibrium is that $\lambda_i > 0$. Given (31) and the fact that $\underline{\theta} = a$, this fact immediately implies

Prop. 6-nP. Note first that necessarily, i's CIPO net impact level $g_i(k)$ strictly exceeds the CISE level $\breve{g}(k)$. Now consider a piecewise affine profile s such that $s_i(\cdot) = \max(\underline{a}, \iota(\cdot) + \lambda_i)$, with $\lambda_i \leq 0$. It then follows from Prop. 4 that $\lambda_r < 0$, for all $r \neq i$. But in this case $\Sigma E s_{-i} < (n-1)E\theta$. Letting $\boldsymbol{\iota}_{-i}(\cdot) = (\iota_r(\cdot))_{r \neq i}$, it now follows from (30) that:

$$\begin{aligned} U_i'(s_i(\theta_i), \theta_i, k; s_{-i}) \quad &< \quad U_i'(\theta_i, \theta_i, k; \boldsymbol{\iota}_{-i}(\cdot)) \\ &= \quad \phi_1 f_1\Big(\breve{g}(k), k\Big) - 1/\phi_2 \quad < \quad \phi_1 f_1\Big(g_i(k), k\Big) - 1/\phi_2 \quad = \quad 0 \end{aligned}$$

Conclude that s cannot be an equilibrium. ■

Proof of Proposition 13: We begin by showing that for each $j \neq i$, $\Sigma E \hat{s}_{-j} = \Sigma E s^j_{-j}$. We have

$$Et(\cdot; k, \hat{s}) \underbrace{=}_{(12)} \frac{\phi_0 + n\phi_1 \Sigma E\hat{s} - \breve{g}(\alpha)}{\phi_2} \underbrace{=}_{\text{Prop. 12}} E\breve{t}_{g_i(k)} \underbrace{=}_{(7)} \frac{\phi_0 + n\phi_1 E\theta - g_i(\alpha)}{\phi_2}$$

$$Et(\cdot; k, s^j) \underbrace{=}_{(12)} \frac{\phi_0 + n\phi_1 \Sigma E s^j - \breve{g}(\alpha)}{\phi_2} \underbrace{=}_{(22)} \frac{\phi_0 + n\phi_1 E\theta - E\psi_\alpha}{\phi_2} \underbrace{=}_{\text{footnote 16}} \frac{\phi_0 + n\phi_1 E\theta - g_i(\alpha)}{\phi_2}$$

so that $\Sigma E\hat{s} \quad = \quad E\hat{s}_j + \Sigma E\hat{s}_{-j} \quad = \quad \Sigma E s^j \quad = \quad E s^j_j + \Sigma E s^j_{-j}$

Since $\hat{s}_j = s^j_j$ by construction, it follows that $\Sigma E\hat{s}_{-j} = \Sigma E s^j_{-j}$. Property (30) now implies that s^j_j will be a best reply to s^j_{-j} iff $\hat{s}_j$ is a best reply to $\hat{s}_{-j}$. Since s^j_i is a best reply to s^j_{-j} (by construction) and $\hat{s}_i$ is a best reply to $\hat{s}_{-i}$ (by assumption), we have therefore established that $\hat{s}$ is an equilibrium for the n-person game iff for each $j \neq i$, s^j is an equilibrium for the two-person game between j and i. Uniqueness now follows from the uniqueness of the two-player equilibria and because given $\Sigma E s_{-i}$, i has a unique best reply of the form $\iota(\cdot) + \lambda_i$ (see (29) and (31)). ■

Proof of Proposition 11-nP: From (28), we have:

$$\begin{aligned} &\Delta E u_r(\alpha) \; - \; \Delta E u_j(\alpha) \\ = \quad &\Big(f(\breve{g}(\alpha),\, k_r(\alpha)) \; - \; f(\breve{g}(\alpha),\, k_j(\alpha))\Big) \; - \; \int_{\underline{\theta}}^{\overline{\theta}} \Big(f\big(\psi_\alpha(\vartheta_j),\, k_r(\alpha)\big) \; - \; f\big(\psi_\alpha(\vartheta_j),\, k_j(\alpha)\big)\Big) d\eta(\vartheta_j) \\ = \quad &\int_{k_j}^{k_r} \big\{ f_2(\breve{g}(\alpha),\, k') \; - \; f_2\big(g_i(\alpha),\, k'\big) d\eta(\vartheta_j) \big\}\, dk' \quad > \quad 0 \text{ (from (11) and } f_{12} > 0). \end{aligned}$$

Since $f_{112} = f_{211} \leq 0$, $f_{21}(g_i(\alpha),\, k') \geq f_{21}(\breve{g}(\alpha),\, k') \geq 0$ while from (20) $\breve{g}'(\alpha) > g_i'(\alpha)$. Hence

$$\begin{aligned} \Delta E u_r'(\alpha) \; - \; \Delta E u_j'(\alpha) \; = \quad &\sum_{\rho=r}^{j} \big(f_2(\breve{g}(\alpha),\, k_\rho(\alpha)) \; - \; f_2(g_i(\alpha),\, k_\rho(\alpha))\big) \\ &+ \int_{k_j}^{k_r} \big\{ f_{21}(\breve{g}(\alpha),\, k')\breve{g}'(\alpha) \; - \; f_{21}(g_i(\alpha),\, k')g_i'(\alpha) \big\}\, dk' \quad > \quad 0 \end{aligned}$$ ■