Pricing American-style Derivatives under the Heston Model Dynamics:

A Fast Fourier Transformation in the Geske–Johnson Scheme

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Introduction and Motivation

- European options: solution available for broad class of AJDs (*Duffie, Pan, Singleton, 2000*)

- Pricing American options has practical importance
- Popular approaches: finite difference schemes, simulation methods

- Efficient methods exist for Black–Scholes dynamics
- Use idea of the Geske–Johnson scheme (1984)

Plan of Talk

- Notation and Model
- Geske–Johnson Scheme: “Bermudan” recursion
- Specifics of:
  - joint characteristic function
  - characteristic function inversion
- Empirical application: pricing of S&P 100 options
- Pros and cons of FFT

Model

Heston's dynamics (under RNPM):

$$dS_t = (r-\delta)\,S_t dt + \sqrt{v_t} S_t dW_{1t} \quad \Leftrightarrow \quad ds_t = \left(r - \delta - \frac{v_t}{2}\right) dt + \sqrt{v_t} dW_{1t}$$

$$dv_t = (\alpha - \beta v_t)\, dt + \gamma \sqrt{v_t} dW_{2t}$$

$\{W_{1t}, W_{2t}\}_{t\geq 0}$: Brownian motions on $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \hat{\mathbb{P}}\right)$

Imperfect correlation: $d\langle W_1, W_2\rangle_t = \rho dt$ and $|\rho| < 1$

Money-market fund: $M_t = M_0 e^{rt}$, where $M_0 > 0$

Geske–Johnson Scheme

Sequence of “Bermudan”-style derivatives: $\{D_n(s_t, v_t, T-t)\}_{n=1}^{\infty}$

D_n can be exercised at times $t_j = t + \frac{j(T-t)}{n},\ j = 1, \ldots, n$

American option: D_∞ European option: D_1

“Bermudan” recursion: $D_n(s_t, v_t, T-t) =$

$$= e^{-r(t_1-t)} \hat{E}\left[\max\left\{EX(s_{t_1}, v_{t_1}, T-t_1), D_{n-1}(s_{t_1}, v_{t_1}, T-t_1)\right\}\right]$$

Exercise value for put: $EX(s_{t'}, v_{t'}, T-t') = (X - e^{s_{t'}})^+$

Linear Richardson extrapolation: $D_\infty \cong 2D_2 - D_1$

Joint Characteristic Function

Conditional ch. f.: $\Psi\left(t\right)=\hat{E}\left[e^{i\left(\zeta_{1}s_{T}+\zeta_{2}v_{T}\right)}|\mathcal{F}_{t}\right]=\Psi\left(\zeta_{1},\zeta_{2};s_{t},v_{t},\tau\right)$

Martingale property: $\hat{E}\left[d\Psi\left(t\right)|\mathcal{F}_{t}\right]=0$

P.d.e.:

$$\Psi_{\tau}=\Psi_{s}\left(r-\delta-\frac{v_{t}}{2}\right)+\Psi_{v}\left(\alpha-\beta v_{t}\right)+\Psi_{ss}\frac{v_{t}}{2}+\Psi_{vv}\gamma^{2}\frac{v_{t}}{2}+\Psi_{sv}\rho\gamma v_{t}$$

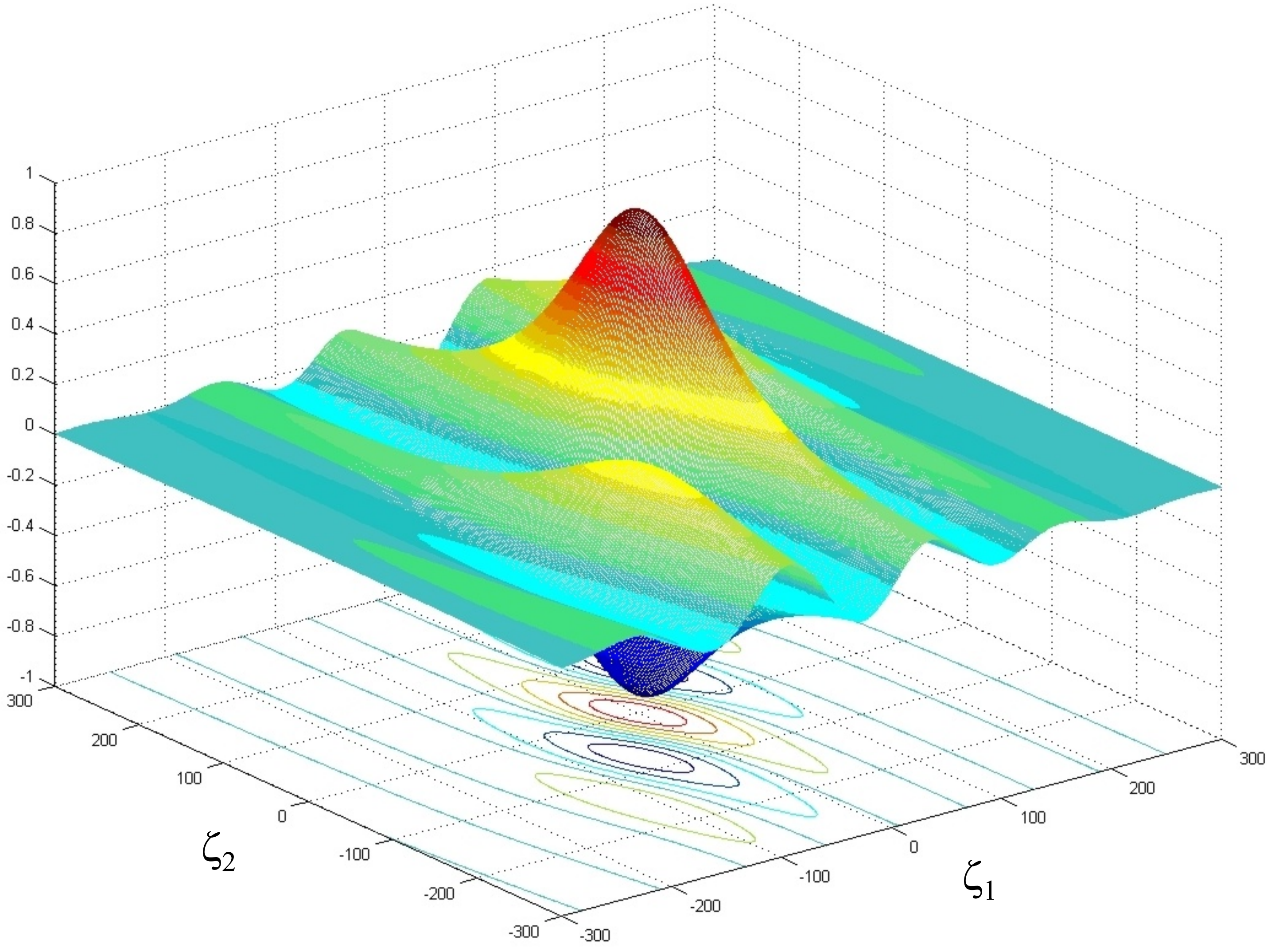
Trial solution:

$$\Psi\left(t\right)=\Psi\left(\zeta_{1},\zeta_{2};s_{t},v_{t},\tau\right)=\exp\left[p\left(\tau;\zeta_{1},\zeta_{2}\right)+q\left(\tau;\zeta_{1},\zeta_{2}\right)v_{t}+i\zeta_{1}s_{t}\right]$$

Solve analytically for: $p\left(\tau;\zeta_{1},\zeta_{2}\right)$ and $q\left(\tau;\zeta_{1},\zeta_{2}\right)$

Solution is rather involved

Characteristic Function: Real Part



Joint Density Function: One Step

Inversion: $f\left(s_T, v_T; s_t, v_t, \tau\right) =$

$$= \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\zeta_1 s_T + \zeta_2 v_T)} \Psi\left(\zeta_1, \zeta_2; s_t, v_t, \tau\right) d\zeta_1 d\zeta_2$$

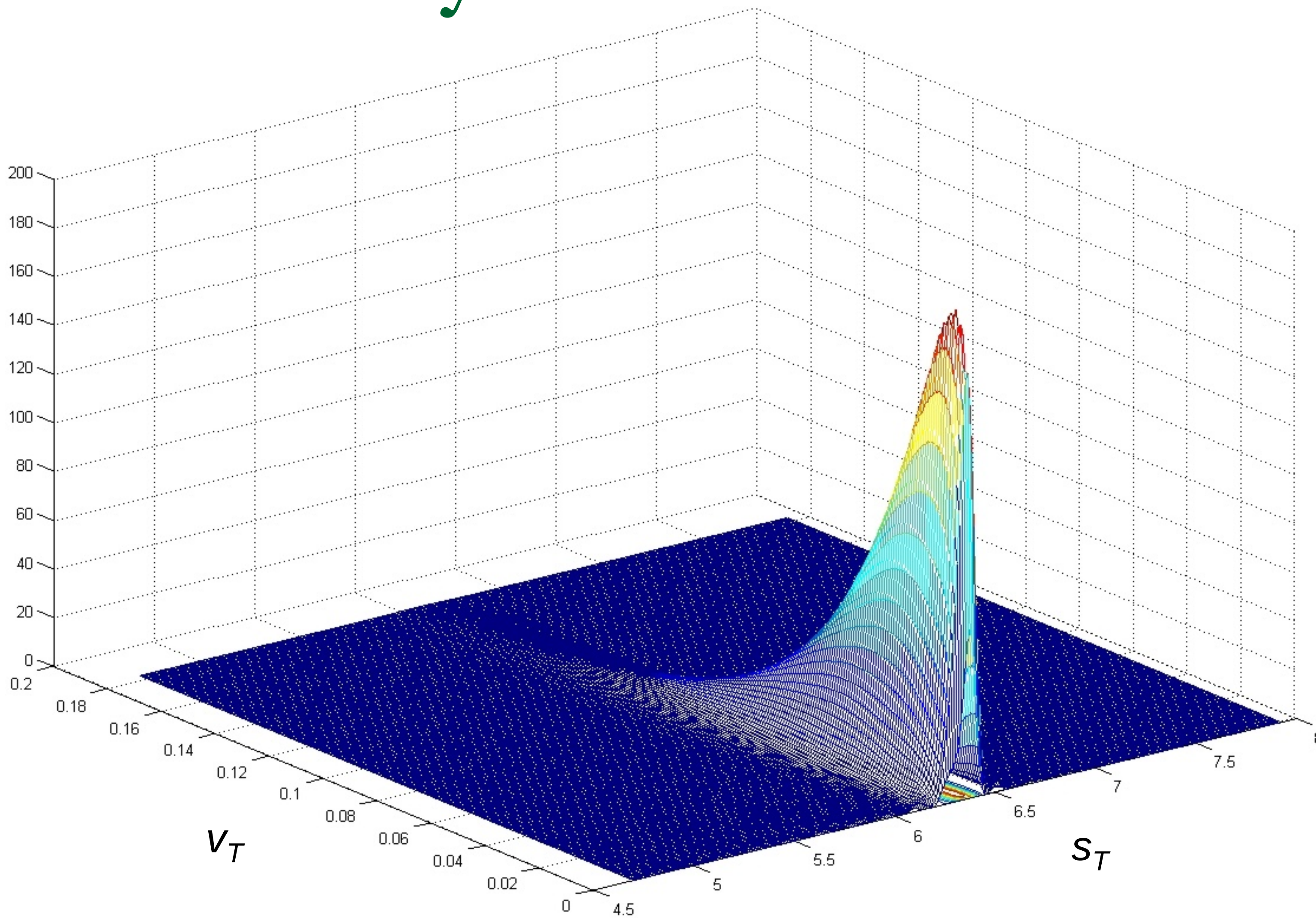
Discrete Fourier transform: $(2\pi)^2 f\left(s_{T,k_1}, v_{T,k_2}\right) \cong$

$$\cong \Delta_1 \Delta_2 e^{-i\left(s_{T,k_1} a_1 + v_{T,k_2} a_2\right)} \sum_{j_2=0}^{N_2-1} \sum_{j_1=0}^{N_1-1} e^{-i \cdot 2\pi \left(k_1 \frac{j_1}{N_1} + k_2 \frac{j_2}{N_2}\right)} \Psi_{j_1, j_2}$$

Apply discrete FFT algorithm to $\sum_{j_2=0}^{N_2-1} \sum_{j_1=0}^{N_1-1} \Rightarrow$

$\Rightarrow$ restore f on s_T, v_T grid from Ψ on ζ_1, ζ_2 grid in **one step**

Joint Density Function



Empirical Application

- Data:
  - CBOE S&P 100 options: American (OEX), European (XEO)
  - Closing prices on June 30th – July 2nd, July 6th – July 9th, 2004
  - July 9th for out-of-sample predictions

- Calibrated parameters (on XEO):

Parameter	Value	Parameter	Value
$v_{t,01}$	0.011392	α	0.353948
$v_{t,02}$	0.010889	β	9.561292
$v_{t,03}$	0.008932	γ	0.763721
$v_{t,04}$	0.016582	ρ	-0.692404
$v_{t,05}$	0.012725		
$v_{t,06}$	0.015280		

Empirical Application: Results

Errors:

	June 30	July 1	July 2	July 6	July 7	July 8	July 9
S_t	553.87	549.01	547.17	543.33	544.25	540.21	542.63
X, type							
400, put	...	...	...	...	0.116577	...	0.170837
420, put	...	...	0.467125	...	...	...	...
440, put	0.272164	0.365445	0.337645	...	...	...	...
450, put	...	...	0.578615	...	0.476928	0.448424	...
460, put	0.477434	0.636137	...	0.555188	0.561046	0.419271	...
480, put	0.233909	0.738647	...	...	0.561046	...	0.926725
490, put	...	...	0.795218	...	...	...	...
500, put	...	0.852520	0.544044	0.434873	0.339729	0.603371	0.735753
510, put	0.694260	0.274149	...	...	0.101801	0.229474	0.419345
520, put	-0.503378	-0.343453	0.302446	-0.048642	-0.521728	-0.126598	-0.174747
530, put	-0.591617	...	...	0.175511	-0.740207	-0.090279	0.525561
540, put	-2.429237	-0.154515	-1.027132	-1.014578	-1.725362	...	-0.131715
550, put	-2.538264	0.520023	-0.366053	-0.103850	-0.295665	-1.805092	...
560, put	-1.522215	...	...	-1.752594	...	0.002115	-1.306124
580, put	...	-1.011618	...	0.074978	...	...	...

RMSEs:

	June 30	July 1	July 2	July 6	July 7	July 8	July 9
July	0.563594	0.729401	0.555327	0.640847	0.529694	1.097807	0.271871
August	1.316759	0.408102	0.490070	0.367468	0.686766	0.419591	0.311062
September	1.203947	0.849578	0.659790	0.878324	1.394570	1.029171	0.617068
October	0.442581	1.059348	0.551232	0.387529	1.425867	0.633056	0.563490
December	2.024635	2.129737	1.723453	1.578420	1.229747	0.984973	1.707936

Conclusion

- Advantages of FFT:
  - p.d.f. is recovered in one step
  - allows to apply equivalent-martingale approach
  - fast

- Limitations of FFT:
  - large RAM needed for speed and accuracy
  - little flexibility in choosing grids