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Martin J. Corless (S'79) was born in Dublin, Ireland, on July 24, 1955. He received the B.E. degree in mechanical engineering from University College, Dublin, Ireland, in 1977. He is currently pursuing the Ph.D. degree in mechanical engineering at the University of California, Berkeley.

At present, his main research interest is in the control of uncertain systems.

George Leitmann received the B.S. and M.A. degrees in physics from Columbia University, New York. For five years thereafter he worked on exterior ballistics problems for the U.S. government. In 1954 he returned to graduate school and in 1956 he obtained the Ph.D. in engineering science from the University of California, Berkeley.

He joined the faculty of the University of California, Berkeley, in 1957 and is presently Professor of Engineering Science and Associate Dean for Graduate Affairs. His current interests are in deterministic control of uncertain systems, guaranteed avoidance control, and differential games. He is an Associate Editor of the *Journal of Optimization Theory and Applications* and of the *Journal of Mathematical Analysis and Applications*, and on the Editorial Board of *Optimal Control Applications and Methods* and of *Acta Astronautica*. He is a member of the Scientific Council of the International Center for Mechanical Sciences. He is the author of various books including *The Calculus of Variations and Optimal Control* (Plenum, 1981); *Quantitative and Qualitative Games*, with Blaquière and Gérard (Academic Press, 1969); *An Introduction to Optimal Control* (McGraw-Hill, 1966).

Dr. Leitmann is a Foreign Member of the Academy of Science of Bologna, Italy, and a Corresponding Member of the International Academy of Astronautics. He is the recipient of the Pendray Aerospace Literature Award of the AIAA, of a von Humboldt U.S. Senior Scientist Award, and of the Levy Medal of the Franklin Institute.

An Exact Sequential Solution Procedure for a Class of Discrete-Time Nonlinear Estimation Problems

ROBERT KALABA, SENIOR MEMBER, IEEE, AND LEIGH TESFATSION, MEMBER, IEEE

Abstract—**An exact procedure is developed for sequentially updating the optimal solution for a general discrete-time nonlinear least-squares estimation problem as the process length increases and new observations are obtained. The optimal sequential estimation equations are derived by means of an imbedding on two physically meaningful parameters, namely, the duration of the dynamical process and the value of the final observation. The optimal sequential estimation equations are contrasted with the approximate sequential estimation equations which would be obtained via extended Kalman filtering.**

I. Introduction

During the 1960's the theory of Sridhar filtering for continuous-time nonlinear processes was developed and applied in a series of studies [1]–[3]. The basic problem under consideration was the sequential least-squares estimation of state variables $x(t)$ generated by a noisy nonlinear dynamical system

$$\dot{x}(t)=F(x(t))+\epsilon(t), \qquad t\in[0,T] \tag{1}$$

where observations $y(t)$ were obtained in the form

$$y(t)=x(t)+\eta(t), \qquad t\in[0,T]. \tag{2}$$

Linear approximations were used to derive sequential least-squares estimates. The form of the resulting sequential estimation equations was

$$\frac{d\hat{x}}{dT}(T)=F(\hat{x}(T))+2P(T)[y(T)-\hat{x}(T)] \tag{3a}$$

$$\frac{dP}{dT}(T)=2P(T)F'(\hat{x}(T))-2P(T)P(T)-\frac{1}{2k} \tag{3b}$$

where k denotes a weighting factor in the least squares criterion function. Equations (3) are analogous in form to

Manuscript received April 13, 1981. This work was supported by the National Science Foundation under Grant ENG77-28432. The work of R. Kalaba was also supported in part by the National Institutes of Health under Grant GM23732-03.

The authors are with the Department of Economics, University of Southern California, Los Angeles, CA 90007.

0018-9286/81/1000-1144$00.75 ©1981 IEEE

the well-known extended Kalman filter equations based on successive dynamical equation linearization and application of ordinary Kalman filter estimation [4].

In the intervening years much progress has been made in developing and refining approximate sequential estimation schemes for nonlinear systems [5]–[14]. However, parallel progress that has been made [15] in the use of invariant imbedding methods to treat nonlinear integral equations suggests that certain perceived difficulties in the use of imbedding techniques to obtain exact sequential estimation solutions for nonlinear systems could possibly be overcome by a more judicious selection of the boundary value parameters used for the imbedding.

In the present paper, imbedding techniques are used to develop an exact procedure for sequentially updating the optimal solution for the discrete-time analog of the basic Sridhar nonlinear least squares estimation problem. The imbedding is based on two physically meaningful parameters, namely, the duration T of the dynamical process and the value y_T of the final observation. The numerical instability problems which can arise when imbedding is based on artificially introduced parameters are thus avoided. More importantly, this choice of imbedding parameters allows the exact derivation of the optimal sequential estimation equations, without any need for approximations.

Although no statistical assumptions are used for the modeling and observational error terms, the optimal sequential estimation equations obtained in the linear case are shown to be analogous in form to the standard Kalman filter state estimation equations, which presuppose mutually and serially uncorrelated error terms. In contrast, the optimal sequential state estimation equations obtained in the nonlinear case are shown *not* to be analogous in form to the standard extended Kalman filter state estimation equations. Briefly, as will be clarified later, optimal sequential state estimation requires the explicit back updating of previous state estimates in addition to incremental observational adjustments.

The estimation problem and associated two-point boundary value problem are developed in Sections II and III. The sequential estimation equations are presented and interpreted in Section IV. Their derivation is given in Section V. The final Section VI summarizes the main results of the paper, and briefly outlines several extensions to be developed in future papers.

II. The Estimation Problem

Consider a nonlinear one-dimensional dynamical system described by

$$x_{t+1}=F(x_t)+\epsilon_t, \qquad t=0,\cdots,T-1 \tag{4}$$

where ϵ_t represents an unknown modeling error. The problem under consideration is the estimation of the state variables x_t on the basis of observations y_t obtained in the form

$$y_t=x_t+\eta_t, \qquad t=0,\cdots,T \tag{5}$$

where η_t represents an unknown observational error.

No statistical assumptions will be used for the error terms ϵ_t and η_t. Rather, we consider the problem of minimizing the least squares criterion function

$$W(x,\epsilon)\equiv\left[\sum_{t=0}^{T}(x_t-y_t)^2+k\sum_{t=0}^{T-1}\epsilon_t^2\right] \tag{6b}$$

with respect to $x_0,\cdots,x_T,\epsilon_0,\cdots,\epsilon_{T-1}$, subject to the restrictions

$$\mathbf{0}=V(x,\epsilon)\equiv\begin{pmatrix} x_1-F(x_0)-\epsilon_0 \\ \vdots \\ x_T-F(x_{T-1})-\epsilon_{T-1}\end{pmatrix} \tag{6b}$$

where k is a fixed positive scalar weight. Thus, state and error estimates are to be obtained at each time T by minimizing a sum of weighted squared residual errors.

III. Derivation of an Associated Two-Point Boundary Value Problem

There are several ways to derive a two-point boundary value problem representation for the necessary conditions which must be satisfied by any optimal solution for (6). The representation established below proved to be particularly useful in the subsequent derivation of a sequential estimation procedure, for, as seen in Section V, it allows one to convert the two-point boundary value problem into an initial value problem via an imbedding on the two physically meaningful parameters T and y_T.

Specifically, defining the Lagrangian function by $L(x,\epsilon,\mu)\equiv W(x,\epsilon)+\mu V(x,\epsilon)$, where $\mu\equiv(\mu_1,\cdots,\mu_T)$ is a vector of Lagrange multipliers, the Euler–Lagrange first-order conditions $\mathbf{0}=\partial L/\partial(x,\epsilon,\mu)$ can be expressed in the form

$$0=2[x_t-y_t]+\mu_t-F'(x_t)\mu_{t+1}, \qquad t=0,\cdots,T-1, \tag{7a}$$

$$0=\mu_0 \tag{7b}$$

$$0=x_T+\tfrac{1}{2}\mu_T-y_T \tag{7c}$$

$$0=2k\epsilon_t-\mu_{t+1}, \qquad t=0,\cdots,T-1 \tag{7d}$$

$$\mathbf{0}=V(x,\epsilon). \tag{7e}$$

Straightforward substitution then leads to the equivalent two-point boundary value representation for problem (7),

$$0=2[x_t-y_t]+\mu_t-F'(x_t)\mu_{t+1}, \qquad t=0,\cdots,T-1 \tag{8a}$$

$$0=x_{t+1}-F(x_t)-\frac{1}{2k}\mu_{t+1}, \qquad t=0,\cdots,T-1 \tag{8b}$$

$$0=\mu_0 \tag{8c}$$

$$y_T=x_T+\tfrac{1}{2}\mu_T. \tag{8d}$$

Since the rank of $\partial V/\partial(x,\epsilon)$ is T, independently of the trajectory point (x,ϵ) at which it is evaluated, (8) is necessary for a trajectory of values x_t and $\epsilon_t\equiv\mu_{t+1}/2k$ to solve the original cost minimization problem (6). In subsequent sections it is assumed that (8) has a unique solution $(x_t, \mu_t)_{t=0}^{T}$ for each $T>0$ and $y_T \in (-\infty, \infty)$. For $T=0$, the unique solution is defined to be $x_0 = y_0$ and $\mu_0 = 0$. Thus, (8) is sufficient as well as necessary for a solution to (6), assuming a solution to (6) exists.

Finally, it is also assumed in subsequent sections that for each $T \geqslant 0$ and $y_T \in (-\infty, \infty)$ the solution to (8) has a unique continuation over the interval $[T, T+1]$.

IV. The Sequential Estimation Equations

The filtering and smoothing equations presented below provide an exact sequential procedure for updating the initial and terminal solution values x_0 and (x_T, μ_T) for the nonlinear two-point boundary value problem (8) when the duration of the process is increased from T to $T+1$ and an additional observation y_{T+1} is obtained. Modifications can easily be introduced to allow the sequential updating of solution values x_t and μ_t at fixed intermediate time points t. (See Section V.)

Let $x(t, \beta_T, T)$ and $\mu(t, \beta_T, T)$ denote the tth period solution values for (8) when the duration of the process is T, $T \geqslant t \geqslant 0$, and the final observation takes on the value β_T, $-\infty < \beta_T < \infty$. Define

$$\rho_T(\beta_T) \equiv x(T, \beta_T, T), \qquad T \geqslant 0, \ -\infty < \beta_T < \infty \tag{9a}$$

$$\hat{x}(t|T) \equiv x(t, y_T, T), \qquad T \geqslant t \geqslant 0 \tag{9b}$$

$$\hat{\mu}(t|T) \equiv \mu(t, y_T, T), \qquad T \geqslant t \geqslant 0. \tag{9c}$$

A procedure will now be given for sequentially updating the smoothing solutions $\hat{x}(0|T)$ and filtering solutions $\hat{x}(T|T)$ and $\hat{\mu}(T|T)$ as the duration of the process increases from T to $T+1$ and an additional observation y_{T+1} is obtained, $T \geqslant 0$. A justification for the procedure is provided in Section V.

At time $T=0$, obtain the first observation y_0 and store the functions $\rho_0(\beta_0) = \beta_0$ and $x(0, \beta_0, 0) = \beta_0$, $-\infty < \beta_0 < \infty$. Output the initial state estimate

$$\hat{x}(0|0) = \rho_0(y_0) = y_0. \tag{10}$$

At time $T \geqslant 0$, in storage are y_T, $\rho_T(\cdot)$, and $x(0, \cdot, T)$. For each β_T in $(-\infty, \infty)$, calculate and store

$$\beta_{T+1} = F(\rho_T(\beta_T)) + \left[\frac{k+1}{k}\right]\left[\frac{\beta_T - y_T}{F'(\rho_T(\beta_T))}\right] \tag{11a}$$

$$\rho_{T+1}(\beta_{T+1}) = \left[\frac{k}{k+1}\right] F(\rho_T(\beta_T)) + \left[\frac{1}{k+1}\right]\beta_{T+1} \tag{11b}$$

$$x(0, \beta_{T+1}, T+1) = x(0, \beta_T, T). \tag{11c}$$

Obtain an additional observation y_{T+1}, and output the updated filtering and smoothing solutions

$$\hat{x}(T+1|T+1) = \rho_{T+1}(y_{T+1}) \tag{12a}$$

$$\hat{\mu}(T+1|T+1) = 2[y_{T+1} - \rho_{T+1}(y_{T+1})] \tag{12b}$$

$$\hat{x}(0|T+1) = x(0, y_{T+1}, T+1). \tag{12c}$$

As clarified in Section V, the uniqueness properties assumed for the solution of (8) for each $T \geqslant 0$ and $y_T \in (-\infty, \infty)$ guarantee that the map $\beta_T \mapsto \beta_{T+1}$ defined by (11a) is one-to-one and onto. Thus, $\rho_{T+1}(\beta_{T+1})$ and $x(0, \beta_{T+1}, T+1)$ in (11b) and (11c) are well-defined functions of β_{T+1} over $(-\infty, \infty)$, as required for the feasibility of (12) and subsequent step calculations.

The basic recurrence relation (11a) has an interesting interpretation. For each possible value β_{T+1} for the observation y_{T+1} at time $T+1$, (11a) yields the value β_T which the observation y_T at time T would have had to equal in order for all previous state and multiplier estimates to remain optimal. In other words, given $y_{T+1} = \beta_{T+1}$, then $y_T = \beta_T$ if and only if $\hat{x}(t|T) = \hat{x}(t|T+1)$ and $\hat{\mu}(t|T) = \hat{\mu}(t|T+1)$ for all $t \leqslant T$. (See Section V.) Thus, $[\beta_T - y_T]$ is a proxy measure for the amount of back updating required at time $T+1$ when an additional observation β_{T+1} is obtained.

The second basic recurrence relation (11b) also has an interesting interpretation. For each possible value β_{T+1} for the observation y_{T+1} at time $T+1$, the corresponding state estimate $\rho_{T+1}(\beta_{T+1}) \equiv x(T+1, \beta_{T+1}, T+1)$ for time $T+1$ is a weighted average of β_{T+1} and the state value $F(\rho_T(\beta_T))$ generated by the state function $F(\cdot)$ evaluated at the updated state estimate $\rho_T(\beta_T) = \hat{x}(T|T+1)$ for time T, i.e., the state estimate for time T based on the $T+2$ observations $y_0, \cdots, y_T, \beta_{T+1}$. (See Section V.)

For explicit comparison with Kalman filter state estimation, suppose the dynamical model takes the linear form

$$x_{t+1} = bx_t + \epsilon_t, \qquad t = 0, \cdots, T-1 \tag{13a}$$

$$y_t = x_t + \eta_t, \qquad t = 0, \cdots, T \tag{13b}$$

where b is a nonzero constant. Then [4], the Kalman filter state estimate for time $T+1$ takes the form

$$\begin{aligned}\tilde{x}(T+1|T+1) &= \tilde{x}(T+1|T) + K_{T+1}[y_{T+1} - \tilde{x}(T+1|T)] \\ &= b\tilde{x}(T|T) + K_{T+1}[y_{T+1} - b\tilde{x}(T|T)] \\ &= [1 - K_{T+1}]b\tilde{x}(T|T) + K_{T+1}y_{T+1}\end{aligned} \tag{14}$$

where the filter gain K_{T+1} is a function of the statistical characteristics assumed for the error terms ϵ_t and η_t. Analogously, the optimal state estimate (12a) for time $T+1$ in the present nonstatistical context takes the form

$$\begin{aligned}\hat{x}(T+1|T+1) &= \left[\frac{k}{k+1}\right] b\hat{x}(T|T+1) + \left[\frac{1}{k+1}\right] y_{T+1} \\ &= [1 - d_{T+1}]b\hat{x}(T|T) + d_{T+1}y_{T+1}\end{aligned} \tag{15}$$

where

$$d_{T+1} \equiv \left[\frac{1 + kb^2 d_T}{1 + k + kb^2 d_T}\right] \tag{16}$$

and d_T is a recursively defined coefficient for the state estimator $\rho_T(\cdot)$ which has the linear form

$$\rho_T(\beta_T) = d_T\beta_T + c_T, \qquad -\infty < \beta_T < \infty.$$

However, for the general dynamical model (4) with nonlinear state function $F(\cdot)$, the extended Kalman filter state estimate for time $T+1$ takes the form

$$\begin{aligned}x^*(T+1|T+1) &= x^*(T+1|T)+N_{T+1}[y_{T+1}-x^*(T+1|T)]\\ &= F(x^*(T|T))+N_{T+1}[y_{T+1}-F(x^*(T|T))]\\ &= [1-N_{T+1}]F(x^*(T|T))+N_{T+1}y_{T+1} \quad (17)\end{aligned}$$

where the filter gain N_{T+1} is a function of both the statistical properties assumed for (ϵ_t, η_t) and the nominal state trajectory selected for the state function linearization [4]. In contrast, the optimal state estimate (12a) for time $T+1$ takes the form

$$\hat{x}(T+1|T+1)=\left[\frac{k}{k+1}\right]F(\hat{x}(T|T+1))+\left[\frac{1}{k+1}\right]y_{T+1} \quad (18)$$

where $F(\hat{x}(T|T+1))$ cannot be expressed as a linear combination of $F(\hat{x}(T|T))$ and y_{T+1}. Thus, optimal state estimation for nonlinear state functions $F(\cdot)$ requires the explicit back updating of the previous state estimate, in addition to an incremental observational adjustment.

V. Derivation of the Sequential Estimation Equations

Consider the two-point boundary value problem (8) for arbitrary duration time $T\geq 0$ and arbitrary inhomogeneous value $\beta_T\in(-\infty,\infty)$ for the terminal boundary condition,

$$0=2[x_t-y_t]+\mu_t-F'(x_t)\mu_{t+1}, \qquad t=0,\cdots,T-1 \quad (19a)$$

$$0=x_{t+1}-F(x_t)-\frac{1}{2k}\mu_{t+1}, \qquad t=0,\cdots,T-1 \quad (19b)$$

$$0=\mu_0 \quad (19c)$$

$$\beta_T=x_T+\tfrac{1}{2}\mu_T. \quad (19d)$$

Consider now a similar two-point boundary value problem for the longer duration time $T+1$ with inhomogeneous term $\beta_{T+1}\in(-\infty,\infty)$

$$0=2[x_t-y_t]+\mu_t-F'(x_t)\mu_{t+1}, \qquad t=0,\cdots,T-1 \quad (20a)$$

$$0=x_{t+1}-F(x_t)-\frac{1}{2k}\mu_{t+1}, \qquad t=0,\cdots,T-1 \quad (20b)$$

$$0=\mu_0 \quad (20c)$$

$$\beta_{T+1}=x_{T+1}+\tfrac{1}{2}\mu_{T+1}. \quad (20d)$$

The sequential estimation equations presented in Section IV establish a recursive relationship between the solution for problem (20) and the solution for (19), assuming these solutions are unique and have unique continuations for each β_T and β_{T+1} in $(-\infty,\infty)$ and each $T>0$.

To rigorously derive the sequential estimation equations, it is useful to slightly modify the notation introduced in Section IV. Thus let $x(t,\beta,T)$ and $\mu(t,\beta,T)$ denote the tth period solution values for problem (19) when the duration of the process is T, $T\geq t\geq 0$, and the inhomogeneous term is β, $-\infty<\beta<\infty$. Suppose β_{T+1} is a given value for the inhomogeneous term in process (20) of duration $T+1$. By the assumed uniqueness of solutions for (19) and (20), there exists some value β_T which satisfies the functional equations

$$x(t,\beta_{T+1},T+1)=x(t,\beta_T,T), \qquad 0\leq t\leq T, \quad (21a)$$

$$\mu(t,\beta_{T+1},T+1)=\mu(t,\beta_T,T), \qquad 0\leq t\leq T. \quad (21b)$$

Specifically, β_T is given by

$$\begin{aligned}\beta_T &= x(T,\beta_{T+1},T+1)+\tfrac{1}{2}\mu(T,\beta_{T+1},T+1) \quad (22a)\\ &= x(T,\beta_T,T)+\tfrac{1}{2}\mu(T,\beta_T,T). \quad (22b)\end{aligned}$$

Here the current time T is considered the running variable, and t is a fixed time point in the past.

In terms of the more precise notation introduced above for x_t and μ_t, (20) at time $t=T$ now take the form

$$\begin{aligned}0=&2[x(T,\beta_{T+1},T+1)-y_T]+\mu(T,\beta_{T+1},T+1)\\ &-F'(x(T,\beta_{T+1},T+1))\mu(T+1,\beta_{T+1},T+1)\end{aligned} \quad (23a)$$

$$0=x(T+1,\beta_{T+1},T+1)-F(x(T,\beta_{T+1},T+1))-\frac{1}{2k}\mu(T+1,\beta_{T+1},T+1) \quad (23b)$$

$$\beta_{T+1}=x(T+1,\beta_{T+1},T+1)+\tfrac{1}{2}\mu(T+1,\beta_{T+1},T+1). \quad (23c)$$

In view of (21), the four terminal values $x(T,\beta_T,T)$, $\mu(T,\beta_T,T)$, $x(T+1,\beta_{T+1},T+1)$, and $\mu(T+1,\beta_{T+1},T+1)$ clearly play a crucial role in (23). Define

$$\rho_T(\beta)\equiv x(T,\beta,T), \qquad T\geq 0, \; -\infty<\beta<\infty \quad (24a)$$

$$\gamma_T(\beta)\equiv \mu(T,\beta,T), \qquad T\geq 0, \; -\infty<\beta<\infty. \quad (24b)$$

Combining (21), (22), and (24), equations (23) reduce to

$$0=2[\beta_T-y_T]-F'(\rho_T(\beta_T))\gamma_{T+1}(\beta_{T+1}) \quad (25a)$$

$$0=\rho_{T+1}(\beta_{T+1})-F(\rho_T(\beta_T))-\frac{1}{2k}\gamma_{T+1}(\beta_{T+1}) \quad (25b)$$

$$\beta_{T+1}=\rho_{T+1}(\beta_{T+1})+\tfrac{1}{2}\gamma_{T+1}(\beta_{T+1}). \quad (25c)$$

The unique continuation property assumed for the solution of problem (8) for each $T\geq 0$ and $y_T\in(-\infty,\infty)$ ensures that $F'(\rho_T(\beta_T))\neq 0$. Thus, given y_T, (25) defines a recurrence relation between $(\beta_{T+1}, \rho_{T+1}(\beta_{T+1}))$ and $(\beta_T, \rho_T(\beta_T))$ which can equivalently be expressed in the form

$$\beta_{T+1}=F(\rho_T(\beta_T))+\left[\frac{k+1}{k}\right]\left[\frac{\beta_T-y_T}{F'(\rho_T(\beta_T))}\right] \quad (26a)$$

$$\rho_{T+1}(\beta_{T+1})=\left[\frac{k}{k+1}\right]F(\rho_T(\beta_T))+\left[\frac{1}{k+1}\right]\beta_{T+1} \quad (26b)$$

$$\gamma_{T+1}(\beta_{T+1})=2[\beta_{T+1}-\rho_{T+1}(\beta_{T+1})]. \quad (26c)$$

Specifically, given y_T, $\rho_T(\beta_T)$, and β_T, (26a) yields β_{T+1}. Given $\rho_T(\beta_T)$, β_T, and β_{T+1}, (26b) yields $\rho_{T+1}(\beta_{T+1})$. Finally, given β_{T+1} and $\rho_{T+1}(\beta_{T+1})$, (26c) yields $\gamma_{T+1}(\beta_{T+1})$. Notice again that T, the current time, is the running variable.

The uniqueness and unique continuation property assumed for the solution of (8) for each $T\geqslant 0$ and y_T in $(-\infty,\infty)$ ensure that the map $\beta_T \mapsto \beta_{T+1}$ defined by (26a) is one-to-one and onto. Thus, $\rho_{T+1}(\beta_{T+1})$ and $\gamma_{T+1}(\beta_{T+1})$ in (26b) and (26c) can be calculated and stored as functions of β_{T+1} over $(-\infty,\infty)$ by varying β_T over $(-\infty,\infty)$. These stored functions can then be used to provide optimal updated filtering solutions

$$\hat{x}(T+1|T+1)\equiv x(T+1, y_{T+1}, T+1)\equiv \rho_{T+1}(y_{T+1}) \tag{27a}$$

$$\hat{\mu}(T+1|T+1)\equiv \mu(T+1, y_{T+1}, T+1)\equiv \gamma_{T+1}(y_{T+1}) \tag{27b}$$

for the original nonlinear two-point boundary value problem (8) when the duration of the process is increased from T to $T+1$ and the additional observation y_{T+1} is obtained.

Secondly, combining (21) and (26a), the following recurrence relations are obtained for each fixed time point $t\geqslant 0$,

$$x(t,\beta_{T+1},T+1)=x(t,\beta_T,T), \qquad T\geqslant t \tag{28a}$$

$$\mu(t,\beta_{T+1},T+1)=\mu(t,\beta_T,T), \qquad T\geqslant t \tag{28b}$$

$$\beta_{T+1}=F(\rho_T(\beta_T))+\left[\frac{k+1}{k}\right]\left[\frac{\beta_T-y_T}{F'(\rho_T(\beta_T))}\right], \qquad T\geqslant t. \tag{28c}$$

As before, $x(t,\beta_{T+1},T+1)$ and $\mu(t,\beta_{T+1},T+1)$ in (28a) and (28b) can be calculated and stored as functions of β_{T+1} over $(-\infty,\infty)$ by varying β_T over $(-\infty,\infty)$. These stored functions can then be used to provide optimal updated smoothing solutions

$$\hat{x}(t|T+1)\equiv \hat{x}(t,y_{T+1},T+1), \qquad T\geqslant t \tag{29a}$$

$$\hat{\mu}(t|T+1)\equiv \mu(t,y_{T+1},T+1), \qquad T\geqslant t \tag{29b}$$

for the values x_t and μ_t of the original nonlinear two-point boundary problem (8) at a fixed time point t as the duration of the process is increased from T to $T+1$, $T\geqslant t$, and the additional observation y_{T+1} is obtained.

Finally, initial conditions for the recurrence relations (26) are provided by

$$\rho_0(\beta_0)=\beta_0, \qquad -\infty<\beta_0<\infty \tag{30a}$$

$$\gamma_0(\beta_0)=0, \qquad -\infty<\beta_0<\infty. \tag{30b}$$

Using the initial conditions (30), (26) can be integrated forward to obtain initial conditions

$$x(t,\beta_t,t)\equiv\rho_t(\beta_t), \qquad -\infty<\beta_t<\infty \tag{31a}$$

$$\mu(t,\beta_t,t)\equiv\gamma_t(\beta_t), \qquad -\infty<\beta_t<\infty \tag{31b}$$

for the recurrence relations (28).

We have now established that the nonlinear two-point boundary value problem (8) of necessity satisfies the initial value problem defined by the recurrence relations (26) and (28) and the initial conditions (30) and (31). Conversely, it is straightforward to establish by induction that a solution of the initial value problem yields a solution to the original nonlinear two-point boundary value problem (8).

VI. Discussion

An initial value problem has been developed that is formally equivalent to the well-known nonlinear two-point boundary value problem associated with the least squares optimization problem outlined in Section II. With new equations to work with, it is hoped that new approaches to establishing existence and uniqueness of solutions for both the optimization problem and the two-point boundary value problem will become apparent. What is needed is a careful detailed investigation of the interrelationship of all three viewpoints.

The sequential nonlinear filtering equations which constitute the initial value problem can be used in practice to locate critical process duration lengths T at which a solution either bifurcates or ceases to exist. Specifically, for any value of T for which the solution blows up, we automatically know there is some difficulty with both the two-point boundary value problem and the optimization problem, and further investigations would have to be made. This is one of the great potential computational advantages that the sequential nonlinear filtering equations have over an iterative solution of the nonlinear two-point boundary value problem or a direct approach to the optimization problem.

In a companion paper [16] a tabular method is developed for numerically implementing the sequential nonlinear filtering equations. The accuracy and efficiency of the method are illustrated by means of several numerical examples of the form

$$x_{t+1}=ax_t+bx_t^2, \qquad t=0,\cdots,T-1$$

$$y_t=x_t+\eta_t, \qquad t=0,\cdots,T.$$

It would be useful to extend the present sequential solution procedure to multidimensional problems with only partially observable states. In [17] we indicate how this might be done for a class of dynamic nonlinear economic models having the form

$$x_{t+1}=f_t(x_t)+u_t, \qquad t+0,\cdots,T-1$$

$$y_t=h_t(x_t)+v_t, \qquad t=0,\cdots,T$$

where x_t in R^n is the state vector, y_t in R^r is an observation on x_t, and u_t and v_t are disturbances which may or may not have *a priori* constraints placed upon them, e.g., $Eu_t=\mathbf{0}$ and $Ev_t=\mathbf{0}$.

Acknowledgment

The authors are grateful to six anonymous referees for their helpful comments.

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Robert Kalaba (SM'59) received the Ph.D. degree in mathematics from New York University, New York.

He spent twenty years at Rand Corporation and for the past dozen years he has been with the University of Southern California, Los Angeles, CA, where he is Professor of Biomedical Engineering, Electrical Engineering, and Economics. He is the author of ten books and nearly 500 papers.

Leigh Tesfatsion (M'77) was born in Connecticut in 1946. She received the Ph.D. degree in economics with a minor in mathematics from the University of Minnesota, Minneapolis, in 1975.

In 1975, she joined the faculty of the Department of Economics of the University of Southern California, Los Angeles, where she is currently Associate Professor. Her research interests include micro-based macroeconomic modeling, nonlinear estimation and control, and system sensitivity analysis.

Quasi-Solutions, Vector Lyapunov Functions, and Monotone Method

V. LAKSHMIKANTHAM, S. LEELA, AND M. N. OĞUZTÖRELI

***Abstract*—The notion of quasi-solution which was introduced earlier has been developed in this paper. Further, the notion of coupled quasi-solutions and coupled maximal and minimal solutions has been introduced. It is shown that the idea of quasi-solutions leads to isolated subsystems, and has obtained error estimates between solutions and quasi-solutions. Also, monotone iterative techniques have been developed to obtain coupled maximal and minimal quasi-solutions.**

Manuscript received June 2, 1980; revised April 10, 1981. This work was supported in part by the U.S. Army under Grant DAAG29-80-C0060.

V. Lakshmikantham is with the Department of Mathematics, University of Texas, Arlington, TX 76019.

S. Leela is with the Department of Mathematics, State University of Arts and Science, Geneseo, NY 14454.

M. N. Oguztoreli is with the Department of Mathematics, University of Alberta, Edmonton, Alta., Canada T6G 2G1.

I. INTRODUCTION

IT IS now well known that the method of vector Lyapunov functions provides an effective tool to investigate the properties of large scale interconnected dynamical and control systems [3]–[5], [11]–[15]. Several Lyapunov functions result in a natural way in the study of such systems by the decomposition and aggregation method [1]–[4], [12]–[14]. However, an unpleasant fact in this approach is the requirement of quasi-monotone property on the comparison system since comparison systems with a desired property like stability exist without satisfying quasi-monotone property. Also, in the study of comparison the-

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