"A Robust Time-varying Style Analysis for Hedge Funds based on Dynamic Quantiles"*

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Abstract

In the original approach by Sharpe (1992), style analysis aims to explain portfolio performances according to the fund exposure to a set of asset classes. Typically, multivariate regression is involved to obtain the fund sensitivity to various benchmarks. However, this approach has been criticized because of the non-robustness of the estimation. Moreover, the analysis framework remains static whereas fund allocation could be dynamic.
To remedy to these drawbacks, we propose a robust time-varying global estimation involving Quantile Regression. The advantage of the Quantile Regression is that the whole shape of the distribution is taken into account rather than the simple mean in the case of OLS. Moreover, Quantile Regression is a robust statistical method that gives less weight to "outliers". We propose a new time-varying framework for quantile regression in order to capture the dynamic style allocation. Our robust approach is finally applied to the main hedge fund strategy indexes from HFR.

Keywords: Style Analysis, Quantile Regression, L-estimator, FLS, Hedge Funds, Time-varying.
JEL Classification: C14, C22, C29, G11.

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1 Introduction

A typical style analysis aims to determine the sensibility of a fund to a set of factors. The emergence and the success of style analysis probably come from the fact that various applications can be proposed from the knowledge of these sensibilities. For an investor, style analysis provides him a better understanding of the fund strategy and its sources of returns. For a fund manager, reference to a specific style leads to a more precise definition of his strategy. It can allows him to define an *ad hoc* benchmark. This is a reason why style analysis is widely used in performance analysis and asset management.

Two main approaches have been proposed when assessing the global style of a portfolio (ter Horst *et al.*, 2004). The first one refers to Holding-based Style Analysis (HBSA) and consists in the style determination of every asset in the portfolio before aggregating them to get the overall style of the portfolio. The second one, often called Return-based Style Analysis (RBSA), was introduced by Sharpe (1988 and 1992). This paper mainly considers RBSA. Such models have been jointly developed with asset pricing models. The first attempts to compare funds have used the Capital Asset Pricing Model (CAPM, Sharpe 1964). The main drawback of single factor models (for instance CAPM) is that they are too restrictive since in most cases, more than one systematic risk factor is needed to describe returns; this is particularly true considering hedge funds with different styles (taking long or short positions on different markets[1]). Consideration of linear multi-factorial models has been proposed to overcome these limits. With the introduction of the Asset Pricing Theory (APT) by Ross (1976), a new generation of multi-factorial models has appeared.

[1]The main hedge fund strategies are briefly presented in the appendix.

The big issue with multi-factorial models lies with the choice of factor and the so called selection bias. Indeed, without enough knowledge on the investment strategy followed by the fund manager, one can miss an important factor; the obtained style analysis could then be biased. For this last approach, the choice of factors is of critical importance. Factor reduction techniques (Principal Component Analysis or Independent Component Analysis) have been proposed to obtain relevant factors without any *a priori* on a fund strategies. From a large set of fund performances these methods aim to extract a small number of independent (at least uncorrelated) variables. These approaches have the advantage of complying with some of the statistical hypothesis required by the considered model but the obtained factors are virtual (they did not correspond to real asset performances) and associated sensibilities are hardly interpretable.

The second class of models uses real assets as factors. Stylized facts such as the "*value premium*" for equity portfolios have lead to the introduction of *ad hoc* factors. For instance, Fama and French (1993) proposed an expanded three-factor model that considers the market exposure factor, the SMB size factor (denoting Small *Minus* Big) and a variable depending on the stock's book-to-market, HML (denoting High book to price *Minus* Low book to price). The existence of a price momentum effect was documented by Jegadeesh and Titman (1993). It reflects strategies that consist in buying stocks which have well performed in the past and selling stocks which have historically performed poorly. Carhart (1997) proposed to integrate a momentum factor (denoting WML for Winners *Minus* Losers) in the so-called Carhart model (1997). But these methods require corrections for drawing inferences (as the model contains inequality constraints). The estimation suffers from the presence of outliers (Chan and Lakonishok, 1992) because it is essentially based on a Least Square estimation procedure.

Hedge funds have received a vast amount of attention over the last decades. They are caracterized by a great opacity regarding their investment strategies. They differ significantly from traditional investments, such as mutual funds, due to the lack of strong regulation on this asset class. Hedge fund managers enjoy a great flexibility in their investments (dynamic asset allocations, leverage, short-selling, derivatives *etc*). There exist various hedge fund strategies which exhibit different statistical properties and risk return characteristics (Fung and Hsieh, 1997 and 1999). The increasing interest in the hedge fund market has lead to a need for sophisticated econometric models having the ability to capture the peculiarities of hedge fund return series.

Linear regression models have been widely used in the hedge fund literature. Many articles investigate the ability of various linear or non-linear risk factors to explain hedge fund returns and evaluate hedge fund performances using different asset pricing models (see, for example, Capocci and Hübner, 2004). Although estimating standard multi-factorial regression models is straightforward, the identification of the relevant risk factors (to be included in the model) is difficult.

Standard regression models describe only the average relationship of hedge fund returns with the set of average risk factors. However, this approach might not be adequate due to the peculiarities of hedge fund returns. Some researches (see, for example, Amin and Kat, 2003) has revealed that, due to their highly dynamic complex nature, hedge fund returns may exhibit a high degree of non-normality, skewness, fat tails and excess *kurtosis*. In presence of these stylized facts, the conditional mean approach may not capture the effect of risk factors to the entire return distribution and may provide non-robust estimates.

Introduced by Koenker and Basset (1978), Quantile Regression can be interpreted as an extension of the Least Squares estimation of conditional mean models to the estimation of a set of conditional quantile functions. The use of Quantile Regression in the style analysis context has originally been proposed by Basset and Chen (2001).

Actually, Quantile Regression can offer an efficient alternative to the standard model as it allows discriminating portfolios that would be otherwise judged equivalent. This method allows us to extract information from the whole asset return distribution rather than the expected value. Quantile Regression should thus provide useful insights as the style exposure could affect returns in different ways at different locations of the portfolio return distribution. Quantile Regression coefficients are then interpretable in terms of portfolio conditional quantiles returns sensitivity to constituent returns.

In addition, the use of different probability levels associated to quantiles allows us to obtain a set of conditional quantile estimators. The latter can be linearly combined in order to construct a L-estimator which then gives rise to a gain in efficiency (Koenker and Portnoy, 1987).

Fund style management is often active, time-varying and depends on market opportunities (Fung *et al.*, 2008; Kosowski *et al.*, 2007; Jagannathan *et al.*, 2010). Bollen and Whaley (2009), propose an optimal change point regression and a stochastic beta model to estimate changes in hedge fund risk dynamics. Billio *et al.* (2010) measure the dynamic risk exposure of Hedge Funds to various risk factors during different market volatility conditions using a regime-switching beta model. Finally, a new time-varying framework based on quantile dynamics can be defined for a better assessement of fund active style management.

The main goal of this paper is to provide and estimate a robust time-varying multi-quantile framework for style analysis. The second section of this paper is devoted to a brief review of style analysis problematics, evolutions and the critical step of the factor selection. We propose to use peer group analysis to select the factors that will be used to investigate hedge fund styles. After having recalled the traditional OLS and Quantile Regression framework, we propose to adapt the Time-varying Flexible Least Square approach to L-estimators based on a Quantile Regression. We thus introduce the time-varying multi-quantile robust approach for style analysis. The fourth section illustrate this robust time-varying approach on hedge fund strategies as defined in the HFR database on monthly data from January 1995 to January 2010. The last section concludes.

2 Style Analysis Approaches

Style analysis stands nowadays out as a major reference in portfolio management. Given that the large variety of management offers, style analysis has become fundamental to decompose and justify performance differences between funds belonging to the same asset class. A vast literature is dedicated to this subject. Several approaches are proposed to estimate the style associated to a fund. In this article, we suppose that a style is defined according to the risk factors to which the fund is exposed. The fund sensibility to risk factors allows us to estimate the future expected returns and verify the coherence of the performances given the fund objectives.

In this section, we first set the general style analysis framework, then we describe the problematic linked to the critical choice of the risk factors. At last, we propose to use a peer group analysis to select among a benchmark universe relevant factors in our robust style analysis framework dedicated to hedge funds.

2.1 Style Analysis Evolution

Style analysis models evolved together with asset evaluation models. Thus, the first attempt of grouping funds uses the systematic risk of the Capital Asset Pricing Model (CAPM; Sharpe, 1964). Since the apparition of the Asset Pricing Theory (APT) by Ross (1976), a new generation of multi-factorial models has been developed. Besides the choice of statistical methods, style analysis models mostly differ from the choice of factors. The definition of the factors is clearly influenced by the observation of market abnormalities. Numerous studies conclude to the existence of a capitalisation bias in equity returns (Lakonishok and Shapiro, 1986). Others (Lakonishok *et al.*, 1994; Fama and French, 1998, for example) show that the Book-to-Market ratio (B/M) should be taken into account for return estimations. These abnormalities give rise to the introduction of four traditional explanatory factors: the styles Growth, Value, Large and Small. If the precise definitions of the styles Large and Small are immediate, the ones of the styles Growth and Value are more uncertain. Roughly, the style Growth characterizes the companies with a significant anticipated organic growth. A Value investment strategy consists in buying securities that are considered underestimated.

For portfolio style analysis, two widely spread approaches have been proposed. The first one, called the Holding-Based Style Analysis (denoted HBSA, *Cf.* Daniel *et al.*, 1997) consists in the analysis of every holding compounding the portfolio. The aggregation of these obtained styles allows us to assess the global style of the portfolio studied. Such an approach requires a large amount of information (breakdown of the portfolio, a forecast ratio and other peculiarities of each holding for instance) and is hard to set up in practice. Moreover, when dealing with hedge funds, the traditional opacity on the investment strategies makes the HBSA potentially unfeasible.

An alternative to the HBSA is found in the Return-Based Style Analysis (denoted RBSA); the goal of this approach is to measure the sensibility of a portfolio onto a set of factors. The seminal work of Sharpe (1988 and 1992) follows a statistical approach to estimate these sensibilities and evaluate the global style of a fund. This approach is hereafter developed in this paper. Thus, style analysis model regresses portfolio returns on the ones of the N constituents, such as:

$$\underset{[T\times 1]}{\mathbf{R}_P} = \widehat{\alpha}\, \underset{[T\times 1]}{\mathbf{1}_{(T\times 1)}} + \underset{[T\times N]}{\mathbf{F}}\, \underset{[N\times 1]}{\widehat{\mathbf{B}}_C} + \underset{[T\times 1]}{\mathbf{e}}\ , \tag{1}$$

where $\mathbf{R}_P$ is the vector of portfolio returns over time, $\mathbf{F}$ the matrix containing the returns over time of the risk factors (portfolio constituents), $\widehat{\alpha}$ and $\widehat{\mathbf{B}}_C = (\widehat{\beta}_1, ..., \widehat{\beta}_N)'$ are estimated constrained parameters, $\mathbf{e} = (\epsilon_1, ..., \epsilon_T)'$ is the vector of residuals and data are observed on T subsequent time periods.

The Principal Component Analysis (PCA), the Independent Component Analysis (ICA) and the Kohonen Self-Organizing Maps (SOM) classification are three different ways of considering the multi-factorial approach presented in equation (1). This methods allow us to express the factors as a portfolio of assets defined by weights, such as (with the previous notations):

$$\underset{[T\times N]}{\widehat{\mathbf{F}}} = \underset{[T\times N]}{\mathbf{R}}\, \underset{[N\times N]}{\widehat{\mathbf{W}}_{\mathbf{x}}} \tag{2}$$

where $\mathbf{R}$ is the $(T \times N)$ matrix of returns, $\widehat{\mathbf{W}}_{\mathbf{x}}$ with $x =$ [PCA, ICA, SOM] stand for $\widehat{\mathbf{W}}_{\mathbf{PCA}}$, $\widehat{\mathbf{W}}_{\mathbf{ICA}}$ and $\widehat{\mathbf{W}}_{\mathbf{SOM}}$ which are respectively the extraction matrices associated to PCA, ICA and SOM methodologies.

Within the PCA the matrix of linear independent factors can be expressed such as:

$$\widehat{\mathbf{W}}_{\mathbf{PCA}} = \underset{\widehat{\mathbf{W}}_{\mathbf{PCA}}}{\text{Argmax}}\, E\left\{(R\widehat{W}_{PCA})^2\right\} \tag{3}$$

where $\mathbf{R}$ is the $(T \times N)$ matrix of assets returns, $\widehat{\mathbf{W}}_{\mathbf{PCA}}$ is the eigenvectors associated to the matrix $\mathbf{RR}'$, with $\mathbf{R}'$ the transpose of $\mathbf{R}$.

PCA aims to find linear combinations of returns (*i.e.* portfolio return time-series) that most explain the global volatility of the underlying assets returns, under the constraint of null-covariances between returns of built factors. The goal of the PCA is therefore to remove second-order dependencies in the data. If higher order dependencies exist between the variables then removing second-order dependencies is insufficient at revealing all structure in the data. Multiple solutions exist for removing higher-order dependencies. For instance, we can transform the data to a more appropriate naive basis. However, in the context of returns decomposition, outliers can be difficult to identify. PCA has been applied to the analysis of hedge funds (Fung and Hsieh, 1997 and 1999), however, the PCA model generated low levels of cross sectional variation in hedge fund returns.

Another method of factorial decomposition, the Independent Component Analysis (ICA, see Hyvarinen *et al.*, 2001) is to impose more general statistical definitions of dependency within a data set: requiring that data along reduced dimensions are statistically independent. This class of algorithms has been demonstrated to succeed in many domains where PCA fails. ICA aims to find linear combinations of returns so that factors are independent grouping asset returns in a class whose representatives are weighted sums of asset return time-series (with weights depending on a distance between the asset return time-series). An application of ICA to explain the factors driving the hedge fund returns can be found in Olszewski (2006). The ICA allows us to express a set of multidimensional observations as a combination of unknown latent variables. These unknown latent variables are called "sources" or "independent components" and are supposed to be statistically independent. To extract the factors you have to get the extraction matrix denoted $\widehat{\mathbf{W}}_{\mathbf{ICA}}$, such as (with the previous notations):

$$\widehat{\mathbf{W}}_{\mathbf{ICA}} = \underset{\widehat{\mathbf{W}}_{\mathbf{ICA}}}{\text{Argmax}}\, NG\left[\widehat{\mathbf{W}}_{\mathbf{ICA}}(\mathbf{R}-\alpha)\right], \tag{4}$$

where $NG(.)$ is a function which measures the non Gaussianity of the returns.

The difference between PCA and ICA that is relevant to hedge funds is that while PCA algorithms use only second order statistical information, ICA algorithms may use higher order statistical information for separating signals. For this reason non-Gaussian signals (or at most, one Gaussian signal) are required (Back and Weigend, 1997) and measured by $NG(.)$.

Kohonen Self-Organizing Maps (SOM) can be used at the same time both to reduce the amount of relevant data by clustering, and for projecting the data non-linearly onto a lower dimensional display. This iterative algorithm has been applied to Hedge Fund classification by Maillet and Rousset (2003). Factors are determined iteratively and adapt theirselves during the SOM algorithm. The $\widehat{w}_{(i,j)_{SOM}}$ components of the $\widehat{\mathbf{W}}_{\mathbf{SOM}}$ matrix can be iteratively determined such as (with the previous notations):

$$\widehat{w}_{(i,j)_{SOM}} = \widehat{w}_{(i-1,j)_{SOM}} - \varepsilon_{i-1}\Lambda\left(BMU_i, \mathbf{F}_j\right)\left[\widehat{w}_{(i-1,j)_{SOM}} - 1\right], \tag{5}$$

with:

$$BMU_i = \underset{\mathbf{F}_j, j\in I}{\text{Argmin}}\left\{\left\|\mathbf{R}_{i,j} - \mathbf{F}_{(i-1,k)}\right\|\right\}, \tag{6}$$

where ε_t is the adaptation gain parameter, which is $]0,1[$-valued, generally decreasing with time. The number of neurons taken into account during the weight update depends on the neighborhood function Λ that also decreases with time and $\|.\|$ is the Euclidean norm.

Most of RBSA models consider a (multi-) factorial approach to model returns and assess the portfolio style. This class of models must have three main desirable properties. First, parameters (sensibility to factors) should be estimated robustly and in a reasonable amount of time. Secondly, an intuitive interpretation has to be provided (factors are real assets for instance). Thirdly, the model has to be parsimonious in terms of involved factors (leads to a robust estimation).

2.2 Style Analysis Approach for Hedge Funds

In front of the reproduction of the empirical works indicating that the other variables explain the excedentary returns and to the difficulty validating the CAPM (for example, Lakonishok and Shapiro, 1986; Chopra and Ritter, 1989; Fama and French, 1992), the evaluation models evolved towards the consideration of others risk factors than the only market risk.

The APT model, introduced by Ross in 1976, supposes that asset returns are influenced by various (macroeconomic) exogeneous factors. The expected return of a portfolio is a function depending on a basket of explanatory factors and on the fund sensibilities with regard to these risk factors. To be valid, these models require to have independent factors. Indeed, the collinearity between these factors can lead to a spurious regression and thus infer in the determination of the sensibilities. To suit these statistical constraints, methods of factorial decomposition were proposed. Ross uses the Principal Component Analysis (PCA) to identify linearly independent factors. Under the hypothesis of normality of asset returns, the main constituents are also independent. The determination of the fund exposure onto the principal orthogonal factors (avoiding the drawbacks of multi-collinearity) is therefore made possible.

If higher order dependencies exist between the variables then removing second-order dependencies the PCA is insufficient at revealing all structure in the data. Another method of factorial decomposition, the Independent Component Analysis (ICA, see Hyvarinen *et al.*, 2001), enables to drop this hypothesis. The ICA allows us to express a set of multidimensional observations as a combination of unknown latent variables. These unknown latent variables are called “sources” or “independent components” and are supposed to be statistically independent. The independence is estimated here beyond the moment of order 2 and integrated directly into the optimization function (*Cf.* Chap and Robin, 2008).
Whatever the factorial decomposition method is chosen, two main issues remain. The implementation of these models requires *a priori* knowledge on the factors to which the considered fund is exposed. The risk factors being abstracted, the interpretation of the fund sensibilities to these factors is not always easy.
Sharpe (1988 and 1992) presents a model adopting this time an approach relative to the APT framework. He considers that the bets and strategies of a manager (preference for an asset class or a given sector) are going to define the style of the manager. The hypothesis of the Sharpe style model is that differences of behavior are going to be echoed directly on the managed funds returns. He uses real factors of the economy and considers a basket of market indicators (shares, bonds). By means of a multivariate regression under constraints, he determines the sensibilities of the fund to the factors. In spite of the choice of the representative benchmarks and the problems of statistical order connected to the potential collinearity of the used factors, this model became a reference in style analysis. Besides, the use of real factors leads to a direct interpretation of the implemented strategies by the manager.

About representative factors of hedge fund strategy, it is reasonable to assume that the risk sources associated with hedge funds are fairly similar to those associated with traditional assets. Multi-factorial models for hedge fund analysis have been proposed (see for instance, Schneeweis and Spurgin, 1999; Schneeweis *et al.*, 2002; and Capocci and Hübner, 2004). Some of these factors are: equity related factors (market or sector index, traded volume...), equity trading styles (Fama and French factors) or interest rate factors (T-bill rate, slope of the yield curve, credit risk *premium*...). In the empirical part of this work, we model the hedge fund returns by using different information variables and pricing factors in the lines of Agarwal and Naik (2000 and 2004). Thanks to a clustering method based on a peer group analysis, we select the relevant factors among a benchmark database, composed initially by: equity related factors (the Russel 3000 equity index, the MSCI World excluding the USA, the MSCI Europe-EMU, the MSCI Emerging Market index, the MSCI Emerging Market excluding Asia, the TOPIX), equity trading styles (Carhart and Fama and French factors on the US; Small *Minus* Big - SMB, High *Minus* Low - HML, Momentum factor, Long-term and Short-term reversal factors), interest rate factors (the US LIBOR 3 Months, the 10 Year Treasury rate, the J.P. Morgan U.S. Aggregate Bond Index, the J.P. Morgan EMU all maturities, the credit Suisse High Yield index), Commodity factors (the S&P global commodity index and the CRB All Commodities), and a factor linked to derivatives (the evolution of equity Implied Volatility Index - VIX).

Some results show that an HBSA approach does not infer better results that the RBSA approach (ter Horst *et al.*, 2004). Although this model is widely spread, it is the object of severe criticisms. The fact that the considered factors are generally strongly collinear constitutes one of them; because any regression on these risks to be fuzzy. The standard RBSA models presented until suffer from two major limits.
The analysis is static; the fund exposure to factors are supposed constants over the time. For a fund managed passively, this approach is valid. But since we consider funds adopting an active management, a static setting is not anymore viable.
The second criticism is connected to the necessity of specifying *a priori* the risk factors. A preliminary knowledge of the analyzed funds is required to choose factors. For instance, Sharpe (1988) chooses real economic factors (equities and bonds benchmarks) without in-

tegrating the styles Growth, Value, Large and Small. It is necessary to wait for the three factors model of Fama and French (1998) to consider these last ones.

Brown and Goetzmann (1997) adopt an original approach. Exploiting the strong link between factorial analysis methods and classification algorithms, they suggest collecting funds in homogeneous groups. For each of these groups is identified a risk factor (defined as the average of the funds of each group). This model, easy to implement, has a strong explanatory power. Indeed, factors arising from the universe of funds, it is not necessary to know *a priori* the styles. The bias, even if they were not observed yet, will be directly integrated into the risk factors. Factors being determined directly from the funds, this model allows us to obtain risk factors reflecting, among others, the active management strategies. Adopting a similar approach, Maillet and Rousset (2003) and Aaron *et al.* (2004) suggest to analyze the style of funds by exploiting a classification method called Kohonen Self-Organizing Maps (SOM). This algorithm allows the simultaneous determination of the homogeneous groups of funds, and the risk factors. An advantage of the maps of Kohonen by report to the algorithm which was used by Brown and Goetzmann (1997), is the preservation of the topology. The representative factors of the groups of funds are ordered on the map according to their similitude.

Following this field of the literature, we use the Self-Organizing Maps (Kohonen, 2000) to build a more robust style analysis model, selecting appropriate benchmarks (factors) in our approach. The method of classification, SOM allows the simultaneous generation of specific risk factors and homogeneous groups of strategies and benchmarks.

2.3 Factors for Style Analysis

To perform our robust style analysis approach, we use the Kohonen approach to analyze a set of risk factors and avoid to get "dependent" explanatory variables.

The algorithm of the Kohonen map allows us to obtain non linear classifications without any *a priori* on data to classify. Various financial applications of SOM were proposed: from the simple exploration of fund universe, to the forecast of future returns. In spite of its robustness, the stochastic property of the algorithm can be subject to convergence problems.

We thus choose a robust version of the Kohonen map (Guinot *et al.*, 2006). We briefly recall hereafter (details are provided in the appendix 6.2), the SOM algorithm and its robust version. We show then, how the robust map can be used within the framework of a style analysis model.

We briefly present herein the data mining technique we apply for grouping together elements (hedge fund track records and factors for instance)[2]. Self-Organizing Map is a clustering method belonging to Artificial Neural Networks. SOM can be used at the same time both to reduce the amount of relevant data by clustering, and for projecting the data non-linearly onto a lower dimensional display. Due to its unsupervised learning and topology preserving properties, the SOM algorithm has proven to be especially suitable in visual analysis of high dimensional sets. They have already been applied in various fields in general, and in finance in particular, for clustering elements sharing some similarities. Without any pretension of exhaustivity, some examples of SOM financial applications are to be found in Deboeck and Kohonen (1998), Resta (2001), Maillet and Rousset (2003), Moreno *et al.* (2006), and Ben Omrane and de Bodt (2007). For further details on this data-mining technique, see Kohonen (2000) and Guinot *et al.* (2006).

The SOM is a method that represents statistical data sets in an ordered way as a natural groundwork on which the distributions of the individual indicators in the set can be displayed and analyzed. It is based on the unsupervised learning process where the training is entirely data-driven and no information about the input data is required (see Kohonen, 2000).

The SOM consists[3] of a network, compound of n neurons, units or code vectors organized on a regular low-dimensional grid. If $I = [1, 2, ..., n]$ is the set of units, the neighborhood structure is provided by a set of neighborhood function Λ defined on I^2. The network state at time t is given by:

$$\mathbf{m}(t) = [\mathbf{m}_1(t), \mathbf{m}_2(t), ..., \mathbf{m}_n(t)], \tag{7}$$

where $\mathbf{m}_i(t)$ is the T-dimensional weight vector of the unit i.

For a given state $\mathbf{m}$ and an input $\mathbf{x}$, the winning unit $i_w(\mathbf{x}, \mathbf{m})$ is the unit whose the weights $\mathbf{m}_{i_w(\mathbf{x},\mathbf{m})}$ is the closest to the input $\mathbf{x}$.

If the quality of SOM classifications is generally satisfactory, the stochastic property of the

[2]An illustration of the algorithm is provided in Figure (16) of the appendix

[3]The SOM algorithm is presented in the appendix.

algorithm does not guaranteed the systematic convergence of the classification. Since two successive SOM learning may not provide the same results, a model risk can arise when using a simple Kohonen algorithm to develop style model. To avoid such convergence problems, we apply a modified version of the Kohonen algorithm as proposed by Guinot *et al.* (2006): robust maps (Robust Self-Organizing Maps, RSOM). To limit the network dependences to the learning data model, it is classic to apply a Bootstrap process with resampling techniques.

The Bootstrap technique is applied here to the SOM by estimating a probability for every individual being in the same group. This probability is empirically estimated during the SOM learning step performed on the resampled temporal series.

The classification algorithm uses only the individuals present in the resampled database (60 % of the original individuals). We generalize this approach by adding an edition without replacement of the observations (60 % of the original observations). At the end of the first step, individuals spread during the process of re-sampling are classified from their distances in vector codes. So, in every stage of the classification, we obtain the probability to be in the same group for every individual (including those spread during the re-sampling).

Thus, we perform the Robust SOM classification onto the benchmark universe (presented above) and the main HFR strategies and we try to select one risk factor *per* group to limit "dependencies" between risk factors.

3 About Style Analysis Modeling

In this section, after having recalled the traditional Least Square and Quantile Regression framework, we propose to adapt the Time-varying Flexible Least Square approach to L-estimators based on Quantile Regression. We thus introduce the time-varying multi-quantile robust approach for style analysis.

3.1 Least Square Constrained Regression Models

The classical model is based on a constrained least square regression model (see Sharpe 1988 and 1992).

The use of a Least Square model focuses on the conditional expectation of portfolio return distribution. It can be formulated as follows (with the previous notations):

$$E(\mathbf{R}_P|\mathbf{F}_t) = \widehat{\alpha} + \mathbf{F}_t\widehat{\mathbf{B}}_C. \tag{8}$$

Estimated compositions are then interpretable in terms of sensitivity of portfolio expected returns to constituent returns. In the classical regression context, the β_i coefficient represents the impact of a change in the returns of the i^{th} constituent on the portfolio expected returns, holding the values of the other constituent returns constant. Using the Least Square model, portfolio style is then determined by estimating the style exposure influence on expected returns. Although the estimated Least Square coefficients are in common practice interpreted as a composition at the last time, it is worthwhile stressing that they reflect the average effect over the time period studied, according to the regression framework. Estimated Least Square coefficients are to be compared to average portfolio composition. Even if the Least Square framework has been sometimes presented within a dynamic estimation method (using for example a rolling window of returns to perform estimations), it can not be considered as a real time-varying framework.

For parameter estimation of a linear model, such as beta risk, the assumption about the error distribution. If the error term has a Gaussian distribution, the OLS estimator of the parameters has a minimum variance of the entire class of unbiased estimators (see Rao, 1973). Moreover, using the Jensen's inequality (1906), the optimality of the OLS procedure under Gaussian conditions can be established for any convex loss function (see Rao, 1973). When normality of the error term cannot be assumed, the OLS method provides the best unbiased parameter estimator according to the linear model only if attention is restricted to those parameters that are linear functions of the dependent variable. In many situations, however, this set may be unnecessarily restrictive. Moreover, outliers can have a potent effect, completely altering least squares estimates (Ruppert and Carroll, 1980; Koenker, 1982). Statistically, a fat-tailed distribution may be modeled as arising from a mixture of normal distributions. For example, the underlying data may come from a standard normal distribution, but are contaminated by aberrant observations from another normal distribution with a higher variance. Such a distribution have heavier tails than a normal distribution. The financial literature confirms that the distribution of daily stock returns exhibits "fatter tails" than a normal distribution[4]. Besides, hedge fund returns are definitively not normal. The empirical evidence suggests that the distribution of residuals departs from normality and is likely to be characterized by fat tails. Roll (1988), suggests an economic model that is consistent with stock returns being generated by a mixture of distributions. He basically assumes that stock returns are related to extreme values, which are linked to news events, thereby substantially increasing the *kurtosis* of the return distribution. Damodaran (1985) also argues that the *kurtosis* of a firm's return process reflects the frequency of information released about the firm.

Robust statistical methods provide an alternative to least squares. Such estimators give less weight to "outlier" observations, for example, by minimizing the sum of absolute deviations (the method of minimum absolute deviations, MAD) instead of the sum of squared deviations.
Extracting information at other places other than the expected value should also provide useful insights as the style exposure could affect returns in different ways at different locations of the portfolio return distribution.

3.2 Quantile Regression Models

Quantile Regression may be used as an extension of the Classical Least Squares estimation of conditional mean models to the estimation of a set of conditional quantile functions.
Exploiting Quantile Regression (Koenker, 2005), it provides a more detailed comparison of financial portfolios. Actually, Quantile Regression coefficients are interpretable in terms of sensitivity of portfolio conditional quantiles returns to benchmark constituent returns (Bassett and Chen, 2001).
In a similar way as for the Least Square Model, coefficients of the Quantile Regression model can be interpreted as the change rate of a specified conditional quantile of the portfolio return distribution for a unit change in a precise constituent returns.

[4]For example, Fama (1965) fits a stable Paretian distribution to daily returns and finds a characteristic exponent less than two; Praetz (1972) and Blattberg and Gonedes (1974) provide evidences in favor of the student-t distribution; Kon (1984) finds that returns on the 30 Dow Jones stocks can be described as a mixture from 2 to 4 normal distributions.

The Quantile Regression model is more powerful than the standard Least Squares regression because it can identify dependence of various parts of the distribution from explanatory variables. A portfolio style depends on how a factor influences the entire return distribution, and this influence cannot be described by a single number. The single number given by the Least Squares regression may obscure the tail behavior (which could be of a prime interest to a risk manager). With the Quantile Regression, we can estimate, for instance, the impact of explanatory variables on the 99^{th} percentile of the loss distribution. Portfolios having exposures to derivatives may have very different regression coefficients of the mean value and tail quantiles. For instance, let us consider the strategy of writing naked deep out-of-the-money options. This strategy in most of the cases behaves like a bond paying some interest, however, in rare cases the strategy loses some amount of money (that may be quite significant). Therefore, the mean value and the 99^{th} percentile may have very different regression coefficients for the explanatory variables.

The Quantile Regression (QR) model for a given conditional quantile τ can be written as follows (with the previous notations):

$$Q_t^{\tau}(\mathbf{R}_P|\mathbf{F}_t) = \mathbf{F}_t\widehat{\mathbf{B}}_C^{\tau} + \epsilon_{\tau,t}, \tag{9}$$

where $Q^{\tau}(.)$ is the τ quantile.
The use of Quantile Regression then offers a more complete view of relationships among portfolio returns and constituent returns. Besides the Least Square coefficients, Quantile Regressions coefficients provide information on the different level of turnover in the portfolio constituents over time.
Moreover, exploiting the conditional quantile estimates shown, it is straightforward to estimate the portfolio conditional return distribution.
The obtained distribution is strictly dependent on the values used for the co-variates. It is then possible to use different potential *scenarii* in order to evaluate the effect on the portfolio conditional return distribution, carrying out a “what-if” analysis. In this way, it is possible to evaluate how the entire shape of the conditional density of the response changes with different values of the conditioning co-variates, without confining oneself to the classical

regression assumption. The co-variates affect only the location of the response distribution, but not its scale or shape.

The conditional quantiles are estimated through an optimization function minimizing a sum of weighted absolute deviation where the choice of the weight determines the particular conditional quantile to estimate. Sensitivity parameters are estimated using the Quantile Regression minimization (denotes QR Sum) presented by Koenker and Bassett (1978):

$$\widehat{\mathbf{B}}_C^{\tau *} = \underset{\widehat{\mathbf{B}}_C^{\tau} \in \mathbb{R}^N}{\text{Argmin}} \left\{ \sum_{t=1}^{T} \left\{ \left| R_{P,t} - \widehat{Q}_t^{\tau} \left(\mathbf{F}_t, \widehat{\mathbf{B}}_C^{\tau} \right) \right| \left| \tau - \mathbb{1}_{\left\{ R_{P,t} < \widehat{Q}_t^{\tau} \left(\mathbf{F}_t, \widehat{\mathbf{B}}_C^{\tau} \right) \right\}} \right| \right\} \right\}, \tag{10}$$

with $\widehat{Q}_t^{\tau} \left(\mathbf{F}_t, \widehat{\mathbf{B}}_C^{\tau} \right)$ is the quantile estimation according to a specific style analysis model, $\widehat{\mathbf{B}}_C^{\tau}$ is a vector of estimated sensitivity parameters and $\mathbb{1}_{\{.\}}$ is the indicator function.

The use of absolute deviations ensures that conditional quantile estimates are robust. Moreover the peculiarity of the method is "non-parametric" in the sense that it does not assume any specific probability distribution of the observations. In the following, we use a semi-parametric approach as we assume a linear model in order to compare the Quantile Regression estimates with the classical style model.

Since the QR function is the sum of the absolute values of the residuals, deviant observations are given less importance than under a squared error *criterion*. More generally, large (small) values of the "weight" τ attach a heavy penalty to observations with large positive (negative) residuals. Each fitted regression line (corresponding to a different value of τ) passes through at least two data points, with at most τT sample observations lying below the fitted line, and at least $(T-2)\tau$ observations lying above the line. For example, when $\tau = .50$, the median fitted residual is zero: half of the data points lie above the line, while half lie below. Varying τ from 0 to 1 yields a set of "regression quantile" estimates $\widehat{\mathbf{B}}_C^{\tau}$, analogous to the quantiles of any sample of data, that is, the set of order statistics. The characterization above suggests, intuitively, the following features of these regression quantiles. Specifically, the effect of large positive or negative outlying observations will tend to be concentrated in the regression quantiles corresponding to extreme (high or low) values of τ. Note, however, that no observations are discarded in the course of computing these statistics. Moreover, the behavior of returns in the sample determines the variation in the regression quantiles as τ changes. From this perspective, choosing an estimate of $\widehat{\mathbf{B}}_C^{\tau}$ corresponding to one value of

τ, such as the Minimum Absolute Deviation estimate, ignores potentially useful information in the sample. Accordingly, the performance of the Minimum Absolute Deviation estimator may be improved by an estimator that incorporates several regression quantiles. In the statistical literature, considerable attention has been devoted to the problem of obtaining robust estimates of the population mean *via* linear combinations of sample quantiles (trimmed means). In the same spirit, regression quantiles serve as the basis for the robust estimators of regression parameters that we consider. The general form of such Trimmed Regression Quantile (TRQ) estimators is:

$$\widehat{\mathbf{B}}_C^{\theta} = (1-2\theta)^{-1} \int_{\theta}^{1-\theta} \widehat{\mathbf{B}}_C^{\tau} d\tau, \tag{11}$$

where $.00 < \theta < .50$.

This estimator is a weighted average of the regression quantile statistics and, hence, belongs to the class of L-estimators[5].

Each regression quantile is weighted by its (data-dependent) "relative frequency" of occurrence, given by its corresponding interval of τ-values. The form of the estimator suggests that it is analogous to a trimmed mean, with trimming proportion θ: the "extreme" quantiles, where the influence of outlying observations should be most heavily concentrated, are deleted. As the sample size goes to infinity, another intuitively natural interpretation of $\widehat{\mathbf{B}}_C^{\theta}$ is possible: consider fitting the θ-th, and $(1-\theta)$-th, regression quantile lines through the data. Then exclude all observations lying on or below the θ-th regression quantile line (corresponding to large negative outliers), as well as all observations lying on or above the $(1-\theta)$-th quantile line (corresponding to large positive outliers). The remaining observations are then used to calculate the ordinary least squares estimator; in large samples, the resulting "trimmed least squares" estimator is equivalent to $\widehat{\mathbf{B}}_C^{\theta}$.

Although the discussion has concentrated on estimation, statistical inference concerning the trimmed regression quantile estimator $\widehat{\mathbf{B}}_C^{\theta}$ is also possible. In large samples, $\widehat{\mathbf{B}}_C^{\theta}$ is consistent and normally distributed with variance-covariance matrix $\sigma_\theta^2 \mathbf{F}'\mathbf{F}$, where $\mathbf{F}$ is the matrix of regressors (see Koenker and Portnoy, 1987). A consistent estimator of σ_θ^2 is given

[5] L-estimators are obtained as linear combinations of order statistics. Examples include the median and trimmed means. A trimmed mean is simply the sample mean, after some proportion θ of the observations at each extreme of the sample are deleted (Koenker, 2005).

by:

$$\widehat{\sigma}_\theta^2 = (1-2\theta)^{-2}\left\{\frac{\sum_{t=1}^{T}\epsilon_{\theta,t}^2}{(T-2)} + \theta[\overline{\mathbf{F}}(\widehat{\mathbf{B}}_C^{\theta} - \widehat{\mathbf{B}}_C^{\tau})^2] + (1-\theta)[\overline{\mathbf{F}}(\widehat{\mathbf{B}}_C^{(1-\theta)} - \widehat{\mathbf{B}}_C^{\tau})^2]\right\}, \tag{12}$$

where $\sum_{t=1}^{T}\epsilon_{\theta,t}^2$ is the sum of squared residuals from the Trimmed Least Squares estimator, based on a sample of T observations, $\overline{\mathbf{F}}$ is a column vector containing the sample means of the regressors, while $\widehat{\mathbf{B}}_C^{\tau}$ is the vector of parameter estimates for the τ-th regression quantile[6].

Thus, the use of different values of probability associated to quantile allows us to obtain a set conditional quantile estimators that can be easily linearly combined to construct L-estimator, in order to gain in efficiency.

The previous presented style analysis methods are purely static. The obtained coefficients (betas) fixed over the period of analysis; such an approach is viable when we consider a portfolio managed passively, but is going to fail if we estimate a manager adopting an active management. Even if Quantile Regression framework allows us to have a more precise view according to the location of the fund returns. The previous models do not allow us to detect precisely the changes of strategy used by the manager through time. Nevertheless, from 1996, Ferson and Schadt ends in changes of styles of investment according to the economic anticipations. It seems then essential to take into account the style dynamics of the investment (see Chan *et al.*, 2002).

In next section, we see how to adapt these methods in order to take into consideration the time-varying dynamics of style analysis.

3.3 Flexible Time-varying Regressions

For any given data and any given linear model proposed to explain the style associated to a fund, each possible estimated model generates two conceptually-distinct types of discrepancy terms, dynamic and measurement.

The dynamic discrepancy terms reflect time variation in successive coefficient vectors (relative to a null of constancy), and the measurement discrepancy terms reflect differences between actual observed outcomes and theoretically predicted outcomes based on the null of

[6] Simulation evidences in Koenker and Portnoy (1987) and Koenker (2005) suggest that the asymptotic approximation is not unreasonable, even in samples of 25 to 50 observations.

a linear regression model. The dynamic and measurement errors are separately aggregated into sums of residual squared errors.

In a series of studies summarized in Kalaba and Tesfatsion (1996), a multicriteria Flexible Least Squares (FLS) approach is developed to model estimation that encompasses a wide range of views regarding the appropriate interpretation and treatment of theory-data discrepancy terms. Stressing minimal reliance on stochastic priors for discrepancy terms, they develop Flexible Least Square Time-Varying Linear Regression (FLS-TVLR).

The standard linear regression model involves a response variable $r_{P,t}$ and N predictor variables $f_1, ..., f_N$, which usually form a predictor column vector $\mathbf{F}_t = (f_{1,t}, ..., f_{N,t})'$. The model postulates that $r_{P,t}$ can be approximated well by $\mathbf{F}_t'\mathbf{B}$, where $\mathbf{B}$ is a N-dimensional vector of regression parameters. In Ordinary Least Square regression (OLS), estimates $\widehat{\mathbf{B}}$ of the parameter vector are values that minimize the cost function $C(.)$, such as (with the previous notations):

$$C(\mathbf{B}) = \sum_{t=1}^{T}(r_{P,t} - \mathbf{F}_t'\mathbf{B})^2. \tag{13}$$

When both the response variable $r_{P,t}$ and the predictor vector $\mathbf{F}_t$ are observations at time t of co-evolving data streams, it may be possible that the linear dependence between $r_{P,t}$ and $\mathbf{F}_t$ changes and evolves, dynamically, over time. Flexible Least Squares (FLS) were introduced by Kalaba and Tesfation (1989) as a generalization of the standard linear regression model in order to apply time-variant regression coefficients. Together with the usual regression assumption that residuals are small, the FLS model also postulates that the regression coefficients may evolve slowly over time. FLS does not require the specification of probabilistic properties for the residual error. This is a favorable aspect of the method for applications in temporal data mining, where we are usually unable to precisely specify a model for the errors, besides any assumed model would not hold true at all times. Montana *et al.* (2009) show that FLS performs well even when there are large and sudden changes in the regression coefficients through time.

The FLS approach consist of minimizing a penalized version of the OLS cost function presented in equation (13), such as (with the previous notations):

$$C(\mathbf{B}_{FLS,t},\delta) = \sum_{t=1}^{T}(r_{P,t} - \mathbf{F}_t'\mathbf{B}_{FLS,t})^2 + (1-\delta)\delta^{-1}\sum_{t=1}^{T}\xi_t, \tag{14}$$

with:

$$\xi_t = (\mathbf{B}_{FLS,t+1} - \mathbf{B}_{FLS,t})'(\mathbf{B}_{FLS,t+1} - \mathbf{B}_{FLS,t}), \tag{15}$$

where $0 \leq \delta \leq 1$ is a scalar to be determined.

Kalaba and Tesfatsion (1988) propose an algorithm that minimizes this cost with respect to every $\mathbf{B}_t$ in a sequential way. This procedure requires all data points until time T to be available, so the coefficient vector $\mathbf{B}_T$ should be computed first. The estimate of $\mathbf{B}_{FLS,T}$ can be obtained sequentially as such (with the previous notations):

$$\widehat{\mathbf{B}}_{FLS,T} = (\mathbf{G}_{T-1} + \mathbf{F}_T\mathbf{F}_T')^{-1}(\mathbf{H}_{T-1} + \mathbf{F}_T r_{P,t}), \tag{16}$$

with:

$$\begin{cases} \mathbf{G}_t = (1-\delta)\delta^{-1}(\mathbf{I}_N - \mathbf{M}_t) \\ \mathbf{H}_t = (1-\delta)\delta^{-1}\mathbf{E}_t \\ \mathbf{M}_t = (1-\delta)\delta^{-1}(\mathbf{G}_{t-1} + (1-\delta)\delta^{-1}\mathbf{I}_N + \mathbf{F}_t\mathbf{F}_t')^{-1} \\ \mathbf{E}_t = \delta(1-\delta)^{-1}\mathbf{M}_t(\mathbf{H}_{t-1} + \mathbf{F}_t r_{P,T}), \end{cases}$$

where $\mathbf{G}_t$ and $\mathbf{H}_t$ have dimensions $N \times N$ and $N \times 1$, $\mathbf{I}_N$ is the $N \times N$ identity matrix.

All remaining coefficient vectors $\widehat{\mathbf{B}}_{FLS,T-1}, ..., \widehat{\mathbf{B}}_{FLS,1}$ are estimated going backwards in time, such as (with the previous notations):

$$\widehat{\mathbf{B}}_{FLS,t} = \mathbf{M}_t\widehat{\mathbf{B}}_{FLS,t+1} + \mathbf{E}_t. \tag{17}$$

The procedure relies on the specification of the regularization parameter $(1-\delta)\delta^{-1}$, this positive scalar penalizes the dynamic component of the cost function defined in equation (23) and can de interpreted as a smoothness parameter that forces the time-varying vector towards or away from the fixed-coefficient OLS solution. Then, with δ set very close to zero, near total weight is given to minimizing the static part of the FLS cost function (equation 14). This is the smoothest solution and results in standard OLS estimates. As δ moves away from zero, greater priority is given to the dynamic component of the cost, which results in time-varying estimates.

We propose to adapt the previous time-varying FLS approach to Quantile Regression framework in order to apply time-variant Quantile Regression coefficients in a robust style analysis framework. This method is denoted FQR for Flexible Quantile Regression in the following.
Following FLS approach, FQR consist of minimizing a penalized version of the aggregated Quantile Regression cost function, such as (with the previous notations):

$$C(\mathbf{B}_{FQR,t}, \delta) = (1-2\theta)^{-1} \sum_{\tau=\theta}^{1-\theta} QR_t^{\tau} \tau + (1-\delta)\delta^{-1} \sum_{t=1}^{T} \xi_t, \tag{18}$$

with:

$$\xi_t = (\mathbf{B}_{FQR,t+1} - \mathbf{B}_{FQR,t})'(\mathbf{B}_{FQR,t+1} - \mathbf{B}_{FQR,t}), \tag{19}$$

and:

$$QR_t^{\tau} = \left\{ \sum_{t=1}^{T} \left\{ \left| r_{P,t} - \widehat{Q}_t^{\tau}\left(\mathbf{F}_t, \widehat{\mathbf{B}}_{FQR,t}^{\tau}\right) \right| \; \left| \tau - \not\Vdash_{\left\{ r_{P,t} < \widehat{Q}_t^{\tau}\left(\mathbf{F}_t, \widehat{\mathbf{B}}_{FQR,t}^{\tau}\right) \right\}} \right| \right\} \right\}, \tag{20}$$

where $0 \leq \delta \leq 1$ is a scalar to be determined, QR_t^{τ} is the scalar defined by the QR Sum associated to the τ quantile at time t and $.00 < \theta < .50$.

We have presented the style analysis approach and set the framework of a robust time-varying multi-quantile style analysis approach based on Quantile Regression, L-estimator and time-varying estimation adapted from the traditional FLS methodology.
This style analysis is performed thanks to real risk factors selected using peer group analysis and SOM.
In the next section we apply this style analysis framework on real hedge fund strategies as provided by the HFR database.

4 Robust Time-varying Multi-Quantile Framework for Style Analysis

We construct, in this section, a robust time-varying L-estimator based on the Quantile Regression framework (presented in the previous section) to analyze the style of Hedge Funds. After having recalled the general methodology of our approach, we briefly describe the statistical properties of the Hedge Fund database, and the factors used to analyze them. At last, we combine these estimations within a time-varying framework to get our robust approach.

4.1 Methodology

In order to analyze style of Hedge Funds, we propose first to use a Kohonen map to check for the adequacy of the factors we use. After having performed Least Square and Quantile Regression static studies, we build a robust time-varying L-estimator based on the Quantile Regression framework and an adaptation of the FLS approach to analyze the style of Hedge Funds.

We first propose to highlight the link between a large variety of benchmarks representing equity factor, classical trading strategies and the representative hedge fund strategies considered. We investigate the statistical characteristics of hedge fund strategies under study. We observe a significant non normality in the funds returns under study, a peer group analysis through a traditional Principal Component analysis should be avoided. We propose instead to apply the Self-organizing Map algorithm to perform a robust non linear clustering. Thanks to this clustering method based on a peer group analysis, we select the "real" factors that we need among a database of benchmarks(in line with previous studies; Agarwal and Naik, 2000 and 2004), composed initially of: equity related factors (the Russel 3000 equity index, the MSCI World excluding the USA, the MSCI Europe-EMU, the MSCI Emerging Market index, the MSCI Emerging Market excluding Asia, the TOPIX), equity trading styles (Carhart and Fama and French factors on the US; Small Minus Big - SMB, High Minus Low - HML, Momentum factor, Long-term and Short term reversal factors), interest rate factors (the US LIBOR 3 Months, the 10 Year Treasury rate, the J.P. Morgan U.S. Aggregate Bond Index, the J.P. Morgan EMU all maturities, the Credit Suisse High Yield index), commodity factors (the S&P global commodity index and the CRB All Commodities), factors linked to derivatives (the evolution of equity Implied Volatility Index - VIX). Then, we check whether the projection of the chosen benchmarks on the map is consistent with the factor location. At last, we project the PCA main factors in the map and observe if they are spread in the whole map. Thus, we perform the robust SOM classification onto the benchmark universe and the main HFR strategies and we try to select one risk factor by group to limit "dependencies" between risk factors.

After having checked for the adequacy of the factors, we first follow Basset and Chen (2001) and perform the traditional Quantile Regression approach for style. A particular attention is paid to the stability of the parameters through quantiles under study. This analysis offers a more complete view of relationships among portfolio returns and benchmark returns. We then combine these Quantile Regression estimates thanks to a L-estimator, such as (with the previous notations):

$$\widehat{\mathbf{B}}_C^{\theta} = (1-2\theta)^{-1} \int_{\theta}^{1-\theta} \widehat{\mathbf{B}}_C^{\tau} d\tau, \tag{21}$$

where $.00 < \theta < .50$.

At last, we adapt the previous time-varying FLS approach to a multi-Quantile Regression framework in order to apply time-varying Quantile Regression coefficients in a robust style analysis framework. This method is denoted FQR for Flexible Quantile Regression. It minimizes a penalized version of the aggregated Quantile Regression cost function, such as (with the previous notations):

$$C(\mathbf{B}_{FQR,t}, \delta) = (1-2\theta)^{-1} \sum_{\tau=\theta}^{1-\theta} QR_t^{\tau} \tau + (1-\delta)\delta^{-1} \sum_{t=1}^{T} \xi_t, \tag{22}$$

with:

$$\xi_t = (\mathbf{B}_{FQR,t+1} - \mathbf{B}_{FQR,t})'(\mathbf{B}_{FQR,t+1} - \mathbf{B}_{FQR,t}), \tag{23}$$

and:

$$QR_t^{\tau} = \left\{ \sum_{t=1}^{T} \left\{ \left| r_{P,t} - \widehat{Q}_t^{\tau}\left(\mathbf{F}_t, \widehat{\mathbf{B}}_{FQR,t}^{\tau}\right) \right| \left| \tau - \mathbb{1}_{\left\{ r_{P,t} < \widehat{Q}_t^{\tau}\left(\mathbf{F}_t, \widehat{\mathbf{B}}_{FQR,t}^{\tau}\right) \right\}} \right| \right\} \right\} \tag{24}$$

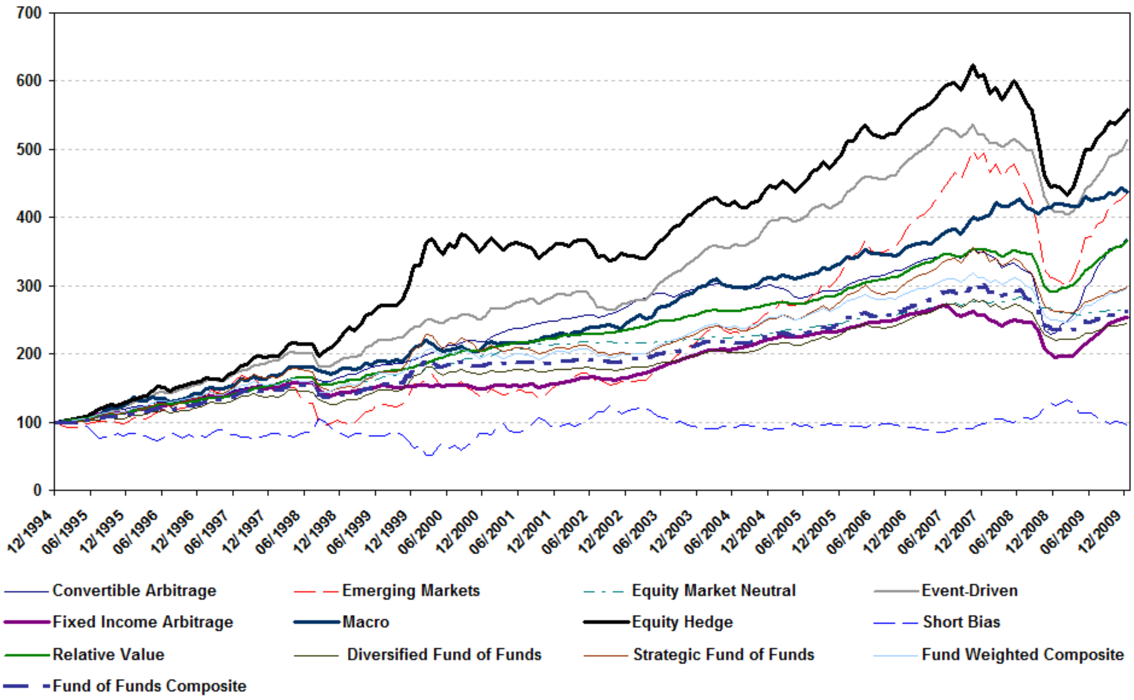
where $0 \leq \delta \leq 1$ is a scalar to be determined, QR_t^{τ} is the scalar defined by the QR Sum associated to the τ quantile at time t and $.00 < \theta < .50$.

We provide, comment and estimate the robust time-varying multi-quantile framework for style analysis using Quantile Regression, L-estimator and the consistency of factors location.

4.2 Hedge Funds and Factors: Some Empirical Evidences

We illustrate the proposed Quantile Regression approach using hedge fund indices data from Hedge Fund Research (HFR). The HFR indices are equally weighted average returns of hedge funds and are computed on a monthly basis. In this section, we use directional strategies that bet on the direction of the markets, as well as non-directional strategies whose bets are related to diversified arbitrage opportunities rather than to the movements of the markets. In particular, we consider the main HFR single strategy indices: Convertible Arbitrage, Emerging Markets, Event-Driven, Fixed Income Arbitrage, Equity Hedge, and Short Bias. Our study of these hedge fund strategies uses monthly returns from the 1^{st} January 1995 to the 1^{st} January 2010. This period includes two major crises and several market events which have affected hedge fund returns and caused a large variability in the return series. We model the hedge fund returns by using different information variables pricing factors in the lines of Agarwal and Naik (2000 and 2004).

Figure 1: Hedge Fund Strategies



Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

We observe that hedge fund strategies are very heterogeneous: there are some strategies with relatively high average returns and high volatilities, such as Equity Hedge, Emerging Markets and Event Driven, while Fixed Income Arbitrage has relatively low average returns and standard deviations. The study of the standard deviations of hedge fund returns indicates major differences between strategies; in particular, the variability of the returns of Short Bias and Emerging Markets is higher. Differences are also apparent in higher order mo-

ments. In particular, Convertible Arbitrage and Fixed Income Arbitrage have high negative skewness and high *kurtosis*, while Short Bias has low positive skewness and relatively small *kurtosis*. Most of the studied hedge fund strategies have negative skewness. This suggests that extreme negative price falls are more likely to happend than extreme price increases for the respective hedge fund strategies. Also, hedge fund strategies have large *kurtosis* indicating fat tails. The hedge fund returns exhibit a high degree of non-normality which can be attributed to the special nature of the investment strategies adopted. The deviation from normality is confirmed by the results of the Jarque-Bera, Lilliefors and Anderson and Darling normality tests (presented in table 1): for the hedge fund return series under study, the normality hypothesis is rejected at a 5% level of significance.
This departure from Normality affects traditional style analysis models (such as traditional constrained regression or Principal Components Analysis models as proposed by Sharpe, 1992 or Brown and Goetzmann 1997).

Table 1: Characteristics of the Hedge Fund Strategies

	Annualized Mean (*)	Standard deviation (*)	Skewness	*Kurtosis*	Jarque Bera (**)	Lilliefors (**)	Anderson Darling (**)
Convertible Arbitrage	9.43	7.54	-3.12	26.25	.10	.10	.00
Emerging Markets	11.51	14.64	-1.06	4.44	.10	.10	.00
Equity Market Neutral	6.77	3.25	-.18	1.55	.38	2.06	.27
Event-Driven	11.80	7.09	-1.42	4.67	.10	.20	.00
Fixed Income Arbitrage	6.62	6.04	-2.35	11.20	.10	.10	.00
Macro	10.57	6.57	.40	.55	2.97	4.69	1.45
Equity Hedge	12.66	9.62	-.24	2.17	.10	4.50	.65
Short Bias	1.57	19.47	.27	2.69	.10	.38	.01
Relative Value	9.15	4.51	-3.13	17.24	.10	.10	.00
Diversified Fund of Funds	6.37	6.51	-.54	3.96	.10	.10	.00
Strategic Fund of Funds	7.97	9.04	-.64	4.05	.10	.60	.00
Fund Weighted Composite	7.81	7.34	-.76	2.86	.10	3.45	.12
Fund of Funds Composite	6.85	6.26	-.78	4.08	.10	.27	.00

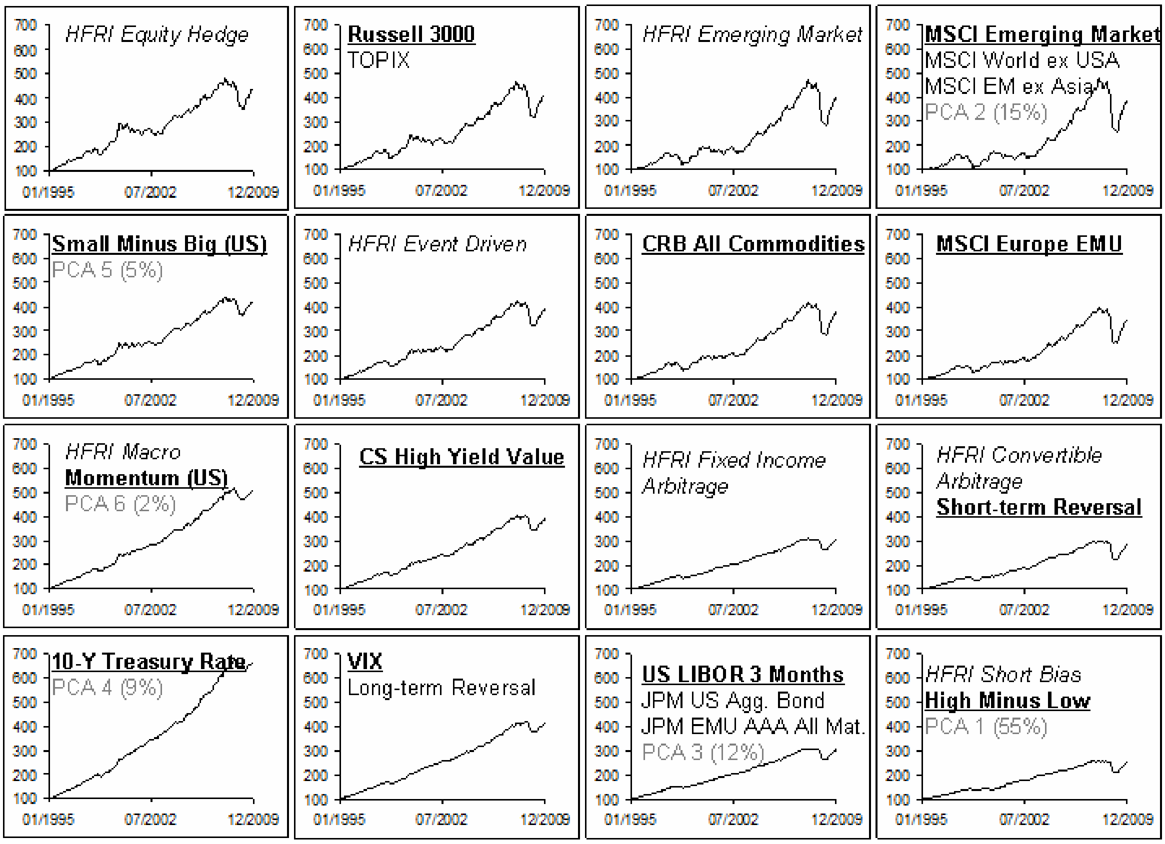
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. The sign (*) indicates that Mean and standard-deviation are annualized and expressed in %. The sign (**) indicates that we present the P-values of Normality tests expressed in %.

Because of this non-normality of factors and asset returns, a particular attention must be paid when doing data mining. We propose to apply the Self-organizing Map algorithm to perform a robust non linear clustering (*Cf.* Maillet and Rousset, 2003). As in Maillet and

Merlin (2010) we choose to proceed with a 4x4 sized map.

Figure 2 illustrates the organized map after the unsupervised learning process within a database composed of the factors and the market index time-series.

Figure 2: Factor Classification *via* SOM



Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

The obtained Map seems well structured. Low volatility strategy factors are gathered on the bottom of the map whereas volatile strategy factors stand on the top of the map. More precisely, we have on the bottom side, HML, US LIBOR 3 months and 10 years Treasury Rate and most of the arbitrage strategies are on the top of the map. On the upper side of the map, we found most of the traditional equity factor.

Moreover, projecting the main PCA factors (explaining 98% of the database variance), we see that these PCA factors are well spread all over the classification map and are next to some of the selected risk factors.

We choose to elect one representative "real" factor by class. The HFR strategies are represented in italic on the SOM representation (figure 2), the main benchmarks of the universe are in normal character and the selected "real" risk factors are in bold and underlined. In what follows, we have chosen to select the twelve main risk factors in order to perform our style analysis. The projection of the chosen market index on the map is consistent with the factor location. Every representative index from HFR stands in the same location than the corresponding risk factors.

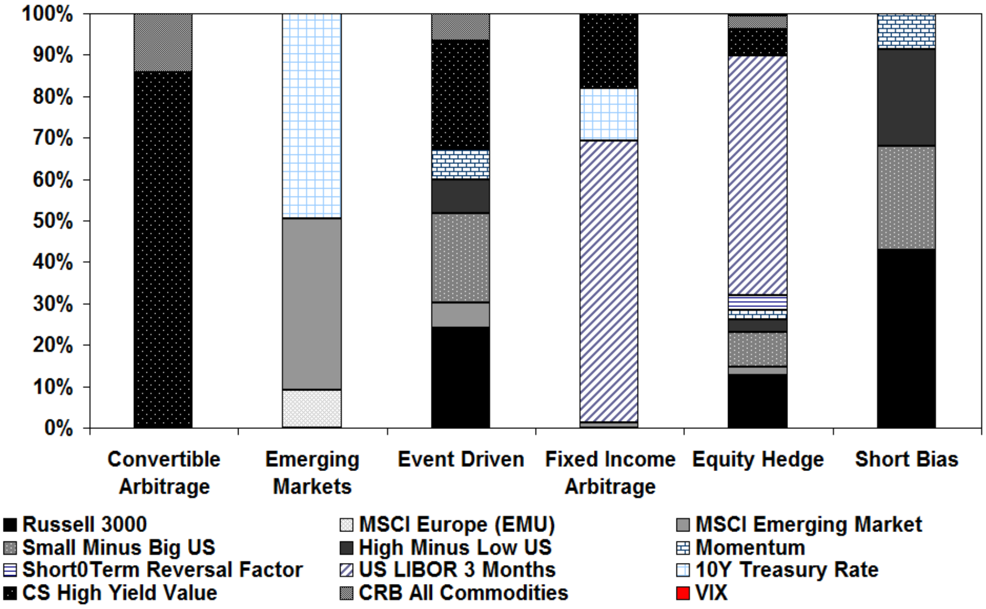
4.3 Investigating the Robust Time-varying Multi-Quantile Framework for Style Analysis

We then explore the impact of these risk factors on the entire conditional distribution of fund returns. Unlike the standard conditional mean regression method, which only examines how the risk factors affect the returns on average, the Quantile Regression approach is able to uncover how this dependence varies across quantiles of returns. Thus, this approach provides useful insights on the distributional dependence of studied fund returns on risk factors.

These empirical results show that the relationship between hedge fund returns and the selected risk factors changes across the distribution of conditional returns. Therefore, the Quantile Regression approach provides a better way to understand this relationship compared to the standard conditional mean regression method. This analysis is therefore of great interest to identify the risk factors associated with particular fund behviours. Aggregating Quantile views into a L-estimator (as defined in previous section), we can investigate the stability of hedge fund sensitibities to these risk factors.

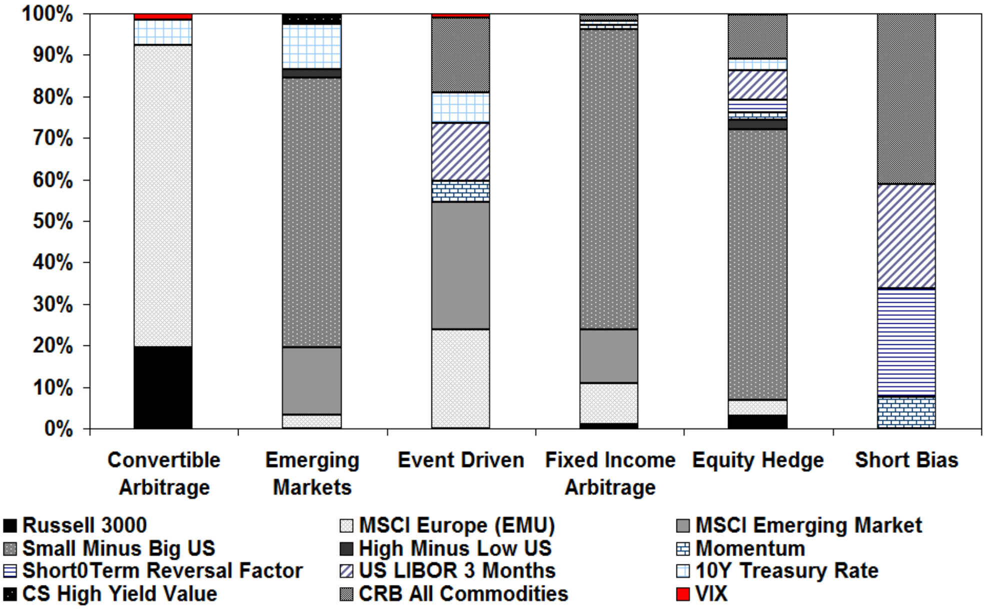
We observe in tables (2) to (7) that there are differences between the median regression, the L-estimator and the conditional mean regression. A larger number of factors are generally used to explain the conditional L-estimator than those needed in the conditional mean regression, while some of the factors are usually different between the two models. The estimates of the model parameters (*i.e.* the alphas and the betas coefficients) are also quite different. These observations may be attributed to the fact that by aggregating quantile sensitibities, the L-estimator provides more robust and more efficient estimates when dealing with hedge fund returns whose distributions deviate from normality.

Figure 3: Hedge Fund Strategies Risk Factor Expositions according to the Median Regression



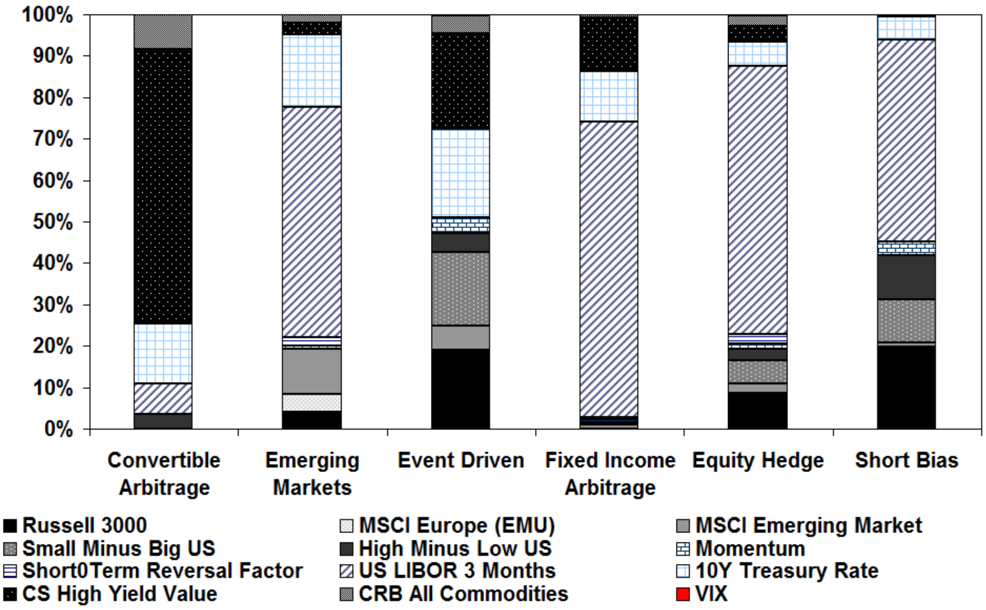
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 4: Hedge Fund Strategies Risk Factor Expositions according to the Mean Regression



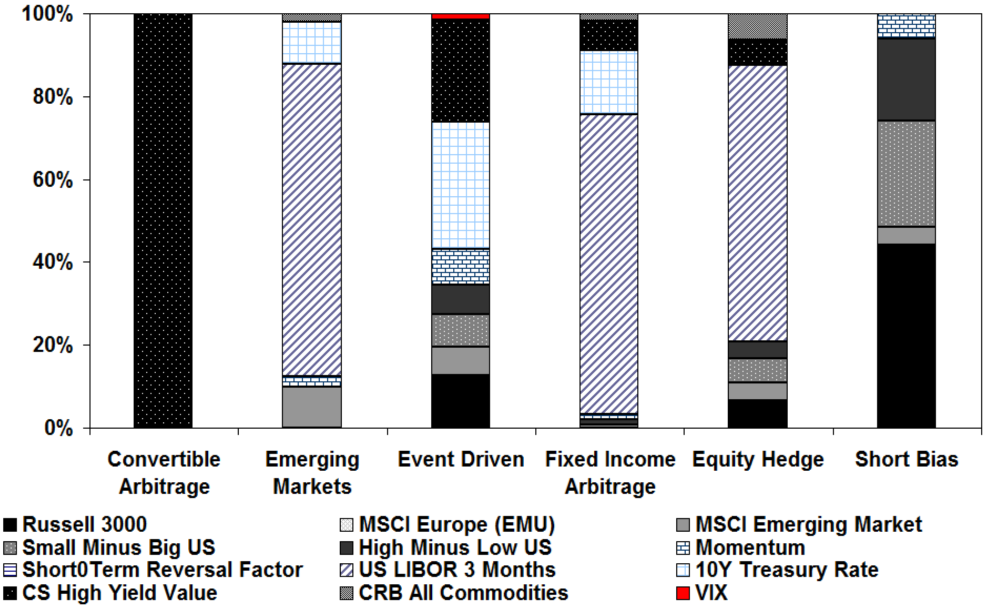
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 5: Hedge Fund Strategies Risk Factor Expositions according to the L-estimator



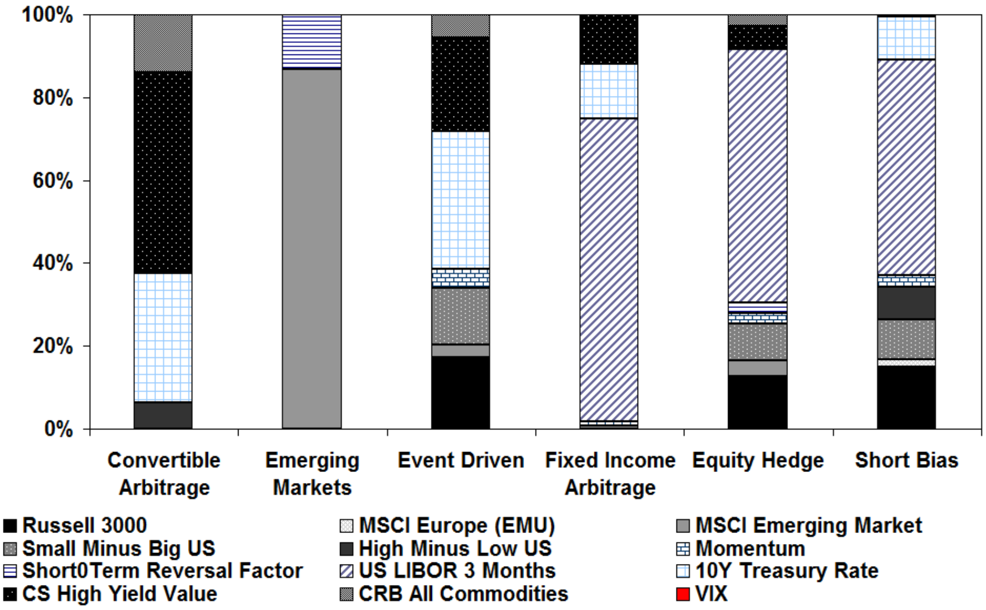
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 6: Hedge Fund Strategies Risk Factor Expositions according to the First Decile Regression



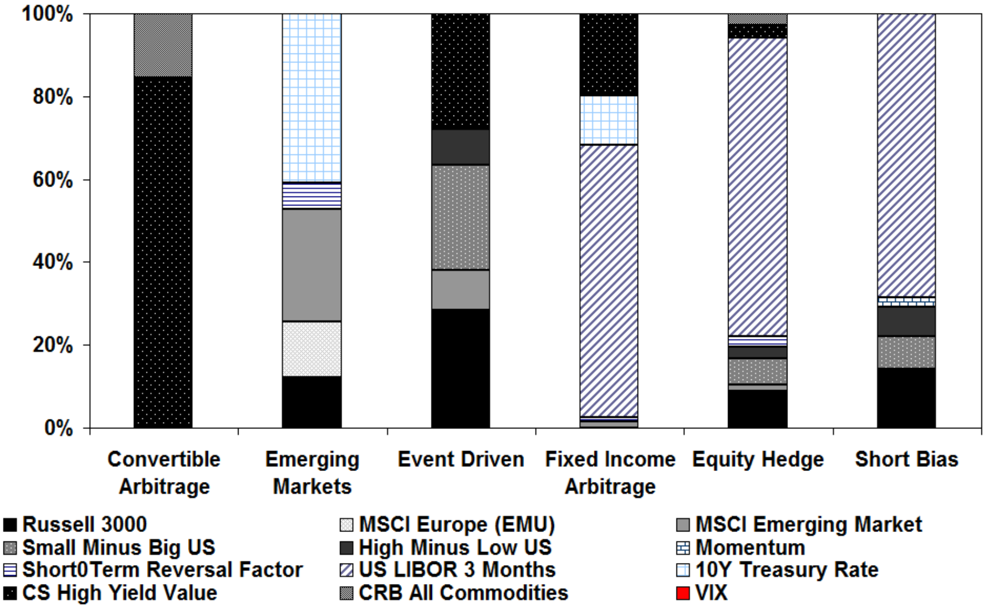
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 7: Hedge Fund Strategies Risk Factor Expositions according to the First Quartile Regression



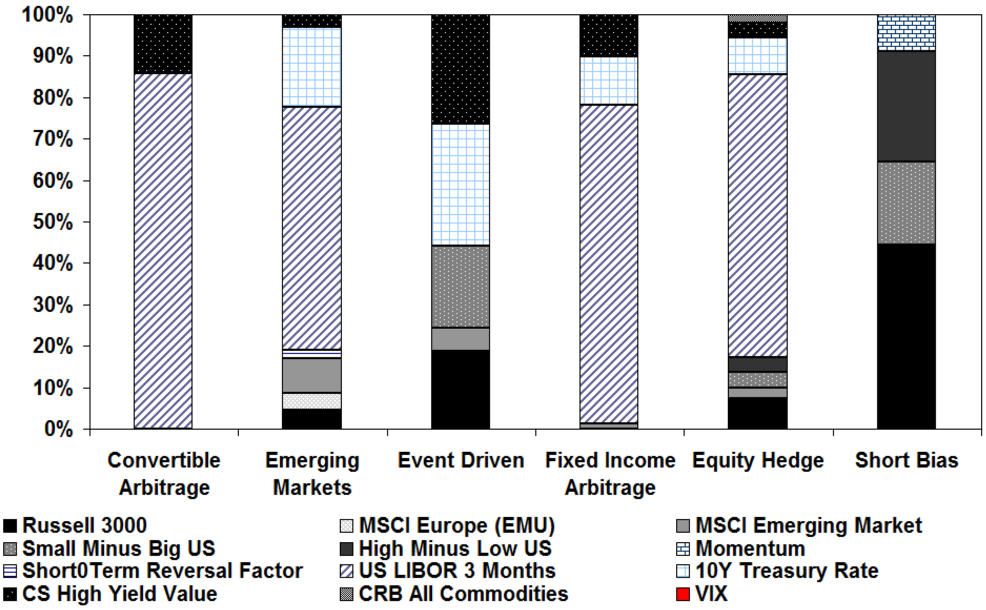
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 8: Hedge Fund Strategies Risk Factor Expositions according to the Third Quartile Regression



Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 9: Hedge Fund Strategies Risk Factor Expositions according to the Ninth Decile Regression



Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Table 2: Robust Quantile Regression on Convertible Arbitrage Strategy

	OLS	10%	25%	50%	75%	90%	L
Alpha	-.02	-.03	-.02	-	.01	.02	.02
P-value	1.04%	.04%	.01%	-	9.49%	.04%	.01%
Russell 3000	.14	-	-	-	-	-	-
P-value	.32%	-	-	-	-	-	-
MSCI Europe (EMU)	.52	-	-	-	-	-	-
P-value	.00%	-	-	-	-	-	-
MSCI Emerging Market	-	-	-	-	-	-	-
P-value	-	-	-	-	-	-	-
Small Minus Big US	-	-	-	-	-	-	-
P-value	-	-	-	-	-	-	-
High Minus Low US	-	-	-.07	-	-	-	-.05
P-value	-	-	2.04%	-	-	-	2.08%
Momentum	-	-	-	-	-	-	-
P-value	-	-	-	-	-	-	-
Short-Term Reversal Factor	-	-	-	-	-	-	-
P-value	-	-	-	-	-	-	-
US LIBOR 3 Months	-	-	-	-	-	-1.74	-.11
P-value	-	-	-	-	-	9.13%	9.31%
10-Y Treasury Rate	.04	-	.33	-	-	-	.21
P-value	8.11%	-	2.56%	-	-	-	2.61%
CS High Yield Value	-	.67	.52	.43	.36	.29	.96
P-value	-	.00%	.00%	.00%	.00%	.00%	.00%
CRB All Commodities	-	-	.15	.07	.07	-	.12
P-value	-	-	.01%	8.51%	7.04%	-	.01%
VIX	-.01	-	-	-	-	-	-
P-value	7.29%	-	-	-	-	-	-
R^2 / *Pseudo*-R^2	63.99%	48.36%	35.65%	27.50%	22.01%	31.55%	65.29%

Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. OLS and L stands respectively for traditional Least Square regression and L-estimator. The probabilities in the first line of the table stand for the probabilities associated to the return quantiles of the hedge fund strategy under study. The standard errors, associated P-Values and the *Pseudo*-R^2 for the Quantile Regressions are based on 1,000 bootstrap replications.

Table 3: Robust Quantile Regression on Emerging Markets Strategy

	OLS	10%	25%	50%	75%	90%	L
Alpha	-	-.03	-.02	-.02	-	-.02	-.04
P-value	-	.68%	3.87%	6.11%	-	7.30%	.69%
Russell 3000	-	-	-	-	.23	.24	.47
P-value	-	-	-	-	.28%	1.04%	.28%
MSCI Europe (EMU)	.15	-	-	-.11	-.25	-.22	-.49
P-value	6.54%	-	-	4.17%	.00%	.00%	.00%
MSCI Emerging Market	.74	.50	.55	.51	.50	.45	1.25
P-value	.29%	.00%	.00%	.00%	.00%	.00%	.00%
Small Minus Big US	-2.96	-	-	-	-	-	-
P-value	5.19%	-	-	-	-	-	-
High Minus Low US	-.09	-	-	-	-	-	-
P-value	2.52%	-	-	-	-	-	-
Momentum	-	.13	-	-	-	-	.08
P-value	-	2.04%	-	-	-	-	2.08%
Short-Term Reversal Factor	-	-	-.08	-	-.12	-.11	-.25
P-value	-	-	6.85%	-	1.62%	7.48%	1.65%
US LIBOR 3 Months	-	-3.82	-	-	-	-3.12	-6.36
P-value	-	4.09%	-	-	-	3.55%	3.63%
10-Y Treasury Rate	.50	.51	-	.61	.76	1.03	2.01
P-value	.00%	7.30%	-	1.87%	.22%	.00%	.00%
CS High Yield Value	-.09	-	-	-	-	.16	.32
P-value	5.91%	-	-	-	-	9.21%	9.40%
CRB All Commodities	-	.10	-	-	-	-	.23
P-value	-	6.63%	-	-	-	-	6.77%
VIX	-.02	-	-	-	-	-	-
P-value	1.14%	-	-	-	-	-	-
R^2 / *Pseudo*-R^2	82.35%	64.56%	61.16%	56.51%	56.35%	58.56%	84.03%

Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. OLS and L stands respectively for traditional Least Square regression and L-estimator. The probabilities in the first line of the table stand for the probabilities associated to the return quantiles of the hedge fund strategy under study. The standard errors, associated P-Values and the *Pseudo*-R^2 for the Quantile Regressions are based on 1,000 bootstrap replications.

Table 4: Robust Quantile Regression on Event Driven Strategy

	OLS	10%	25%	50%	75%	90%	L
Alpha	-	-.03	-.02	-	-	-	.00
P-value	-	.01%	.27%	-	-	-	.01%
Russell 3000	-	.15	.20	.20	.19	.16	.46
P-value	-	.84%	.00%	.00%	.00%	.06%	.00%
MSCI Europe (EMU)	.23	-	-	-	-	-	-
P-value	.00%	-	-	-	-	-	-
MSCI Emerging Market	.30	.08	.04	.05	.07	.05	.14
P-value	1.33%	.08%	6.43%	1.20%	.07%	4.23%	.07%
Small Minus Big US	-	.09	.16	.18	.17	.17	.43
P-value	-	2.90%	.00%	.00%	.00%	.00%	.00%
High Minus Low US	-	.09	-	.06	.06	-	.11
P-value	-	2.90%	-	3.98%	4.88%	-	2.95%
Momentum	.05	.10	.05	.06	-	-	.09
P-value	3.00%	.08%	2.24%	1.44%	-	-	.08%
Short-Term Reversal Factor	-	-	-	-	-	-	-
P-value	-	-	-	-	-	-	-
US LIBOR 3 Months	.14	-	-	-	-	-	-
P-value	.01%	-	-	-	-	-	-
10-Y Treasury Rate	.07	.37	.38	-	-	.25	.52
P-value	.07%	5.77%	1.44%	-	-	1.37%	1.39%
CS High Yield Value	-	.29	.26	.21	.19	.23	.56
P-value	-	.00%	.00%	.00%	.00%	.01%	.00%
CRB All Commodities	.17	-	.06	.05	-	-	.11
P-value	.01%	-	6.11%	9.48%	-	-	6.23%
VIX	-.01	-.02	-	-	-	-	.00
P-value	1.82%	6.33%	-	-	-	-	6.46%
R^2 / *Pseudo*-R^2	80.23%	61.26%	57.27%	54.95%	53.50%	52.72%	81.87%

Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. OLS and L stands respectively for traditional Least Square regression and L-estimator. The probabilities in the first line of the table stand for the probabilities associated to the return quantiles of the hedge fund strategy under study. The standard errors, associated P-Values and the *Pseudo*-R^2 for the Quantile Regressions are based on 1,000 bootstrap replications.

Table 5: Robust Quantile Regression on Fixed Income Arbitrage Strategy

	OLS	10%	25%	50%	75%	90%	L
Alpha	-	-.04	-.02	-.01	-	-	-.02
P-value	-	.00%	.01%	1.95%	-	-	.00%
Russell 3000	.05	-	-	-	-	-	-
P-value	6.06%	-	-	-	-	-	-
MSCI Europe (EMU)	.43	-	-	-	-	-	-
P-value	.00%	-	-	-	-	-	-
MSCI Emerging Market	.56	.04	.03	.03	.04	.05	.10
P-value	.00%	8.54%	4.45%	4.79%	.58%	.12%	.13%
Small Minus Big US	-3.15	-	-	-	-	-	-
P-value	.01%	-	-	-	-	-	-
High Minus Low US	-	.08	-	-	-	-	.03
P-value	-	9.02%	-	-	-	-	9.20%
Momentum	.04	.08	.04	-	-	-	.08
P-value	3.83%	1.50%	5.56%	-	-	-	1.54%
Short-Term Reversal Factor	-	-	-	-	-.03	-	-.04
P-value	-	-	-	-	5.47%	-	5.58%
US LIBOR 3 Months	-	-4.36	-2.63	-1.81	-1.64	-3.53	-6.54
P-value	-	.01%	.01%	.80%	.28%	.00%	.00%
10-Y Treasury Rate	.04	.94	.47	.34	.30	.53	1.11
P-value	.71%	.00%	.00%	.08%	.03%	.00%	.00%
CS High Yield Value	-	.43	.43	.48	.49	.46	1.20
P-value	-	.00%	.00%	.00%	.00%	.00%	.00%
CRB All Commodities	.06	.11	-	-	-	-	.07
P-value	3.74%	2.71%	-	-	-	-	2.76%
VIX	-.02	-	-	-	-	-	-
P-value	.03%	-	-	-	-	-	-
R^2 / *Pseudo*-R^2	80.20%	60.65%	54.70%	51.05%	51.25%	55.48%	81.84%

Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. OLS and L stands respectively for traditional Least Square regression and L-estimator. The probabilities in the first line of the table stand for the probabilities associated to the return quantiles of the hedge fund strategy under study. The standard errors, associated P-Values and the *Pseudo*-R^2 for the Quantile Regressions are based on 1,000 bootstrap replications.

Table 6: Robust Quantile Regression on Equity Hedge Strategy

	OLS	10%	25%	50%	75%	90%	L
Alpha	-	-.02	-.01	-	-	-	-.02
P-value	-	.25%	1.77%	-	-	-	.25%
Russell 3000	.09	.18	.32	.35	.34	.31	.82
P-value	.77%	.00%	.00%	.00%	.00%	.01%	.00%
MSCI Europe (EMU)	.12	-	-	-	-	-	-
P-value	1.87%	-	-	-	-	-	-
MSCI Emerging Market	-	.12	.09	.06	.06	.10	.21
P-value	-	.00%	.00%	.90%	2.09%	1.27%	.00%
Small Minus Big US	1.99	.16	.22	.24	.24	.16	.53
P-value	2.50%	.01%	.00%	.00%	.00%	.71%	.00%
High Minus Low US	-.07	-.11	-	-.08	-.11	-.14	-.27
P-value	.28%	.02%	-	2.01%	1.83%	2.73%	.02%
Momentum	.05	-	.07	.07	-	-	.12
P-value	3.71%	-	.41%	1.69%	-	-	.42%
Short-Term Reversal Factor	-.09	-	-.06	-.10	-.10	-	-.22
P-value	.66%	-	.93%	.01%	.22%	-	.01%
US LIBOR 3 Months	.21	1.86	1.53	1.63	2.76	2.84	6.13
P-value	.00%	5.89%	9.40%	9.50%	.70%	.55%	.56%
10-Y Treasury Rate	.09	-	-	-	-	.37	.56
P-value	.03%	-	-	-	-	2.25%	2.30%
CS High Yield Value	-	.17	.14	.18	.12	.14	.36
P-value	-	.28%	.38%	.13%	9.91%	9.34%	.13%
CRB All Commodities	.32	.17	.07	.09	.10	.09	.24
P-value	.00%	.02%	4.96%	1.66%	2.06%	9.52%	.02%
VIX	-.01	-	-	.02	-	-	.03
P-value	2.76%	-	-	1.59%	-	-	1.62%
R^2 / *Pseudo*-R^2	86.77%	69.52%	66.35%	63.54%	61.38%	64.63%	88.54%

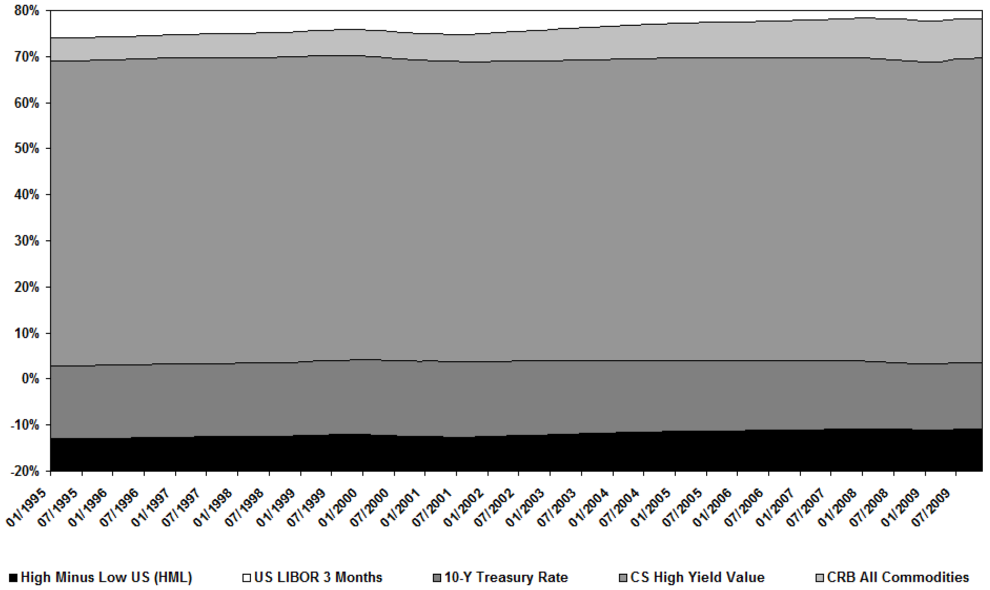
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. OLS and L stands respectively for traditional Least Square regression and L-estimator. The probabilities in the first line of the table stand for the probabilities associated to the return quantiles of the hedge fund strategy under study. The standard errors, associated P-Values and the *Pseudo*-R^2 for the Quantile Regressions are based on 1,000 bootstrap replications.

Table 7: Robust Quantile Regression on Short Bias Strategy

	OLS	10%	25%	50%	75%	90%	L
Alpha	-	-.02	-.03	-	-	-	-.01
P-value	-	4.26%	.20%	-	-	-	.20%
Russell 3000	-	-1.03	-.96	-.82	-.85	-.87	-2.22
P-value	-	.00%	.00%	.00%	.00%	.00%	.00%
MSCI Europe (EMU)	-	-	.11	-	-	-	.07
P-value	-	-	6.69%	-	-	-	6.82%
MSCI Emerging Market	-	.10	-	-	-	-	.10
P-value	-	1.95%	-	-	-	-	1.99%
Small Minus Big US	-	-.60	-.61	-.48	-.48	-.40	-1.19
P-value	-	.00%	.00%	.00%	.00%	.00%	.00%
High Minus Low US	-	.46	.51	.45	.43	.52	1.22
P-value	-	.00%	.00%	.00%	.00%	.00%	.00%
Momentum	.17	.14	.17	.16	.14	.17	.40
P-value	.44%	9.53%	.94%	.61%	.86%	.22%	.22%
Short-Term Reversal Factor	.58	-	-	-	-	-	-
P-value	.00%	-	-	-	-	-	-
US LIBOR 3 Months	-.55	-	-3.32	-	4.14	-	5.60
P-value	.00%	-	9.06%	-	2.66%	-	2.72%
10-Y Treasury Rate	-	-	.66	-	-	-	.65
P-value	-	-	1.48%	-	-	-	1.52%
CS High Yield Value	-	-	-	-	-	-	-
P-value	-	-	-	-	-	-	-
CRB All Commodities	-.90	-	-	-	-	-	-
P-value	.00%	-	-	-	-	-	-
VIX	-	-	-.03	-	-	-	-.05
P-value	-	-	6.19%	-	-	-	6.32%
R^2 / *Pseudo*-R^2	85.30%	66.11%	62.31%	60.73%	65.05%	67.03%	87.04%

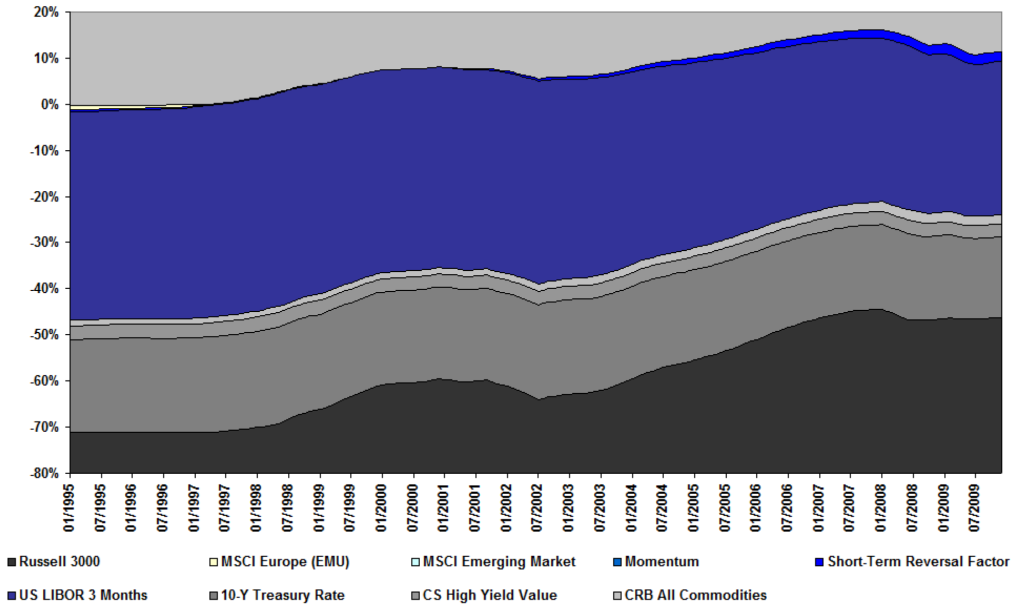
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors. OLS and L stands respectively for traditional Least Square regression and L-estimator. The probabilities in the first line of the table stand for the probabilities associated to the return quantiles of the hedge fund strategy under study. The standard errors, associated P-Values and the *Pseudo*-R^2 for the Quantile Regressions are based on 1,000 bootstrap replications.

Figure 10: Time-varying Risk Factor Expositions of the Convertible Arbitrage Strategy



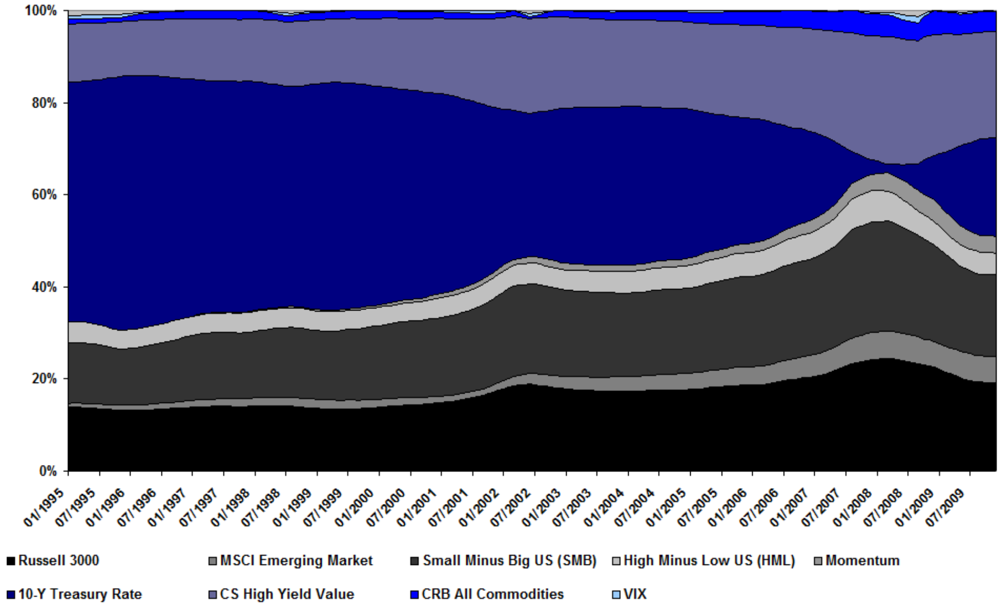
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 11: Time-varying Risk Factor Expositions of the Emerging Markets Strategy



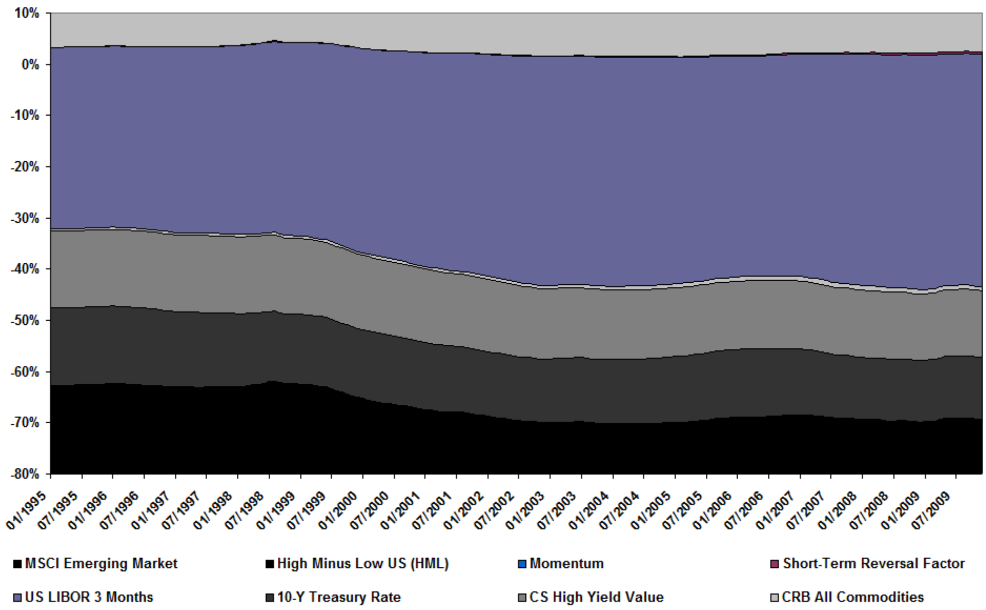
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 12: Time-varying Risk Factor Expositions of the Event Driven Strategy



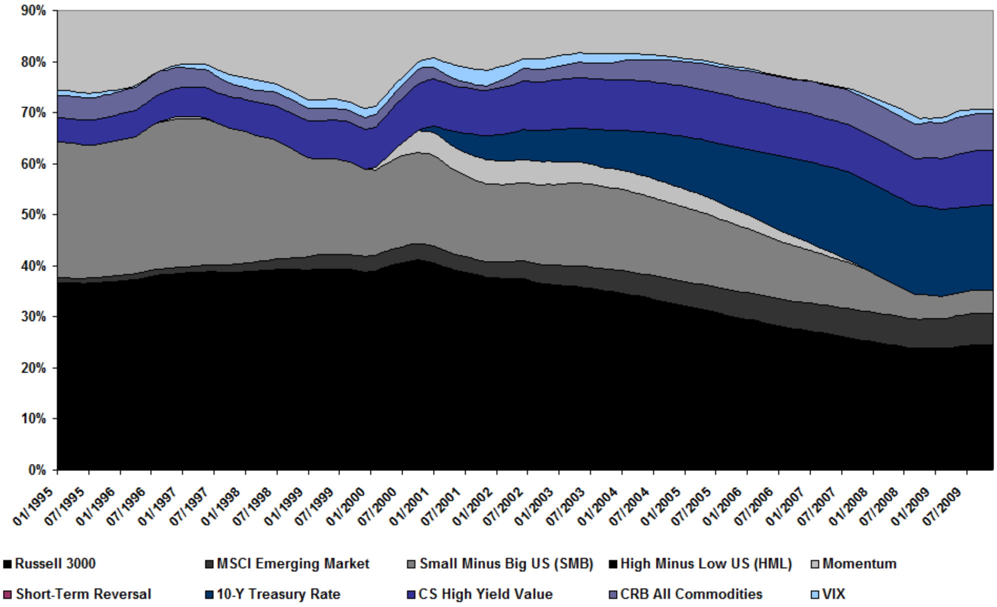
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 13: Time-varying Risk Factor Expositions of the Fixed Income Arbitrage Strategy



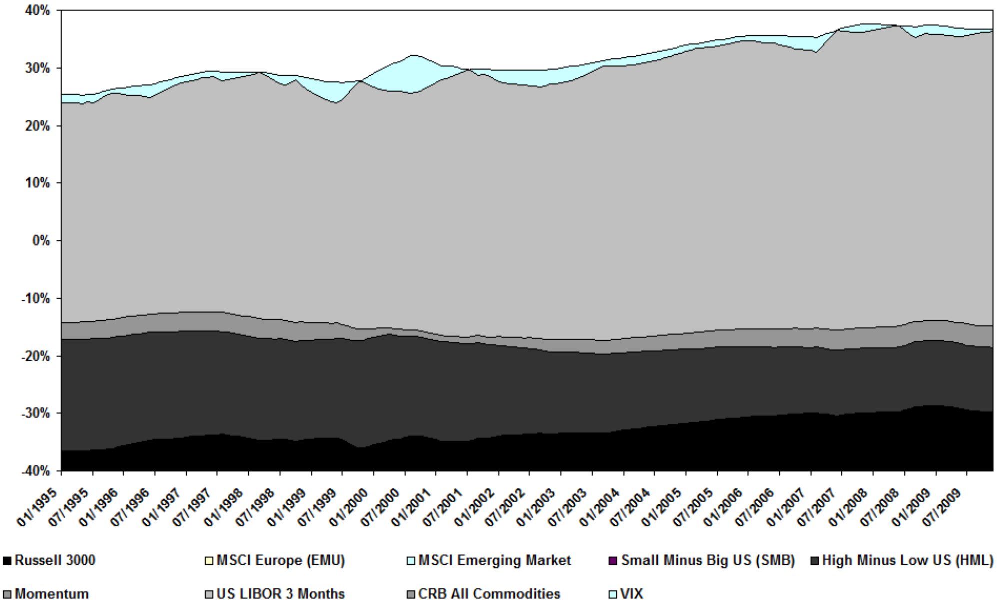
Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 14: Time-varying Risk Factor Expositions of the Macro Strategy



Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Figure 15: Time-varying Risk Factor Expositions of the Short Bias Strategy



Source: *Bloomberg*, HFR index prices in USD monthly data from January 1995 to January 2010. Computation by the authors.

Moreover, hedge fund strategies employ different trading tools, each of them involving particular financial assets and, therefore, each strategy is expected to be on average explained by a number of risk factors associated with specific strategies. Hedge funds are definitively actively managed. We apply the time-varying framework for a multi-quantile estimator that we have introduced in the previous section as illustrated on figures (10) to (15).
In extreme economic conditions (for example, in financial distress, political instability or in periods of economic prosperity) the financial asset classes affect each other. Hence, in such cases hedge fund strategies, which are flexible with respect to the type of securities they hold and the type of positions they take, are likely to be affected by a larger variety of economic risk factors. This is reflected on the dependence of higher and lower hedge fund returns on this wide class of factors and by the evolution of the exposition according to time and market evolutions.

This time-varying multi-quantile approach seems to be particularly useful in cases where the distribution of returns is characterized by large skewness, *kurtosis*, fat tails, or in general deviates from normality. In those cases, the conditional mean regression method is not adequate, while this Quantile Regression approach provides more robust and more efficient results. Deviations from normality are very common in most of fund returns data. Due for example to the highly dynamic complex trading strategies of funds, the return series usually display special characteristics, such as skewness, *kurtosis* and non-normality. Therefore, this time-varying multi-Quantile Regression approach may also provide a useful tool for the analysis of other specific funds than hedge funds.

Moreover these empirical evidences confirm that, in the conditional mean regression models, fewer factors are included than those required to explain the entire conditional distribution of returns using Quantile Regression. The Quantile Regression technique provides considerably stronger insight on the distributional dependence of hedge fund returns on factors compared to the standard conditional mean regression method. Indeed, our findings show that the Quantile Regression approach is very useful for the analysis of hedge funds as it underlines the factors that are not taken into consideration in the conditional mean model. This observation is consistent with the fact that the mean regression only reflects the relationship of conditional returns with factors on average, while the Quantile Regression approach captures the change of this relationship across the entire distribution of conditional returns. The latter approach to modeling hedge fund returns is able to capture the special characteristics of the data, and provides a more sophisticated decomposition of the returns into the part that can be replicated by related risk factors and the residual which is attributed to the managers' skill across different states represented by certain conditional quantiles. This may have an impact on the ranking of hedge fund strategies based on their alphas and the construction of style portfolios.

We have applied our time-varying multi-quantile framework approach to fund single strategy indices and compared it with the standard conditional mean regression method. We have found substantial differences with respect to this subset of factors between these methods and, in particular, we have found that a larger number of factors are required to model the upper and lower quantiles of fund returns than those needed to explain the conditional mean or median.

5 Conclusion

In this paper, we have proposed a complete robust conditional multi-Quantile Regression approach for the style analysis of fund single strategy return series. We first proposed to use "peer group" analysis within the SOM algorithm (associated to the traditional factor decomposition methods as ICA or PCA) reducing the selection bias on the "real" risk factors. This analysis also explores the impact of a number of risk factors on the entire conditional distribution of hedge fund returns and aggregates these views into a robust multi-quantile estimator. At last, we adapt the FLS methodology to our multi-quantile model in order to perform a time-varying analysis (more coherent with active management) taking into account the evolution of the sensibilities to the risk factors.

Unlike the standard conditional mean regression method, which only examines how the risk factors affect returns on average, our Quantile Regression approach is able to uncover how this dependence varies across time and return quantiles. Thus, this approach provides useful insights on the distributional dependence of fund returns on risk factors. Moreover, the aggregation of these views is less sensitive to "outliers" and allows us to get a robust estimate of the style of funds studied. It is also of great interest to identify the risk factors associated with fund investing, in the context of Quantile Regression.

The proposed time-varying multi-Quantile Regression approach is particularly useful in cases where the return distribution is characterized by large skewness, *kurtosis*, fat tails, or in general deviates from normality, due for example to the highly dynamic complex trading strategies of some funds. In those cases, the conditional mean regression method may not be adequate, while this multi-quantile approach provides more robust and more efficient estimates. Deviations from normality are very common in most of the fund studied. We have thus applied the proposed approach to hedge fund single strategy indices and compared it with the standard conditional mean regression method.

We have found substantial differences with respect to the most relevant subsets of factors between the two methods and, in particular, we have found that a larger number of factors are required to model the upper and lower quantiles of hedge fund returns than those needed to explain the conditional mean or median.

To investigate potential economic impacts of our approach, we could build style allocation portfolios based on the best performing (according to the ranking of alphas for the best, under different model selection methods, mean and Quantile Regression models) fund strategies and classify them using a SOM classification based on quantiles of these strategies.

At last, there exist several potential applications of our conditional time-varying multi-quantile approach. An interesting application of the proposed method could be in risk management of fund strategies, by computing stress *scenarii* for risk measures as the Expected Shortfall (ES). Actually, this method can be applied to qualify the nature of the risk by identifying the relevant risk factors associated to the ES and perform a "what if" analysis using the estimated time-varying risk exposures.

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6 Appendix

6.1 HFR^{TM} Main Classification - Strategies Definitions

- Market neutral hedge funds take long and short positions in such a way that the impact of the overall market is minimized. Market neutral can imply dollar neutral, beta neutral or both. - Dollar neutral strategy has zero net investment (i.e., equal dollar amounts in long and short positions). - Beta neutral strategy targets a zero total portfolio beta (i.e., the beta of the long side equals the beta of the short side). While dollar neutrality has the virtue of simplicity, beta neutrality better defines a strategy uncorrelated with the market return. Many practitioners of market-neutral long/short equity trading balance their longs and shorts in the same sector or industry. By being sector neutral, they avoid the risk of market swings affecting some industries or sectors differently than others.

- Convertible arbitrage is sometime considered as a sub strategy of market neutral strategies. Funds following this strategy invest in convertible bonds. These are some hybrid assets that combine a traditional bond plus a call option on the same company. Long positions on convertible are generally hedged by short positions on the company stock (the hedge ratio will typically be between 30% and 100%. This strategy will typically perform when equity market volatility increases or when embedded option price is not well evaluated. Leverage used for this kind of strategy can goes up to 6.

- Emerging market hedge funds invest in equity or debt of emerging markets that tend to have higher inflation and volatile growth. Short selling is not permitted in many emerging markets, and, therefore, effective hedging is often not available, although Brady debt can be partially hedged *via* U.S. Treasury futures and currency markets, but their expected volatility is very high.

- Event driven strategies tend to take advantage of special situations such as merger, acquisitions, IPO, collapse, debt restructuration or a change in credit notation. Two main sub strategies are Event Driven: distressed securities and merger arbitrage. In the case of a merger arbitrage strategy (also named risk arbitrage or event driven equity), manager is looking for opportunities born from merger or acquisition between companies. Positions are generally taken before any formal announcement of a merger. The manager looking for opportunities born from the spread between the market valuation and the theoretical value of the company being merged or acquired. A distressed security strategy invests in debt of companies that are likely to collapse. The ratings of these bonds are typically CCC. Hedge fund manager seek opportunities born from a decline of stock or bond prices. The rational behind this strategy is that when a company is likely to collapse, market prices will go bellow theoretical prices due to market irrationality.

- Fixed income arbitrage aims to take advantage of the spread between assets issued by a same entity. Bonds used by this kind of strategy are typically rated investment grade (corresponding to a BBB rating from Fitch and Standard and Poor's or Baa for the Moody's metric). The arbitrage can be found between junior and senior debts. Typically, the fund will be long for the senior debt and short for the junior debt. Long positions on a company debt can sometime be hedge by short positions on stock (when a downgrade of the company is anticipated. Liquidity and credit risk of the debt are compensated better by the short position on equity than others debts.

- High yield arbitrage strategy seeks to deliver positive returns by taking advantage of high credit risk bonds. Assets used are generally rated below investment grade. They considered as speculative investment because of the high probability to default (compare with a AAA-rated debt). This probability to default is compensated by a high return (high yield). Depending on the horizon (maturity) of the fund, funds manager will invest in high yield bonds if a long term strategy is set. In a case of a short term strategy, the fund will by some debt those credit rating is expected to be upgraded. Indeed, a decrease of credit risk will go with a decrease of spread (yield) and thus an increase of the debt price. It is to be notice that this strategy is in fact not an arbitrage strategy. Main risks can not be hedged. High yield bonds are not liquid enough to allow short positions. Funds following this strategy have generally a long bias on high yield risk. The fund manager has to manage in priority credit and liquidity as well as interest risks.

- The Global Marco strategy is based on a global macroeconomic approach that aims to detect/anticipate future market trends. Global Macro manager generally adopt a top down approach that rely on a fundamental analysis (unemployment rate, political situation, money markets *etc.*). A complementary micro analysis is often conducted for new fundamental tendency detection and their impact on asset classes. Thus Global Macro funds can be highly concentrated on specific asset class. This opportunistic approach can leads to high performance but with a poor risk management.

- Long/short equity hedge funds combine long positions with short sales (generally in the same sectors of the market). These strategies may reduce the Market risk, but effective stock analysis and stock picking is essential to obtaining meaningful results. For example, a long/short manager might purchase a portfolio of core stocks that occupy the S&P 500 and hedge by selling (shorting) S&P 500 Index futures. If the S&P 500 goes down, the short position will offset the losses in the core portfolio, limiting overall losses. The leverage may be used to enhance returns. These funds have usually low (or no) correlation to the market. They use sometimes market index futures to hedge out systematic (market) risk.

- Dedicated short bias hedge funds are specialized in the short sale of over-valued securities. Because losses on short-only positions are theoretically unlimited (because the stock can rise indefinitely), these strategies are particularly risky.

- Managed Futures funds invest in the global currency, interest rate, equity, metal, energy and agricultural markets. They do this through the use of futures, forwards and options.

- Fund of Funds mixes and matches hedge funds. This blending of different strategies and asset classes aims to provide a high level of diversification and a more stable long-term investment return than any of the individual funds. Returns, risk, and volatility can be controlled by the mix of underlying strategies and funds.

6.2 The SOM Algorithm

The SOM algorithm is recursively defined by the followings steps (see Figure 16):

1. Draw randomly an observation $\mathbf{x}$.

2. Find the winning unit $i_w(\mathbf{x}, \mathbf{m})$ also called the Best Matching Unit (noted BMU) such that:
$$BMU = i_w\left[\mathbf{x}\left(t+1\right), \mathbf{m}\left(t\right)\right] = \underset{\mathbf{m}_i, i \in I}{\text{Argmin}} \left\{ \left\| \mathbf{x}\left(t+1\right) - \mathbf{m}_i\left(t\right) \right\| \right\}, \tag{25}$$
where $\|.\|$ is the Euclidean norm.

3. Once the BMU is found, the weight vectors of the SOM are updated so that the BMU and this neighbors are moved closer to the input vector. The SOM update rule is:

$$\mathbf{m}_i\left(t+1\right) = \mathbf{m}_i\left(t\right) - \varepsilon_t \Lambda\left(BMU, i\right)\left[\mathbf{m}_i\left(t\right) - \mathbf{x}\left(t+1\right)\right], \forall i \in I, \tag{26}$$

where ε_t is the adaptation gain parameter, which is $]0,1[$-valued, generally decreasing with time.

The number of neurons taken into account during the weight update depends on the neighborhood function Λ that also decreases with time.

Following Boucher *et al.* (2010), the Bootstrap technique principle is applied to SOM by estimating a probability for every individual to be in the same group. This probability is empirically estimated during the learning SOM step made on the resampled temporal series.

The matrix $\mathbf{P}$ contains the empirical probability of two individuals to be in the same group at the end of a classification. When the matrix $\mathbf{P}$ is built, the first phase is ended. During the second phase, the algorithm SOM is again executed a large number of time (by keeping this time all the observations). To every map obtained M_i, we build a table $\mathbf{P}_{M_i}$ similar to the previous one (its values are 1 if two individuals are neighbor and 0 otherwise). The robust selected map (denoted below $RMap$) is the one which minimizes the distance between both neighborhood structures:

$$RMap = \underset{M_i, i \in I}{\text{Argmin}} \left\{ \left\| P - P_{M_i} \right\|_{Frob} \right\}, \tag{27}$$

with $\|A\|_{Frob} = \frac{1}{n^2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} a_{[i,j]}^2}$, and n is the dimension of the squared matrix $\mathbf{A}$, which elements are $a_{[i,j]}, \forall\left(i,j\right) \in I^2$.

Figure 16: Illustration of the Kohonen Iterative Process.