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INDIVIDUAL TRACKING OF AN EIGENVALUE AND EIGENVECTOR OF A PARAMETERIZED MATRIX*

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1. INTRODUCTION

LET $M(\alpha)$ BE an $n \times n$ complex matrix-valued differentiable function of a parameter α varying over some simply connected region C^0 of the complex plane C. In a previous paper [1] a complete initial value system of differential equations was developed for both the eigenvalues and right and left eigenvectors of $M(\alpha)$, assuming $M(\alpha)$ has n distinct eigenvalues for each α in C^0. In the present paper it is shown that an initial value system can also be developed for tracking a single eigenvalue of $M(\alpha)$ together with one of its corresponding right or left eigenvectors. The computational feasibility of the initial value system is illustrated by numerical example.

2. VARIATIONAL EQUATIONS FOR INDIVIDUAL TRACKING

Consider the system of equations

$$M(\alpha)x = \lambda x, \tag{1a}$$

$$x^T x = 1, \tag{1b}$$

where superscript T denotes transpose. Any solution $(\lambda(\alpha), x(\alpha))^T$ for system (1) in $C \times C^n$ yields an eigenvalue and corresponding unit normalized right eigenvector for $M(\alpha)$, respectively.

If the Jacobian Matrix for system (1),

$$J(\alpha) \equiv \begin{bmatrix} \lambda(\alpha)I - M(\alpha) & x(\alpha) \\ x(\alpha)^T & 0 \end{bmatrix}, \tag{2}$$

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is nonsingular at α^0, then, totally differentiating (1) with respect to α, one obtains

$$\begin{bmatrix} \dot{x}(\alpha) \\ \dot{\lambda}(\alpha) \end{bmatrix} = J(\alpha)^{-1} \begin{bmatrix} \dot{M}(\alpha)x(\alpha) \\ 0 \end{bmatrix}, \tag{3}$$

where a dot denotes differentiation with respect to α. Letting $A(\alpha)$ denote the adjoint $\text{Adj}(J(\alpha))$ of $J(\alpha)$ and $\delta(\alpha)$ denote the determinant $\text{Det}(J(\alpha))$ of $J(\alpha)$, system (3) can be expanded [2] into a complete initial value differential system of the form

$$\begin{bmatrix} \dot{x}(\alpha) \\ \dot{\lambda}(\alpha) \end{bmatrix} = \frac{A(\alpha)}{\delta(\alpha)} \begin{bmatrix} \dot{M}(\alpha)x(\alpha) \\ 0 \end{bmatrix}, \tag{4a}$$

$$\dot{A}(\alpha) = [A(\alpha)\,\text{Trace}(A(\alpha)B(\alpha)) - A(\alpha)B(\alpha)A(\alpha)]/\delta(\alpha), \tag{4b}$$

$$\dot{\delta}(\alpha) = \text{Trace}(A(\alpha)B(\alpha)), \tag{4c}$$

with initial conditions

$$x(\alpha^0) = x^0, \tag{4d}$$

$$\lambda(\alpha^0) = \lambda^0, \tag{4e}$$

$$A(\alpha^0) = \text{Adj}(J(\alpha^0)), \tag{4f}$$

$$\delta(\alpha^0) = \text{Det}(J(\alpha^0)), \tag{4g}$$

where $B(\alpha) \equiv \mathrm{d}J(\alpha)/\mathrm{d}\alpha$, and λ^0 and x^0 denote an arbitrarily selected eigenvalue and corresponding unit normalized right eigenvector for $M(\alpha^0)$, respectively.

The Jacobian $J(\alpha)$ defined by (2) is an interesting bordered matrix whose properties do not appear to have been previously explored. It is shown in a companion paper [3] that $J(\alpha)$ is nonsingular if $M(\alpha)$ is a positive matrix, $\lambda(\alpha)$ is the Perron root of $M(\alpha)$, and $x(\alpha)$ is a corresponding right eigenvector with elements taken to be positive. Alternatively, $J(\alpha)$ is nonsingular if $M(\alpha)$ is real and symmetric, with complete orthonormal system of real eigenvectors given by $x_1(\alpha), \ldots, x_n(\alpha)$, and $\lambda(\alpha) \equiv \lambda_1(\alpha)$ is a simple root of $M(\alpha)$ corresponding to the eigenvector $x(\alpha) \equiv x_1(\alpha)$. Specifically, it is easily verified that the bordered matrix

$$\begin{bmatrix} \lambda_1(\alpha)\mathrm{I} - M(\alpha) & x_1(\alpha) \\ x_1(\alpha)^T & 0 \end{bmatrix}_{(n+1)\times(n+1)} \tag{5}$$

then has $n + 1$ linearly independent eigenvectors

$$\begin{pmatrix} x_1(\alpha) \\ 1 \end{pmatrix}, \begin{pmatrix} x_2(\alpha) \\ -1 \end{pmatrix}, \begin{pmatrix} x_2(\alpha) \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} x_n(\alpha) \\ 0 \end{pmatrix}, \tag{6}$$

with corresponding nonzero eigenvalues

$$1, -1, (\lambda_1(\alpha) - \lambda_2(\alpha)), \ldots, (\lambda_1(\alpha) - \lambda_n(\alpha)), \tag{7}$$

hence the determinant $\delta(\alpha)$ of $J(\alpha)$ is not zero.

In the following section the potential usefulness of the initial value system (4) as a practical tool for numerical work will be illustrated by example.

3. AN ILLUSTRATIVE EXAMPLE

Consider a matrix-valued function $M(\alpha)$ defined over α in R by

$$M(\alpha) = \begin{bmatrix} 1 & \alpha \\ \alpha^2 & 3 \end{bmatrix}. \tag{8}$$

For this simple example, analytical expressions are easily obtained for the eigenvalues $\{\lambda_1(\alpha), \lambda_2(\alpha)\}$ and the corresponding unit normalized right eigenvectors $\{x_1(\alpha), x_2(\alpha)\}$ of $M(\alpha)$. Specifically,

$$\lambda_1(\alpha) = 2 + \gamma(\alpha), \tag{9a}$$

$$\lambda_2(\alpha) = 2 - \gamma(\alpha), \tag{9b}$$

$$x_1(\alpha) = \left(\frac{\alpha}{1 + \gamma(\alpha)} k_1, k_1\right)^T, \tag{9c}$$

$$x_2(\alpha) = \left(\frac{\alpha}{1 - \gamma(\alpha)} k_2, k_2\right)^T. \tag{9d}$$

where

$$\gamma(\alpha) \equiv \sqrt{1 + \alpha^3}, \tag{9e}$$

$$k_1 \equiv 1 \Big/ \sqrt{\frac{\alpha^2}{(1 + \gamma(\alpha))^2} + 1}, \tag{9f}$$

$$k_2 \equiv 1 \Big/ \sqrt{\frac{\alpha^2}{(1 - \gamma(\alpha))^2} + 1}. \tag{9g}$$

Consider the individual tracking of the first eigenvalue $\lambda_1(\alpha) \equiv \lambda(\alpha)$ and its corresponding right eigenvector $x_1(\alpha)^T \equiv (u(\alpha), v(\alpha))^T$. For this special case the differential system (4) reduces to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{\lambda} \end{pmatrix} = \frac{A}{\delta}\begin{pmatrix} v \\ 2\alpha u \end{pmatrix} \tag{10a}$$

$$\dot{A} = [A\,\mathrm{Trace}(AB) - ABA]/\delta \tag{10b}$$

$$\dot{\delta} = \mathrm{Trace}(AB), \tag{10c}$$

with initial conditions

$$u(\alpha^0) = \alpha^0 k_1/[1 + \gamma(\alpha^0)], \tag{10d}$$

$$v(\alpha^0) = k_1, \tag{10e}$$

$$\lambda(\alpha^0) = 2 + \lambda(\alpha^0), \tag{10f}$$

$$A(\alpha^0) = \mathrm{Adj}(J(\alpha^0)), \tag{10g}$$

$$\delta(\alpha^0) = \mathrm{Det}(J(\alpha^0)), \tag{10h}$$

where

$$J(\alpha) = \begin{bmatrix} \lambda(\alpha) - 1 & -\alpha & u(\alpha) \\ -\alpha^2 & \lambda(\alpha) - 3 & v(\alpha) \\ u(\alpha) & v(\alpha) & 0 \end{bmatrix} \tag{10i}$$

and

$$B(\alpha) = \frac{dJ(\alpha)}{d\alpha} = \begin{bmatrix} \dot{\lambda}(\alpha) & -1 & \dot{u}(\alpha) \\ -2\alpha & \dot{\lambda}(\alpha) & \dot{v}(\alpha) \\ \dot{u}(\alpha) & \dot{v}(\alpha) & 0 \end{bmatrix}. \quad (10j)$$

A numerical solution was obtained for the first eigenvalue $\lambda_1(\alpha)$ and corresponding right eigenvector $x_1(\alpha)$ of $M(\alpha)$ by integrating the initial value system (10) from $\alpha^0 = 0.5$ to 2.0. A fourth-order Runge–Kutta method was used for the integration with the α grid intervals set equal to 0.01. As indicated in Table 1, the numerical results agree with the analytical solution to at least six digits. Similar results were obtained for the second eigenvalue $\lambda_2(\alpha)$.

Table 1. Eigenvalue $\lambda_1(\alpha) \equiv \lambda(\alpha)$ and corresponding eigenvector $x_1(\alpha)^T = (u(\alpha), v(\alpha))$ at $\alpha = 2.0$

	Numerical solution	Analytical solution
$\lambda(2)$	5.0	5.0
$u(2)$	0.447 214	0.447 214
$v(2)$	0.894 427	0.894 427

It is clear from (9a) and (9b) that two eigenvalues $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ coalesce at $\alpha = -1.0$ and become complex for $\alpha < -1.0$. The eigenvector $x_1(\alpha)^T$ becomes (0, 1) at $\alpha = 0.0$. The initial value system (10) was integrated from $\alpha^0 = 0.5$ to -1.0 through 0.0 using an integration step size of -0.01 for α. Six digit accuracy was obtained integrating from 0.5 to -0.98. Approximately two digit accuracy was obtained at the point $\alpha = -1.0$ where the roots coalesce.

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