A New Differential Equation Method for Finding the Perron Root of a Positive Matrix*

R. Kalaba†

Department of Economics
University of Southern California
Los Angeles, California 90007

K. Spingarn‡

Space and Communications Group
Hughes Aircraft Company
Los Angeles, California 90009

and

L. Tesfatsion**

Department of Economics
University of Southern California
Los Angeles, California 90007

ABSTRACT

A basic problem in linear algebra is the determination of the largest eigenvalue (Perron root) of a positive matrix. In the present paper a new differential equation method for finding the Perron root is given. The method utilizes the initial value differential system developed in a companion paper for individually tracking the eigenvalue and corresponding right eigenvector of a parametrized matrix.

1. INTRODUCTION

A basic problem in linear algebra [1–4] is the determination of the largest eigenvalue of an $n \times n$ matrix Q. When the entries of Q are nonnegative, the largest eigenvalue is referred to as the Perron or Perron-Frobenius root.

*The work of R. Kalaba and L. Tesfatsion was partially supported by the National Science Foundation under Grant ENG77-28432 and the National Institutes of Health under Grant GM23732-03.

†Professor of Economics and Biomedical Engineering.

‡Senior Staff Engineer.

**Assistant Professor of Economics.

52 Vanderbilt Ave.,
New York, NY 10017 0096-3003/80/070187+7$01.75

In two companion papers [5, 6] it is shown how the eigenvalues and the right and left eigenvectors of a parametrized matrix $M(\alpha)$ can be tracked as functions of a scalar parameter α by integrating a system of ordinary differential equations from initial conditions.[1] In the present paper it is shown how the initial value differential system developed in Ref. [6] can be modified to obtain an initial value differential system for tracking the Perron root and a corresponding unit normalized right eigenvector for a positive matrix Q, i.e., a matrix Q with all positive elements.

For completeness, the initial value system developed in Ref. [6] is outlined in Sec. 2. The modified initial value system for tracking the Perron root is developed in Sec. 3. A numerical example is given in the final Sec. 4.

2. INDIVIDUAL TRACKING OF AN EIGENVALUE AND EIGENVECTOR

Let $M($ be an $n\times n$ complex matrix-valued continuously differentiable function of a parameter α varying over a simply connected region of the complex plane. In a companion paper [6] it is shown that an initial value system can be developed for tracking a single eigenvalue $\lambda(\alpha)$ and corresponding unit normalized right eigenvector $x(\alpha)$ of $M(\alpha)$ if, at the initial point α^0, a certain Jacobian matrix $J(\alpha^0)$ is nonsingular.

Specifically, the differential system takes the form

$$\begin{bmatrix}\dot{x}(\alpha)\\ \dot{\lambda}(\alpha)\end{bmatrix}=\frac{A(\alpha)}{\delta(\alpha)}\begin{bmatrix}M(\alpha)x(\alpha)\\ 0\end{bmatrix}, \tag{1a}$$

$$\dot{A}(\alpha)=\frac{A(\alpha)\operatorname{Trace}(A(\alpha)B(\alpha))-A(\alpha)B(\alpha)A(\alpha)}{\delta(\alpha)}, \tag{1b}$$

$$\dot{\delta}(\alpha)=\operatorname{Trace}(A(\alpha)B(\alpha)), \tag{1c}$$

with initial conditions

$$x(\alpha^0)=x^0, \tag{1d}$$

$$\lambda(\alpha^0)=\lambda^0, \tag{1e}$$

$$A(\alpha^0)=\operatorname{Adj}(J(\alpha^0)), \tag{1f}$$

$$\delta(\alpha^0)=\operatorname{Det}(J(\alpha^0)), \tag{1g}$$

[1] The right and left eigenvectors of a given $n\times n$ matrix M corresponding to an eigenvalue λ are defined to be the nontrivial solutions x and w^T to $Mx=\lambda x$ and $w^TM=\lambda w^T$, respectively, where superscript T denotes transpose.

where

$$J(\alpha)\equiv\begin{bmatrix}\lambda(\alpha)I-M(\alpha) & x(\alpha)\\ x(\alpha)^T & 0\end{bmatrix}, \tag{1h}$$

$$B(\alpha)\equiv\dot{J}(\alpha)=\begin{bmatrix}\dot{\lambda}(\alpha)I-\dot{M}(\alpha) & \dot{x}(\alpha)\\ \dot{x}(\alpha)^T & 0\end{bmatrix}, \tag{1i}$$

and a dot denotes differentiation with respect to α.

The initial conditions (1d)–(1g) are obtained by solving

$$M(\alpha^0)x=\lambda x, \tag{2a}$$

$$x^Tx=1 \tag{2b}$$

for x^0 and λ^0, and then determining the adjoint $A(\alpha^0)$ and determinant $\delta(\alpha^0)$ of the Jacobian matrix

$$J(\alpha)\equiv\begin{bmatrix}\lambda^0 I-M(\alpha^0) & x^0\\ x^{0T} & 0\end{bmatrix} \tag{3}$$

for the system (2).

3. VARIATIONAL EQUATIONS FOR THE PERRON ROOT

The presently proposed procedure for determining the Perron root λ_p of an arbitrary $n\times n$ positive matrix Q is as follows. Define

$$M(\alpha)\equiv[1-\alpha]C+\alpha Q, \qquad 0\leqslant\alpha\leqslant 1, \tag{4}$$

where

$$C\equiv\begin{bmatrix}1 & \cdots & 1\\ \vdots & & \vdots\\ 1 & \cdots & 1\end{bmatrix}_{n\times n}. \tag{5}$$

The Perron root $\lambda(0)$ of $M(0)=C$ is easily determined to be $\lambda(0)=n$, and the unit length normalized right eigenvector corresponding to $\lambda(0)$ may be taken to be $x(0)^T=(1/\sqrt{n},\dots,1/\sqrt{n})$. Clearly the Perron root $\lambda(1)$ of $M(1)$ is the desired Perron root λ_p of Q. Thus, in principle, the Perron root λ_p of Q may be found by integrating system (1) from $\alpha^0=0.0$ to $\alpha=1.0$ using the initial conditions

$$x(0)^T=\left(1/\sqrt{n},\ldots,1/\sqrt{n}\right), \tag{6}$$

$$\lambda(0)=n, \tag{7}$$

$$A(0)=\mathrm{Adj}(J(0)), \tag{8}$$

$$\delta(0)=\mathrm{Det}(J(0)), \tag{9}$$

where

$$J(0)=\begin{bmatrix} \lambda(0)I-M(0) & x(0) \\ x(0)^T & 0 \end{bmatrix} = \begin{bmatrix} nI-C & x(0) \\ x(0)^T & 0 \end{bmatrix}. \tag{10}$$

Due to the nonlinearity of the differential equation system (1), the integration may be stopped prior to reaching $\alpha=1.0$. However, as the following theorem demonstrates, the Jacobian matrix

$$J(\alpha)\equiv\begin{bmatrix} \lambda(\alpha)I-M(\alpha) & x(\alpha) \\ x(\alpha)^T & 0 \end{bmatrix} \tag{11}$$

is nonsingular for $0<\alpha<1$, where $\lambda(\alpha)$ is the Perron root of $M(\alpha)$ and $x(\alpha)$ is a corresponding unit normalized right eigenvector with all components taken to be positive. Thus, $\delta(\alpha)=\mathrm{Det}(J(\alpha))$ is uniformly bounded away from zero over the compact interval $[0,1]$.

Theorem. *Let λ_p be the Perron root of a positive $n\times n$ matrix M, and let x_p be a corresponding right eigenvector with all elements taken to be positive. Then*

$$\mathrm{Det}\begin{pmatrix} \lambda_p I-M & x_p \\ x_p^T & 0 \end{pmatrix}\neq 0. \tag{12}$$

Proof. It is well known that the dominant (Perron) root λ_p of a positive $n\times n$ matrix M is positive and simple, and that the elements of the right and left eigenvectors x_p and w_p corresponding to λ_p can be taken to be positive.

Define $N\equiv\lambda_p I-M$. Suppose (12) is false, i.e., suppose there exists a nonzero vector $(y, s)^T$ satisfying

$$\begin{pmatrix} N & x_p \\ x_p^T & 0 \end{pmatrix}\begin{pmatrix} y \\ s \end{pmatrix}=0. \tag{13}$$

Then

$$Ny+x_p s=0, \tag{14a}$$

$$x_p^T y=0. \tag{14b}$$

Multiplying through (14a) by w_p^T, one obtains $w_p^T Ny+w_p^T x_p s=0$, which in turn implies $s=0$, since $w_p^T N=0$ and $w_p^T x_p>0$. It follows from (14) that $Ny=0$ and $x_p^T y=0$. However, since λ_p is simple, the right eigenvector x_p is unique up to positive linear transformation. Thus $y=0$, a contradiction.

It follows from this proof by contradiction that the condition (12) must be true. ■

4. AN ILLUSTRATIVE NUMERICAL EXAMPLE

Consider the 2×2 positive matrix Q defined by

$$Q\equiv\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}. \tag{15}$$

It is easily verified that, to six decimal places, the Perron root of Q is

$$\lambda_p=6.464102, \tag{16}$$

with corresponding positive unit normalized right eigenvector given by

$$x_p=\begin{bmatrix} u_p \\ v_p \end{bmatrix}=\begin{bmatrix} 0.343724 \\ 0.939071 \end{bmatrix}. \tag{17}$$

To find the Perron root of Q by means of the differential equation method developed in Sec. 3, we first define

$$M(\alpha)\equiv(1-\alpha)\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}+\alpha Q, \qquad 0\leqslant\alpha\leqslant 1. \tag{18}$$

The appropriate initial value system (1) for the problem at hand then reduces to

$$\begin{bmatrix}\dot{u}(\alpha)\\ \dot{v}(\alpha)\\ \dot{\lambda}(\alpha)\end{bmatrix}=\frac{A(\alpha)}{\delta(\alpha)}\begin{bmatrix}v(\alpha)\\ 3u(\alpha)+4v(\alpha)\\ 0\end{bmatrix}, \tag{19a}$$

$$\dot{A}(\alpha)=\frac{A(\alpha)\operatorname{Trace}(A(\alpha)B(\alpha))-A(\alpha)B(\alpha)A(\alpha)}{\delta(\alpha)}, \tag{19b}$$

$$\dot{\delta}(\alpha)=\operatorname{Trace}(A(\alpha)B(\alpha)), \tag{19c}$$

with initial conditions

$$u(0)=1/\sqrt{2}, \tag{19d}$$

$$v(0)=1/\sqrt{2}, \tag{19e}$$

$$\lambda(0)=2, \tag{19f}$$

$$A(0)=\operatorname{Adj}(J(0)), \tag{19g}$$

$$\delta(0)=\operatorname{Det}(J(0)), \tag{19h}$$

where

$$J(0)=\begin{bmatrix}1 & -1 & 1/\sqrt{2}\\ -1 & 1 & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix} \tag{19i}$$

and

$$B(\alpha)=\dot{J}(\alpha)=\begin{bmatrix}\dot{\lambda}(\alpha) & -1 & \dot{u}(\alpha)\\ -3 & \dot{\lambda}(\alpha)-4 & \dot{v}(\alpha)\\ \dot{u}(\alpha) & \dot{v}(\alpha) & 0\end{bmatrix}. \tag{19j}$$

TABLE 1
PERRON EIGENVALUE $\lambda(\alpha)$ AND CORRESPONDING RIGHT EIGENVECTOR $x(\alpha)^T=(u(\alpha)v(\alpha))$ OF $M(\alpha)$, AND DETERMINANT $\delta(\alpha)$ OF $J(\alpha)$[a]

	Eigenvector		Eigenvalue	Determinant
α	$u(\alpha)$	$v(\alpha)$	$\lambda(\alpha)$	$\delta(\alpha)$
0.0	0.707107	0.707107	2.0	−2.0
0.1	0.614441	0.788963	2.41244	−2.42487
0.2	0.545806	0.837912	2.84222	−2.88444
0.3	0.494948	0.868923	3.28226	−3.36452
0.4	0.456485	0.889731	3.72873	−3.85746
0.5	0.426679	0.904403	4.17945	−4.35890
0.6	0.403041	0.915182	4.63311	−4.86621
0.7	0.383904	0.923373	5.08887	−5.37773
0.8	0.368131	0.929774	5.54618	−5.89237
0.9	0.354927	0.934894	6.00468	−6.40937
1.0	0.343724	0.939071	6.46410	−6.92820

[a]The Perron root of Q is given by $\lambda(\alpha)$ at $\alpha=1$.

A fourth-order Runge-Kutta method was used to integrate the system (19) from $\alpha=0$ to $\alpha=1$, with grid intervals equal to 0.01. As indicated in Table 1, the Perron root $\lambda(1)$ obtained for $M(1)=Q$ agrees to at least six digits with the analytically derived Perron root λ_p of Q given by (16).

REFERENCES

1 R. Gault, R. Haskins, J. Milner, and M. Pratt, *Computational Methods in Linear Algebra*, Wiley, New York, 1974.

2 R. Kalaba and K. Spingarn, Numerical approaches to the eigenvalues of Saaty's matrices for fuzzy sets, *Computers and Mathematics with Applications* 4:369–375 (1978).

3 R. Bellman, An iterative method for finding the Perron root of a matrix, *Proc. Amer. Math. Soc.* 6:719–725 (1955).

4 R. Bellman, Selective computation V: The largest characteristic root of a matrix, *Nonlinear Analysis* 6:905–908 (1979).

5 R. Kalaba, K. Spingarn, and L. Tesfatsion, Variational equations for the eigenvalues and eigenvectors of nonsymmetric matrices, *J. Optimization Theory Appl.*, to appear in vol. 33, no. 1 (Jan. 1981).

6 R. Kalaba, K. Spingarn, and L. Tesfatsion, Individual tracking of an eigenvalue and eigenvector of a parameterized matrix, *Nonlinear Analysis*, to appear.