Team Dynamics and the Empirical Structure of U.S. Firms

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TEAM DYNAMICS AND THE EMPIRICAL STRUCTURE OF U.S. FIRMS

ABSTRACT

A model in which purposive agents self-organize into teams is demonstrated to closely reproduce empirical data on the population of U.S. firms. There are increasing returns within teams and agents move between teams or start new teams when it is in their self-interest. Nash equilibria of the team formation game exist but are unstable. Dynamics are studied using agent-based computing at full-scale with the U.S. private sector (120 million agents). There arise stationary distributions of team sizes, growth rates, ages, output, productivity, income, and job tenure, growth rates that decline with age, growth rate variance that falls with size and age, and approximately constant returns to scale at the aggregate level. Job-to-job flows, hiring, unemployment and other labor market phenomena occur for microeconomic reasons, without resort to external shocks. The model quantitatively reproduces a large number of important regularities associated with firms and labor markets.

Keywords: endogenous firm formation, increasing returns, bounded rationality, unstable Nash equilibria, labor market dynamics, power law firm size distribution, heavy-tailed firm growth rate distribution, agent-based model, non-equilibrium economics, path-dependence, economic complexity, evolutionary economics

JEL classification codes: C63, C73, D23, L11, L22

1 Introduction

A model is described in which coalitions self-organize within a large, heterogeneous population of boundedly rational agents, who interact locally in team production environments, out of equilibrium. The resulting agent coalitions, it will be shown, can have a variety of statistical properties characteristic of firms. The model is extremely simple, with many familiar features: purposive agents who have well-defined preferences and choose their behavior; production functions that generate more (less) output when inputs are increased (decreased); and Nash equilibrium of the associated team formation game. Stationary distributions of firm sizes and growth rates, output and ages, job tenure and agent incomes are generated by the model and are demonstrated to closely resemble empirical data on the population of U.S. firms. The model achieves these results using somewhat less familiar concepts and methodology, including agent-level *disequilibrium* (Nash equilibria exist but are dynamically unstable), the *emergence* of firms as *meta-stable meso-structures* (between the agent and aggregate levels), and so-called *agent-based computing* for generating transient, finitely-lived firms. Specifically, the model is individual-based and I eschew the notion of a firm as a unitary actor—if a firm consists of ten people then the behaviors of all ten are explicitly modeled.[1] Among the surprising results of this model is that *no exogenous shocks are necessary* to generate realistic levels of firm volatility and employment variability. Rather, the (deterministic) processes of (a) agents adjusting their work efforts and occasionally migrating between jobs, (b) each firm's output fluctuating as a result of the changing inputs to production and its ever-varying pool of employees, and (c) firm birth and death as a result of entrepreneurial decision-making are, taken together, demonstrated to be *sufficient* to yield realistic (1) firm growth rate variability, (2) job turnover, and (3) firm entry and exit *rates*. That is, purely microeconomic motivations and behaviors are able to generate empirical

[1] Our firms are *multi-agent systems* (e.g., Minar et al. 1996; Jennings et al. 1998; Ferber 1999; Weiss 1999; d'Inverno and Luck 2001; Liu 2001; Wooldridge 2002; Deguchi 2004; Shoham and Layton-Brown 2009).

levels of aggregate economic variability. Other surprising phenomena that arise in this model are large firms that have little or no explicit internal structure. That is, even though management hierarchies are not part of the model it is possible to get quite large firms to arise. Specifically, for model realizations with 120 million agents, roughly the size of the U.S. private sector work force, the largest firm is $O(10^6)$ agents, in reasonable agreement with the size of the largest American firm.

In some ways the model developed here is *more* realistic than existing models from industrial organization, since agent rationality is bounded, workers can migrate between firms, firm sizes fluctuate, and all these dynamics are internally-generated. In other ways the model is *unrealistic*, as it lacks physical capital, uses only very simple compensation systems, models behavior as one-dimensional (basically, deciding how hard to work), completely neglects prices, disregards social norms of work effort, and so on. As to whether this model represents progress over previous approaches, this is an empirical question. If the model can do a reasonable job reproducing the data then it has much to recommend it. If its output cannot be empirically-grounded then it is of little interest. We shall see that the model can be made to match literally dozens of data on U.S. firms, while the best competitor model seems capable of explaining a couple of empirical facts at most. Therefore, independently of how much one does or does not *like* the model specifications, the very strong empirical character of the output seems to argue that the model should be taken seriously.

The point of departure for this model is to treat firms as multi-agent systems. Indisputably, most firms are composed of multiple agents. However, theories of the firm typically neglect its multi-agent character, a point forcefully made by Winter (1993).[2] A model in which realistic firms emerge from the interactions of individuals could shed substantial light on which elements of the received theories are essential and which are of secondary importance. Such a model could also help

[2] Early analyses of the firm as composed of multiple agents are the Carnegie School's behavioral theory of the firm (Cyert and March 1963) and the Marschak-Radner theory of teams (1972). While these are neglected today, each prefigured modern developments, computational in the former case, game theoretic in the latter.

to distinguish descriptive from normative elements in extant theories. For example, are theoretical claims about the core organizing forces of firms, such as transaction costs, statements of necessity or sufficiency? What happens as agent rationality assumptions are progressively relaxed—as complete rationality gives way to bounded rationality and finally to mere purposiveness, do firm-like multi-agent groups form more easily or with more difficulty? Are extant theories of the firm even sufficiently well-specified that one can build more or less complete microeconomic models of them?[3] A working model, in which some firms grow and prosper while others do not, could serve as a laboratory for experimentation. Or it might do much more, leading to new conceptualizations of the firm.

Another motivation for modeling firm formation within a population of agents is to open up employment dynamics to a more methodologically individualist perspective. While it is conventional to talk about labor markets from the point-of-view of individual workers and firms (e.g., job creation and job destruction), models in labor economics are aggregate, written in terms of unemployment pools, vacancy rates, and search intensity (Pissarides 2000; Shimer 2010). A model in which individual agents are explicitly represented within firms, and which captures individual worker migration between firms as agents seek better opportunities, has endogenous labor market dynamics, without the need for shocks coming from outside the economy, whether structural or idiosyncratic.

A further goal of treating the firm as a multi-agent system is to contribute to the development of agent-based computing in the social sciences (cf., Hillebrand and Stender 1994; Epstein and Axtell 1996; Tesfatsion 2002), specifically agent-based economics. Creation of agent models in software requires explicit specification of individual behavior at the micro-level. Such models are spun forward in time and patterns and structures emerge. Today, much is known about how to model *markets* using software agents (cf. Palmer et al. 1994; Chen and Yeh 1997; Kirman and Vriend 2000; LeBaron 2001; Axtell 2005; Cont 2006). Too,

[3] Models without explicit dynamics constitute, at best, partial explanations (Simon 1976).

computational organization theory has progressed as a modeling discipline, typically by taking organizational forms as given (Carley and Prietula 1994; Prietula et al. 1998; Lomi and Larsen 2001). However, little is known about how to get multi-agent organizations to form endogenously.[4] Dynamic multi-agent firms are an important step on the road to the creation of an agent-based macroeconomy in software (Axtell 2006; Delli Gatti et al. 2008).[5]

These distinct motivations—to test the extant theory of the firm, to make labor market dynamics endogenous, and to add to the methodology of agent modeling—are intrinsically related. On one hand, any model of firm formation must have foundations that reside in the decisions of individuals. On the other hand, the dynamics of team formation can be studied via agent computing.

Realizations of our agent model will be compared to data on the universe of U.S. firms. Over the past decade, driven largely by advances in information technology, there have appeared increasing amounts of micro-data on U.S. businesses. The model described below is capable of reproducing many important features of the empirical data: firm size, age and growth rate distributions, including joint and conditional distributions involving these variables and their moments, distributions of job tenure and wages across agents, certain network properties, and a few other quantities. For most of these data the best explanations today are largely phenomenological in nature, with little economic content. Concerning firm sizes, for example, from the early work of Gibrat (1931)[6] and continuing in the efforts of Simon and co-workers (Simon 1955; Simon and Bonini 1958), stochastic growth models have been shown to yield skew firm sizes, following lognormal, Pareto, Yule or similar 'thick-tailed' distributions (Stanley and al. 1995; Kwasnicki 1998; Hashemi 2000; Cabral and Mata 2003; Gabaix and

[4] Padgett (1997) has modeled the formation of networks of complementary skills within an agent population. Luna (2000) investigates problem-solving by teams of neural nets and interprets the results in terms of firms.

[5] Building an entire artificial economy from agents was described by Arthur (in Waldrop [1992]) and Lane (1993; 1993); also Lewin (1997). An early model is Basu and Pryor (1997), where firms are unitary actors.

[6] For reviews of Gibrat's contributions see Steindl (1965) and Sutton (1997).

Ioannides 2004; de Wit 2005; Saichev et al. 2010).[7,8] An early attempt to add some microeconomics to these stochastic process stories is due to Lucas (1978), who derived Pareto-distributed firm sizes from a Pareto distribution of managerial talent. More recently, Luttmer obtains Zipf-distributed firm sizes in a variety of general equilibrium settings, driven by skewed productivity distributions (Luttmer 2007), or by innovation (Luttmer 2010), or by replication of organizational capital (Luttmer 2011), always mediated in some subtle way by firm entry, and always driven by exogenous shocks. He has attempted to explain, with less empirical success, firm ages (Luttmer 2007) and growth rate variability (Luttmer 2011). Overall, today there do not exist models with microeconomic foundations that can explain substantial portions of the emerging microdata on firms. Thus my main empirical goal here is to develop just such a model.

There are four streams of thought among theories of the firm that are most relevant to the model described below. The first is team production (Alchian and Demsetz 1972; Holmstrom 1982; Holmstrom and Tirole 1988), in which increasing returns to scale are treated as the origin of incentive problems by virtue of the difficulty of paying agents their marginal products. Second is the *general equilibrium* view (Kihlstrom and Laffont 1979; Laussel and LeBreton 1995; Prescott and Townsend 2006), closely related to the *coalition formation* perspective (e.g., Ray 2007). Here, agents are treated as heterogeneous, each with unique preferences and abilities. A firm is then a (stable) coalition of such agents. The formation of such coalitions can be considered endogenous (Hart and Kurz 1983; Ray and Vohra 1999) and dynamic (Roth 1984; Seidmann and Winter 1998; Konishii and Ray 2003). However, the number of coalition structures is so vast that it is implausible any realistic firm formation process could ever realize

[7] A generation ago Simon caustically critiqued the inability of the neoclassical theory of the firm—with its U-shaped cost functions and perfectly informed and rational managers—to plausibly explain the empirical size distribution (Ijiri and Simon 1977: 7-11, 138-140; Simon 1997). The transaction cost (e.g., Williamson 1985) and more game theoretic theories of the firm (e.g., Hart 1995; Zame 2007) are also ambiguous empirically, placing few restrictions on size and growth rate distributions, for example.

[8] Sutton's (1998) game theoretic models of bound the extent of intra-industry concentration, constraining the shape of size distributions. He has also studied how growth rate variance depends on size (Sutton 2002).

anything like an optimal firm structure (De Vany 1993a; De Vany 1993b; De Vany 1996a; De Vany 1996b).[9] The third stream of relevant literature is the economics of information processing within organizations, where the firm is modeled as a network of communicating agents (Radner 1993; DeCanio and Watkins 1998; Van Zandt 1998; Van Zandt 1999; Miller 2001). The comparative efficiency of firms having alternative incentive and organizational structures is the primary object of study. The related view of organizations as communication networks is found in Dow (1990) and Bolton and Dewatripont (1994). Fourth, the firm plays a central role within the broad field of *evolutionary economics*, where it is viewed neither as a production function, nor a nexus of contracts, but as a set of operating rules and heuristics (Nelson and Winter 1982; Klepper and Graddy 1990; Hodgson 1993; De Vany 1996; Kwasnicki 1996; Mazzucato 2000; Potts 2000; Bowles 2003). Instead of interpreting firm behavior as optimizing, the evolutionary approach treats firm behavior as *adaptive* and profit *seeking*.

The model described below draws together various threads from these competing theoretical literatures. From the neoclassical tradition the notion of a production function is preserved, albeit in a modified form. The model is written at the level of individual agents and incentive problems commonly studied in the principal-agent literature manifest themselves. The agents in the model work in perpetually novel environments, so contracts are incomplete and transaction costs are implicit. Each firm is a coalition of agents, so the general equilibrium approach is relevant. Finally, the ways in which agents make decisions, and firms grow and decline, is in the spirit of evolutionary economics.

Specifically, my model of firm formation consists of a heterogeneous population of agents with preferences for income and leisure. There are increasing returns, so agents who work together can produce more output per unit effort than by working alone. However, agents act non-cooperatively:[10] they select efforts that

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[9] Work on the computational complexity of optimal coalitions can be viewed as impossibility results (cf. Shehory and Kraus 1993; Klusch and Shehory 1996; Klusch and Shehory 1996; Sandholm et al. 1998).

[10] For a cooperative game theoretic view of firms see Ichiishi (1993).

improve their individual welfare, and may migrate between firms or start-up new firms when advantageous. Analytically, Nash equilibria can be unstable in this environment. Large firms are not stable because each agent's compensation is imperfectly related to its effort level, making free-riding possible. Highly productive agents eventually leave large firms and such firms eventually decline. Agent computing is used to study the non-equilibrium dynamics, in which firms are perpetually forming, growing and perishing. For essentially all of the agents *the non-equilibrium regime provides greater welfare than equilibrium.*

Although the model is situated conceptually within existing theories of the firm, the main results are developed using agent-based computation (Holland and Miller 1991; Vriend 1995; Axtell 2000; Tesfatsion 2002). In agent computing, software objects representing individuals are instantiated along with behavioral rules governing their interactions. The model is then marched forward in time and regularities—often at the macro-level—emerge from the interactions (e.g., Grimm et al. 2005). The shorthand for this is that macro-structure "grows" from the bottom-up. No equations governing the macro level are specified. Nor do agents have either complete information or correct models for how the economy will unfold. Instead, they glean data inductively from the environment and their social networks—i.e., through direct social interactions—and make imperfect forecasts of economic opportunities. (Arthur 1994). This methodology facilitates modeling agent heterogeneity, non-equilibrium dynamics, local interactions (Follmer 1974; Kirman 1997), and bounded rationality (Arthur 1991). As we shall see, aggregate stationarity is attained in the model despite perpetual behavioral adjustments and changing employment arrangements at the agent (microeconomic) level. Thus, agent-level equilibria are not the focal point of the analysis. Pragmatically, this is because microeconomic equilibria are not achieved, but there is a deeper reason as well. In section 4 we will see that the variations that bring the model near agent-level (static) equilibria essentially sever its close connection to data. Indeed, maintaining empirical relevance seems to demand a systematic break from agent-level equilibrium notions, although this is not to say that agents are not 'intendly

rational'—they respond to incentives, are always looking for utility gains, and so on.[11] The apparent need for microeconomic disequilibrium is perhaps why theories of the firm have had vague empirical relevance. Whether this irrelevance result applies to other branches of economics and finance remains to be seen. In section 5 I argue it likely applies to macroeconomics, and that the search for microeconomic *equilibrium* foundations for macro may be quixotic, although micro-foundations in general remain a laudable goal.

2 Team Production and Team Formation

Holmström (1982) formally characterized the equilibria that obtain in team production. These results have been extended in various ways (e.g., Watts 2002). I model a group of agents engaged in team production, each agent contributing a variable amount of effort, leading to variable team output and team instability.[12]

Consider a finite set of agents, A, $|A| = n$, each of whom works with an effort level $e_{i \in A} \in [0, \omega_i]$. The total effort of the group is then $E \equiv \sum_{i \in A} e_i$. The group produces output, O, as a function of E, according to $O(E) = aE + bE^{\beta}$, $\beta > 1$. This represents the group's production function.[13] For $b > 0$ there are increasing returns to effort, while $b = 0$ amounts to constant returns.[14] Increasing returns in production means that agents working together can produce more than they can as individuals.[15] To see this, consider two agents having effort levels e_1 and e_2, with $\beta = 2$. As individuals they produce total output $O_1 + O_2 = a(e_1 + e_2) + b(e_1^2 + e_2^2)$, while working together they make $a(e_1 + e_2) + b(e_1 + e_2)^2$. Clearly this latter quantity is at least as large as the former since $(e_1 + e_2)^2 \geq e_1^2 + e_2^2$. As a compensation rule let us first consider agents sharing total output equally: at the

[11] This state of affairs is roughly comparable to a key finding of the original artificial agent financial market model (Arthur et al. 1997)—the configuration of the model that generated rational expectations equilibria is far from configurations that yield agent dynamics closely reproducing financial market data.

[12] The model is similar to Canning (1995), Huberman and Glance (1998) and Glance *et al.* (1997).

[13] While $O(E)$ relates inputs to outputs, like a standard production function, the inputs are not explicit choices of a decision-maker, since E results from autonomous agent actions. Thus, $O(E)$ cannot be made the subject of a math program, as in conventional production theory, although, it does describe production possibilities.

[14] Increasing returns at the firm level goes back at least to Marshall (1920) and was the basis of theoretical controversies in the 1920s (Sraffa 1926; Young 1928). Recent work on increasing returns is reprinted in Arthur (1994) and Buchanan and Yoon (1994). Colander and Landreth (1999) give a history of the idea.

[15] Increasing returns are justifiable via 'four hands' problems and other ways; these will not be pursued here.

end of each period all output is sold for unit price and each agent receives an O/N share of the total output.[16] Agents have Cobb-Douglas preferences for income and leisure.[17] All time not spent working is spent in leisure, so agent i's utility can be written as a function of its effort, e_i, and the effort of other agents, $E_{\sim i} \equiv E - e_i$ as

$$U_i\left(e_i;\theta_i,\omega_i,E_{\sim i},n\right)=\left(\frac{O\left(e_i;E_{\sim i}\right)}{n}\right)^{\theta_i}\left(\omega_i-e_i\right)^{1-\theta_i}. \qquad (1)$$

2.1 Equilibrium of the Team Formation Game

Consider the individual efforts of agents to be unobservable. From team output, O, each agent i determines E and, from its contribution to production, e_i, can figure out $E_{\sim i}$. Agent i then selects effort $e_i^*\left(\theta_i,\omega_i,E_{\sim i},n\right)=\arg\max_{e_i} U_i\left(e_i\right)$. For $\beta = 2$, in symbols, $e_i^*(\theta_i, \omega_i, E_{\sim i}) =$

$$\max\left[0,\frac{-a-2b\left(E_{\sim i}-\theta_i\omega_i\right)+\sqrt{a^2+4b\theta_i^2\left(\omega_i+E_{\sim i}\right)\left[a+b\left(\omega_i+E_{\sim i}\right)\right]}}{2b\left(1+\theta_i\right)}\right]. \qquad (2)$$

Note that e^* does not depend on n but does depend on $E_{\sim i}$—the effort put in by the other agents. To develop intuition for the general dependence of e_i^* on its parameters, we plot it for $a = b = \omega_i = 1$ in figure 1, as functions of $E_{\sim i}$ and θ_i.

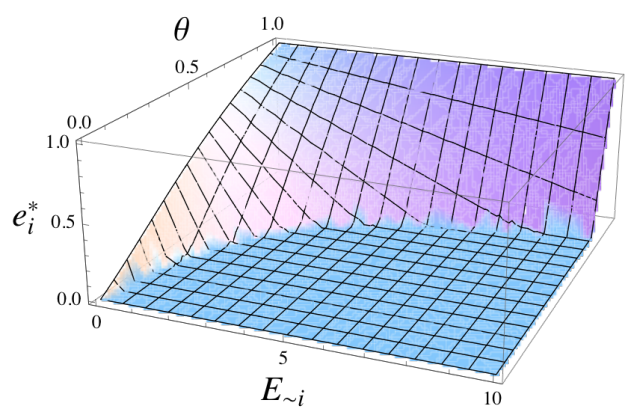


Figure 1: Dependence of e_i^* on $E_{\sim i}$ and θ for $a = b = \omega_i = 1$

The optimal effort level decreases monotonically as 'other agent effort,' $E_{\sim i}$, increases. For each θ_i there exists some $E_{\sim i}$ beyond which it is rational for agent i to

[16] The model yields roughly constant total output, so in a competitive market the price of output would be nearly constant. Since there are no fixed costs, agent shares sum to total cost, which equals total revenue. The shares can be thought of as either uniform wages in pure competition or equal profit shares in a partnership.

[17] In the appendix a more general model of preferences is specified, yielding qualitatively identical results.

put in no effort. In the case of constant returns, e_i^* decreases linearly with slope $\theta_i - 1$.

Equilibrium in a group corresponds to each agent working with effort e_i^* from equation 2, using $E_{\sim i}^*$ in place of $E_{\sim i}$ such that $E_{\sim i}^* = \sum_{j \neq i} e_j^*$. This leads to:

***Proposition 1*:** Nash equilibria exist in any group.

<u>*Proof*</u>: From the continuity of the *RHS*s of (2) and (3) and the convexity and compactness of the space of effort levels, a fixed point exists by the Brouwer theorem. Each fixed point is a Nash equilibrium, since once it is established no agent can make itself better off by working at some other effort level. □

Proposition 2: There exists a set of agent effort levels that Pareto dominate the Nash equilibrium, as well as a subset that are Pareto optimal. These solutions all (a) involve larger amounts of effort than the Nash equilibrium, and (b) are not individually rational.

<u>*Proof*</u>: To see (a) note that $dU_i\left(e_i^*;\theta_i,E_{\sim i}^*,n\right) = \frac{\partial U_i}{\partial e_i} de_i + \frac{\partial U_i}{\partial E_{\sim i}} dE_{\sim i} > 0$, since the first term on the *RHS* vanishes at the Nash equilibrium and

$$\frac{\partial U_i}{\partial E_{\sim i}} = \frac{\theta_i\left[a + 2b\left(e_i + E_{\sim i}\right)\right]\left(\omega_i - e_i\right)^{1-\theta_i}}{n^{\theta_i}\left[\left(e_i + E_{\sim i}\right)\left(a + b\left(e_i + E_{\sim i}\right)\right)\right]^{1-\theta_i}} > 0 .$$

For (b), each agent's utility is monotone increasing on the interval [0, e_i^*), and monotone decreasing on (e_i^*, ω_i]. Therefore, $\partial U_i / \partial e_i < 0 \forall e_i > e_i^*, E_{\sim i} > E_{\sim i}^*$. □ This effort region that Pareto-dominates Nash equilibrium is where firms live.

<u>*Example 1:*</u> Graphical depiction of the solution space, two identical agents

Consider two agents having $\theta = 0.5$ and $\omega = 1$. Solving (2) for e^* with $E_{\sim i} = e^*$ and $a = b = 1$ yields $e^* = 0.4215$, corresponding to utility level 0.6704. Effort deviations by either agent alone are Pareto dominated by the Nash equilibrium. For example, decreasing the first agent's effort to $e_1 = 0.4000$, with e_2 at the Nash level yields utility levels of 0.6700 and 0.6579, respectively. An effort increase to $e_1 = 0.4400$ with e_2 unchanged produces utility levels of 0.6701 and 0.6811, respectively, a loss for the first agent while the second gains. If both agents decrease their effort from the Nash level their utilities fall, while joint increases in effort are welfare-improving. There exist symmetric Pareto

optimal efforts of 0.6080 and utility of 0.7267. However, efforts exceeding Nash levels are not individually rational—each agent gains by putting in less effort.

Figure 2 plots iso-utility contours for these agents as a function of effort. The 'U' shaped lines are for the first agent, utility increasing upwards. The 'C' shaped curves refer to the second agent, utility growing to the right. The point labeled 'N' is the Nash equilibrium. The 'core' shaped region extending above and to the right of 'N' is the set of efforts that Pareto-dominate Nash. The set of efforts from 'P' to 'P' are Pareto optimal, with the subset from 'D' to 'D' being Nash dominant.

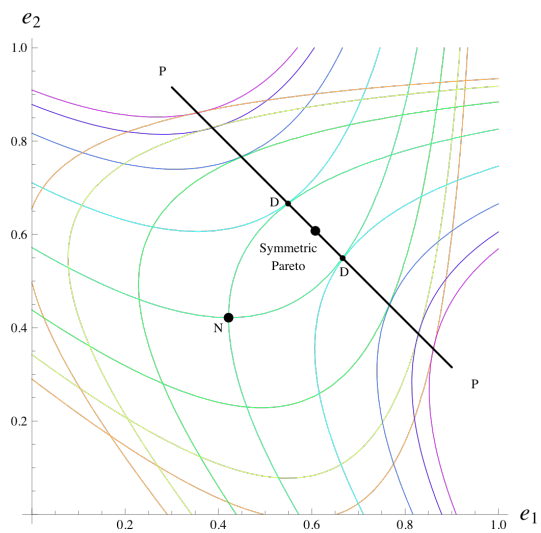


Figure 2: Effort level space for two agents with $\theta = 0.5$ and $a = b = \omega = 1$; colored lines are iso-utility contours, 'N' designates the Nash equilibrium, the heavy line from *P-P* are the Pareto optima, and the segment *D-D* represents the Pareto optima that dominate the Nash equilibrium

For two agents with distinct preferences the qualitative structure of the effort space shown in figure 2 is preserved, but the symmetry is lost. Increasing returns insures the existence of solutions that Pareto-dominate the Nash equilibrium. For more than two agents the Nash equilibrium and Pareto optimal efforts continue to be distinct.

Singleton Firms

The $E_{\sim i} = 0$ solution of (2) corresponds to agents working alone in single agent firms. For this case the expression for the optimal effort level is

$$e^*(\theta,\omega) = \frac{-a + 2b\theta\omega + \sqrt{a^2 + 4b\theta^2\omega(a + b\omega)}}{2b(1+\theta)}. \qquad (3)$$

In the limit of $\theta = 0$, (3) gives $e^* = 0$, while for $\theta = 1$ we have $e^* = \omega$. For $\theta \in (0, 1)$ it can be shown that the optimal effort is greater than for constant returns.

Example 2: Nash equilibrium in a team with free entry and exit

Four agents having θs of {0.6, 0.7, 0.8, 0.9} work together with $a = b = \omega_i = 1$. Equilibrium, from (2), has agents working with efforts {0.15, 0.45, 0.68, 0.86}, respectively, producing 6.74 units of output. The corresponding utilities are {1.28, 1.20, 1.21, 1.32}. If these agents worked alone they would, by (3), put in efforts {0.68, 0.77, 0.85, 0.93}, generating outputs of {1.14, 1.36, 1.58, 1.80} and total output of 6.07. Their utilities would be {0.69, 0.80, 0.98, 1.30}. Working together they put in less effort and receive greater reward. This is the essence of team production.

Now say a $\theta = 0.75$ agent joins the team. The four original members adjust their effort to {0.05, 0.39, 0.64, 0.84}—i.e., all work less—while total output rises to 8.41. Their utilities increase to {1.34, 1.24, 1.23, 1.33}. The new agent works with effort 0.52, receiving utility of 1.23. Joining is individually rational for this agent since its singleton utility is 0.88.

Imagine that another agent having θ of 0.75 joins the group. The new equilibrium efforts among the original 4 group members are {0.00, 0.33, 0.61, 0.83}, while the two newest (twin) agents each put in effort of 0.48. The total output rises to 10.09. The corresponding utilities are {1.37, 1.28, 1.26, 1.34} for the original agents and 1.26 for each of the twins. Overall, even though the new agent induces free riding, the net effect is a Pareto improvement.

Next, an agent with $\theta = 0.55$ (or less) joins. Such an agent will free ride and not affect the effort or output levels, so efforts of the extant group members will not change. However, since output must be shared with one additional agent, all utilities fall. For the 4 original agents these become {1.25, 1.15, 1.11, 1.17}. For the twins their utility falls to 1.12. The utility of the $\theta = 0.9$ agent is now below what it can get working alone (1.17 vs 1.30). Since agents may exit the group freely, it is rational for this agent to do so, causing readjustment to a new equilibrium: the three original agents work with efforts {0.10, 0.42, 0.66}, while the twins put in effort of 0.55 and the newest agent free rides. Output is 7.52, yielding utility of {1.10, 0.99, 0.96} for the original three, 0.97 for the twins, and 1.13 for the free rider. Unfortunately for the group, the $\theta = 0.8$ agent now can do better by working alone—utility of 0.98 versus 0.96, inducing further adjustments: the original two work with efforts 0.21 and 0.49, respectively, the twins put in effort of 0.61, and the $\theta = 0.55$ agent rises out of free-riding to work at the 0.04 level; output drops to 5.80. The utilities of

the originals are now 0.99 and 0.90, 0.88 for the twins, and 1.07 for the newest agent. Now the $\theta = 0.75$ agents are indifferent to staying or starting new singleton teams.

Homogeneous Groups

It is interesting to consider a group composed of agents of the same type (identical θ and ω). In a homogeneous group each agent works with the same effort in equilibrium, determined from (2) above by substituting (n-1) e_i^* for $E_{\sim i}$, and solving for e^*, yielding:

$$e^*(\theta,\omega,n) = \frac{2b\theta\omega n - a\left(\theta + n(1-\theta)\right) + \sqrt{4b\theta^2\omega n\left(a + b\omega n\right) + a^2\left(\theta + n(1-\theta)\right)^2}}{2bn\left(2\theta + n(1-\theta)\right)}. \tag{4}$$

These efforts are shown in figure 3a as a function of θ, with $a = b = \omega = 1$ and various n. Figure 3b plots the utilities for $\theta \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$ versus n.

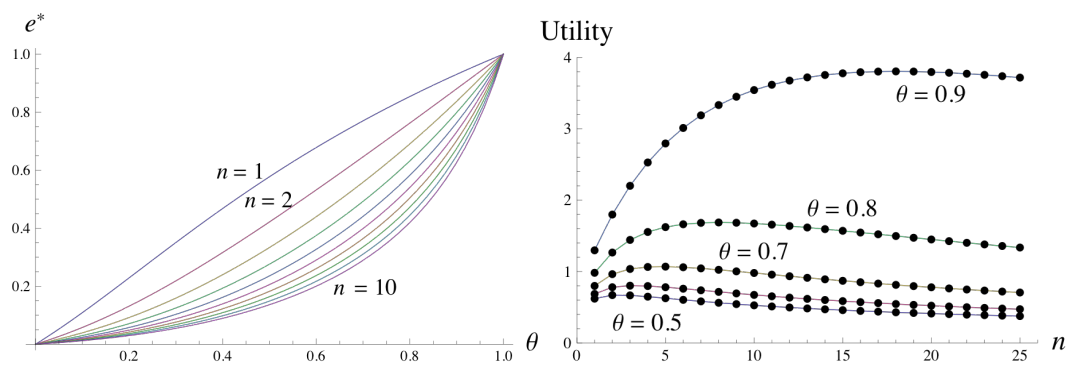


Figure 3: Optimal effort (a) and utility (b) in homogeneous groups as functions of θ and n, with $a = b = \omega = 1$

Note that each curve in figure 3b is single-peaked, so there is an optimal group size for every θ. This size is shown in figure 4a for two values of ω.

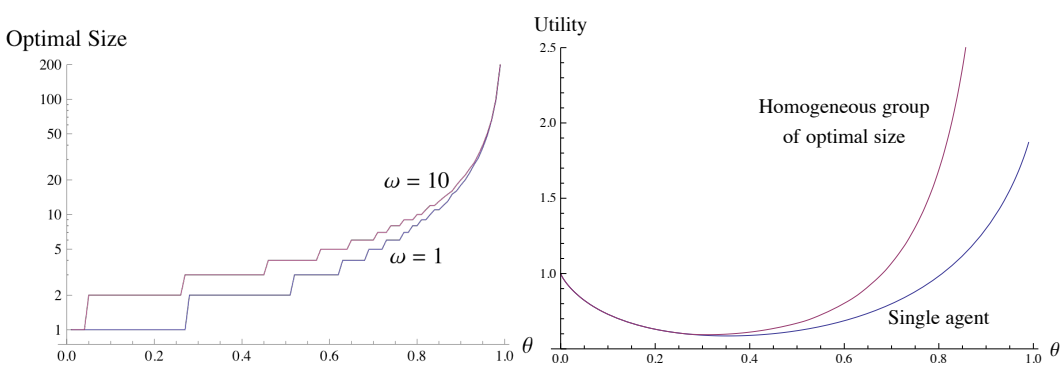


Figure 4: Optimal size (a) and utility (b) in homogeneous groups as functions of θ; $a = b = 1$; $\omega = 1, 10$

Optimal group sizes rise quickly with θ (note log scale). Utilities in such groups are shown in figure 4b. Gains from being in a team are greater for high θ agents.[18]

2.2 Stability of Nash Equilibrium and Dependence on Team Size

A unique Nash equilibrium always exists but for sufficiently large group size it is unstable. To see this, consider a team out of equilibrium, each agent adjusting its effort. As long as the adjustment functions are decreasing in $E_{\sim i}$ then one expects the Nash levels to obtain. Because aggregate effort is a linear combination of individual efforts, the adjustment dynamics can be conceived of in aggregate terms. In particular, the total effort level at time $t + 1$, $E(t+1)$, is a decreasing function of $E(t)$, as depicted notionally in figure 5 for a five agent firm, with the dependence of $E(t+1)$ on $E(t)$ shown as piecewise linear.

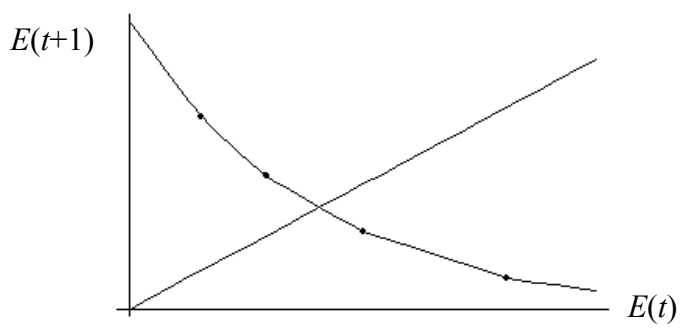


Figure 5: Phase space of effort level adjustment, $n = 5$

The intersection of this function with the 45° line is the equilibrium total effort. However, if the slope at the intersection is less than –1, the equilibrium will be unstable. Every team has a maximum stable size, dependent on agent θs.

Consider the n agent group in some state other than equilibrium at time t, described by the vector of effort levels, $e(t) = (e_1(t), e_2(t), ..., e_n(t))$. Now suppose that at $t + 1$ each agent adjusts its effort level using (2), a 'best reply' to the previous period's value of $E_{\sim i}$,[19]

[18] For analytical characterization of an equal share (partnership) model with perfect exclusionary power see Farrell and Scotchmer (1988); an extension to heterogeneous skills is given by Sherstyuk (1998).

[19] All effort adjustment functions yield qualitatively similar results when they are decreasing in $E_{\sim i}$ and increasing in θ_i; see appendix A. While this is a dynamic strategic environment, agents make no attempt to deduce optimal multi-period strategies. Rather, at each period they myopically 'best respond'. This simple behavior is sufficient to produce very complex dynamics, making anything like sub-game perfection unreasonable in such an environment..

$$e_i(t+1) = \max\left[0, \frac{-a - 2b\left(E_{\sim i}(t) - \theta_i\omega_i\right) + \sqrt{a^2 + 4b\theta_i^2\left(\omega_i + E_{\sim i}(t)\right)\left[a + b\left(\omega_i + E_{\sim i}(t)\right)\right]}}{2b\left(1+\theta_i\right)}\right].$$

Each agent adjusts its effort level, resulting an n-dimensional dynamical system, and:

Proposition 3: All teams are unstable for sufficiently large group size.

Proof: Stability is assessed from the eigenvalues of the Jacobian matrix:[20]

$$J_{ij} \equiv \frac{\partial e_i}{\partial e_j} = \frac{1}{1+\theta_i}\left\{-1 + \theta_i^2 \frac{a + 2b\left(\omega_i + E_{\sim i}^*\right)}{\sqrt{a^2 + 4b\theta_i^2\left(\omega_i + E_{\sim i}^*\right)\left[a + b\left(\omega_i + E_{\sim i}^*\right)\right]}}\right\}, \quad (5)$$

while $J_{ii} = 0$. Since each $\theta_i \in [0, 1]$ it can be shown that $J_{ij} \in [-1,0]$, and J_{ij} is monotone increasing with θ_i,. The *RHS* of (5) is independent of j, so each row of the Jacobian has the same value off the diagonal, i.e., $J_{ij} \equiv k_i$ for all $j \neq i$. Overall,

$$J = \begin{bmatrix} 0 & k_1 & \cdots & k_1 \\ k_2 & 0 & \cdots & k_2 \\ \vdots & & \ddots & \vdots \\ k_n & \cdots & k_n & 0 \end{bmatrix},$$

with each of the $k_i \leq 0$. Stability of equilibrium requires that this matrix's dominant eigenvalue, λ_0, have modulus strictly inside the unit circle. It will now be shown that this condition holds only for sufficiently small group sizes. Call ρ_i the row sum of the ith row of J. It is well-known (Luenberger 1979: 194-195) that $\min_i \rho_i \leq \lambda_0 \leq \max_i \rho_i$. Since the rows of J are comprised of identical entries

$$(n-1)\min_i k_i \leq \lambda_0 \leq (n-1)\max_i k_i. \quad (6)$$

Consider the upper bound: when the largest $k_i < 0$ there is some value of n beyond which $\lambda_0 < -1$ and the solution is unstable. Furthermore, since large k_i corresponds to agents with high θ_i, it is the most productive members of a group who determine its stability. From (6), compute the maximum stable group size, $N^{\max}$, by setting $\lambda_0 = -1$ and rearranging:

[20] Technically, agents who put in no effort do not contribute to the dynamics, so the effective dimension of the system will be strictly less than n when such agents are present.

$$n^{\max} \leq \left\lfloor \frac{\max_i k_i - 1}{\max_i k_i} \right\rfloor, \tag{7}$$

where $\lfloor z \rfloor$ refers to the largest integer less than or equal to z. Groups larger than $n^{\max}$ will never be stable, that is, (7) is an upper bound on group size. □

For either b or $E_{\sim i}$ » a, such as when $a \sim 0$, $k_i \approx (\theta_i - 1)/(\theta_i + 1)$. Using this together with (7) we obtain an expression for $n^{\max}$ in terms of preferences

$$n^{\max} \leq \left\lfloor \frac{2}{1 - \max_i \theta_i} \right\rfloor. \tag{8}$$

The agent with *highest* income preference thus determines the maximum stable group size. Other bounds on λ_0 can be obtained through the column sums of J. Noting the ith column sum by γ_i, we have $\min_i \gamma_i \leq \lambda_0 \leq \max_i \gamma_i$, which means that

$$\sum_{i=1}^{n} k_i - \min_i k_i \leq \lambda_0 \leq \sum_{i=1}^{n} k_i - \max_i k_i . \tag{9}$$

These bounds on λ_0 can be written in terms of the group size by substituting $n\,\bar{k}$ for the sums. Then an expression for $n^{\max}$ can be obtained by substituting $\lambda_0 = 1$ in the upper bound of (9) and solving for the maximum group size, yielding

$$n^{\max} \leq \left\lfloor \frac{\max_i k_i - 1}{\bar{k}} \right\rfloor. \tag{10}$$

The bounds given by (7) and (10) are the same (tight) for homogeneous groups.

<u>*Example 3*</u>*:* Onset of instability in a homogeneous group

Consider a group of agents having $\theta = 0.7$, with $a = b = \omega = 1$. From (8) the maximum stable group size is 6. Consider how instability arises as the group grows. For an agent working alone the optimal effort, from (3), is 0.770, utility is 0.799. Now imagine two agents working together. From (4) the Nash efforts are 0.646 and utility increases to 0.964. Each element of the Jacobian (5) is identical; call this k. For $n = 2$, $k = -0.188 = \lambda_0$. For $n = 3$ utility is higher, and $\lambda_0 = -0.368$. The same qualitative results hold for group sizes 4 and 5, with λ_0 approaching -1. At $n = 6$ efforts again decline but each agent's utility is lower. For $n = 7$ λ_0 is -1.082: the group is unstable—any perturbation of the Nash equilibrium creates dynamics that do not settle down. This is summarized in table 1.

n	*e**	*U(e*)*	*k*	$\lambda_0 = (n\text{-}1)k$
1	0.770	0.799	not applicable	not applicable
2	0.646	0.964	-0.188	-0.188
3	0.558	1.036	-0.184	-0.368
4	0.492	1.065	-0.182	-0.547
5	0.441	1.069	-0.181	-0.726
6	0.399	1.061	-0.181	-0.904
7	0.364	1.045	-0.180	-1.082

Table 1: Onset of instability in a group having $\theta = 0.7$; Nash eq. in groups larger than 6 are unstable

Groups of greater size are also unstable in this sense. For lesser θ instability occurs at smaller sizes, while groups having higher θ can support larger numbers.

These calculations are performed for all θ in figure 7. The maximum stable size is shown (green), with the smallest size at which instability occurs (red).

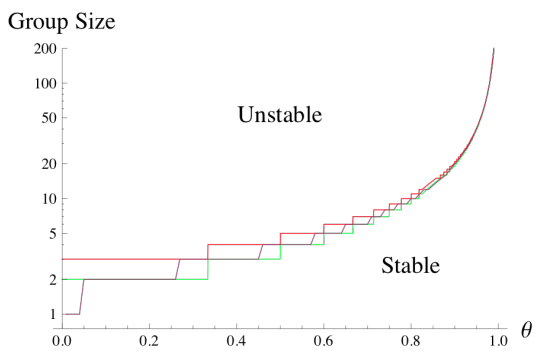


Figure 6: Unstable Nash equilibria in homogeneous groups having income parameter θ

The lower line (magenta) is the optimal group size (figure 4a), very near the stability boundary, meaning optimally-sized firms could be destabilized by the addition of a single agent. This is reminiscent of the 'edge of chaos' literature, for systems poised at the boundary between order and disorder (Levitan et al. 2002).

Unstable Equilibria and Pattern Formation Far From Equilibrium

Unstable equilibria may be viewed as problematical if one assumes agent level equilibria are *necessary* for social regularity.[21] But games in which optimal strategies are cycles have long been known (e.g., Shapley 1964; Shubik 1997).

[21] Osborne and Rubinstein (1994: 5) seem to suggest that any empirical regularity is necessarily an equilibrium. They cite Binmore (1987; 1988), who describes Simon's distinction between *substantive* and *procedural* rationality and admits that the former notion is a static one. He then distinguishes *eductive* and *evolutive* ways that players might arrive at equilibrium, claiming each is "a dynamic process by means of which equilibrium is achieved," (Binmore 1987: 184).

Solution concepts can be defined to include such possibilities (Gilboa and Matsui 1991). Agent level equilibria are *sufficient* for macro-regularity, but not *necessary*. When agents are learning or in combinatorially rich environments, as here, fixed points seem unlikely. Non-equilibrium models in economics include Papageorgiou and Smith (1983) and Krugman (1996).[22]

Firms are inherently dynamic. As they age, agent dynamics shift, some agents leave, new ones arrive, hard work and shirking coexist.[23] Indeed, there is vast turnover: of the largest 5000 U.S. firms in 1982, in excess of 65% of them no longer existed as independent entities by 1996 (Blair et al. 2000)! 'Turbulent' is apropos for such volatility (Beesley and Hamilton 1984; Ericson and Pakes 1995).

3 Computational Implementation with Software Agents

The motivation for a computational version of the model is simple. Since equilibria of the team formation game are unstable, what are its non-equilibrium dynamics? Do the dynamics contain firm formation patterns that are recognizable vis-a-vis actual firms? Such patterns can be difficult to discern analytically, leaving computational models as a practical way of studying them.[24] In what follows we find that such patterns *do* exist and they are closely related to data.

3.1 Set-Up of the Computational Model

In the analytical model above the focus is a single group. In the computational model many groups will form within the agent population. The computational set-up is just like the analytical model. Total output of a firm consists of both constant and increasing returns. Preferences, θ, are heterogeneous across agents. When agent i acts it searches over $[0, \omega_i]$ to find the effort that maximizes its next period utility. Each agent now has a social network consisting of v_i other agents, assigned randomly (Erdös-Renyi graph), and repeats this effort calculation for (a) starting

[22] Non-equilibrium models are better known and well-established in other sciences, e.g., in mathematical biology the instabilities of certain PDE systems are the basis for pattern formation (Murray 1993).

[23] Arguments against firm equilibrium include Kaldor (1972; 1985), Moss (1981) and Lazonick (1991).

[24] Turbulent flow involves transient phenomena on multiple length and time scales. Turbulence has resisted analysis despite the equations being well known. Today computational techniques are the tools of choice.

up a new firm in which it is the only agent, and (b) joining v_i other firms—i.e., it engages in a job search using its social network (Granovetter 1973; Montgomery 1991). The agent chooses the option that yields greatest utility. Since agents evaluate only a small number of firms their information is limited. We use 120 million agents, roughly the size of the U.S. private sector workforce. One period consists of about 5 million agents being activated, and corresponds to one calendar month, calibrated by job search frequency (Fallick and Fleischman 2001). Each agent starts working alone, thus120 million firms initially. The model's 'base case' is table 2.[25]

Model Attribute	Value
number of agents	120,000,000
constant returns coefficient, a	*Uniform* on [0, 1/2]
increasing returns coefficient, b	*Uniform* on [3/4, 5/4]
increasing returns exponent, β	*Uniform* on [3/2, 2]
distribution of preferences, θ	*Uniform* on (0, 1)
endowments, ω	1
compensation rule	equal shares
number of neighbors, v	*Uniform* on [2,6]
agent activations per period	4,800,000 or 4% of total agents
time calibration: one model period	one month of calendar time
initial condition	all agents in singleton firms

Table 2: 'Base case' configuration of the computational model

The model's execution can now be summarized in pseudo-code:

```
• INSTANTIATE agents and firms, INITIALIZE time, statistical objects;
• WHILE time < finalTime DO:
    ○ FOR each agent, activate it with 4% probability:
        ▪ Compute e* and U(e*) in current_firm;
        ▪ Compute e* and U(e*) for starting up a new firm;
        ▪ FOR each firm in the agent's social network:
            • Compute e* and U(e*);
        ▪ IF current_firm is not best choice, leave current
          firm;
            • If start-up firm is best: form start-up;
            • If another firm is best: join other firm;
    ○ FOR each firm:
        ▪ Sum agent inputs and then do production;
        ▪ Distribute output;
    ○ IF in stationary state collect monthly statistics;
    ○ INCREMENT time and reset periodic statistics;
• Collect final statistics.
```

[25] For model attributes with random values, each new agent or firm is given a realization having that specification.

Over time multi-agent firms form, grow and perish and there emerge stationary distributions of firm sizes, firm growth rates, firm ages, job tenure and so on. The essential feature of this model is that it is specified at the level of individuals, thus it is 'agent-based'. It is important to emphasize that it is *not* a numerical model: there are no (explicit) equations governing the macrosystem; the only equations present are for agent decision-making. "Solving" an agent model amounts to iterating it forward and observing the patterns produced at the individual and aggregate levels (cf. Axtell 2000).

3.2 Aggregate Dynamics

Agents work alone initially. As each is activated it discovers it can do better working with another agent to jointly produce output. Over time some groups expand as agents find it welfare-improving to join them, while others contract as their agents discover better opportunities elsewhere. New firms are born as discontented agents form start-ups. Overall, once an initial transient passes, an approximately stationary macrostate emerges.

Number of Firms and Average Firm Size

The number of firms varies over time, due both to firm entry—agents leaving extant firms for start-ups—and the demise of failing firms; figure 7 is typical.

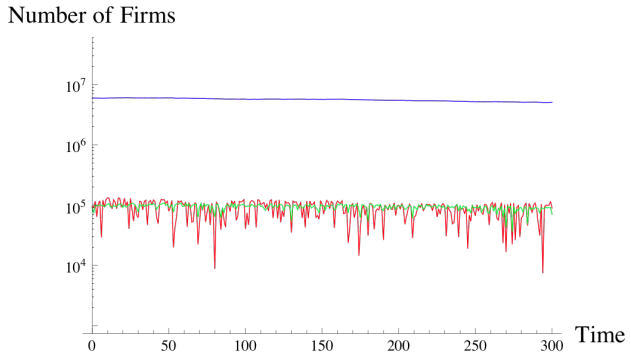


Figure 7: Typical monthly time series for the total number of firms (blue), new firms (green), and exiting firms (red) over 25 years (300 periods); note higher volatility in exits.

About 6 million firms in the U.S. have employees. A comparable number are shown in figure 7. There are about 100K startups with employees in the U.S. monthly (Fairlie 2012), quite close to the green line in figure 7. Note that there is more variability in firm exit than entry. Since the agents are fixed and the number

of firms is almost constant, average firm size does not vary much, as in figure 8.

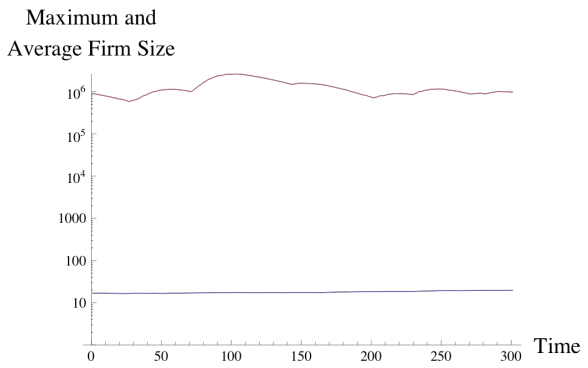


Figure 8: Typical time series for average firm size (blue) and maximum firm size (magenta)

The average firm in the U.S. has about 20 employees, as in figure 8. Also shown there is the size of the largest firm at each time, which fluctuates.

Typical Effort, Output, Income and Utility Levels

Agents who work together can improve upon their singleton utility levels with reduced effort. This is the essence of firms, as shown in figure 9.

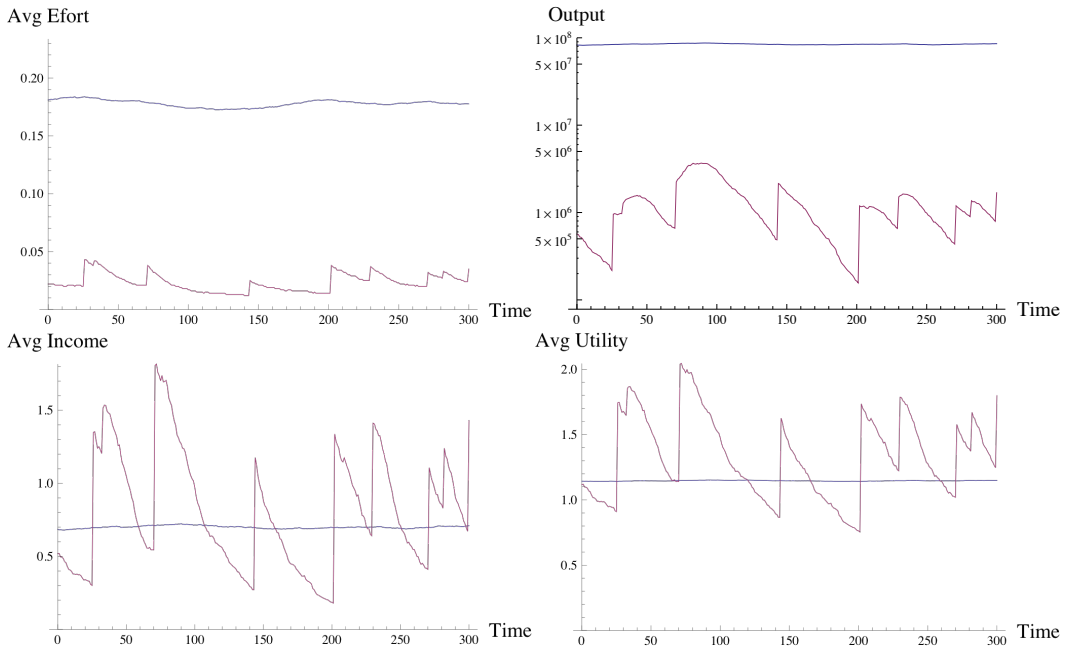


Figure 9: Typical time series for (a) average effort level in the population (blue) and in the largest firm (magenta), (b) total output (blue) and of the largest firm (magenta), (c) average income (blue) and income in the largest firm (magenta), and (d) average utility (blue) and in the largest firm (magenta)

While efforts in large firms fluctuate, the average effort level is quite stable (figure 9a). Much of the dynamism in the 'large firm' time series is due to the identity of

the largest firm changing. Figure 9b shows that overall output is quite stable over time (blue line) while output of the largest firm (red line) varies considerably. Figure 9c shows that the average income in the population overall (blue) is usually exceeded by that in the largest firm (red). Figure 9d shows the same is true of average utility.

Labor Flows

In real economies people change jobs with, what is to some, "astonishingly high" frequency (Hall 1999: 1151). Job-to-job switching, also known as employer-to-employer flow, represents 30-40% of labor turnover, substantially higher than unemployment flows (Davis et al. 1996; Fallick and Fleischman 2001; Davis et al. 2006; Faberman and Nagypál 2008; Nagypál 2008; Davis et al. 2012). Moving between jobs is the main agent decision in our model. In figure 10 the level of monthly job changing occurring in the run of the model described in figures 7-9 is shown in blue, along with measures of jobs created (red) and jobs destroyed (green). Job creation occurs in firms with net monthly hiring, while job destruction takes place when firms lose workers over a month. Note the higher volatility in the job destruction series.

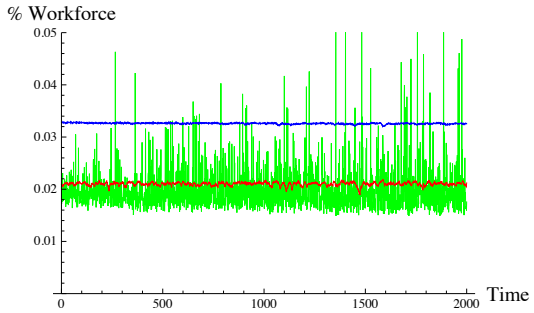


Figure 10: Typical time series for monthly job-to-job changes (blue), job creation (red) and destruction (green)

Overall, figures 7-10 develop intuition about typical dynamics of firm formation, growth and dissolution. They are a 'longitudinal' picture of typical micro-dynamics in the co-evolving populations of agents and firms. We now turn to cross-sectional properties.

3.3 Firms in Cross-Section: Sizes, Ages and Growth Rates

Firms clearly emerge in this model. When one runs it and watches individual firms form, grow, and die, the human eye immediately picks up the 'lumpiness' of the output, with a few big firms, more medium-sized ones, and lots and lots of small ones.[26]

Firm Sizes (by Employees and Output)

At any instant there exist distributions of firm sizes in the model. Since firms are of unit size at $t = 0$ there is a transient period over which firm sizes reach a stationary, skew configuration, with a few large firms and larger numbers of progressively smaller ones. Typical output from the model is shown in figure 11 for firm size measured two ways.

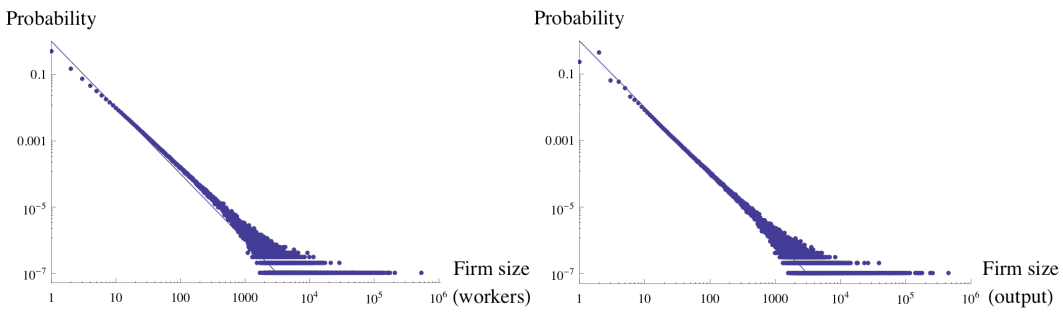


Figure 11: Stationary firm size distributions (probability mass functions) by (a) employees and (b) output

The modal firm size is 1 employee, the median is between 3 and 4, and the mean is 20. Empirical data on U.S. firms have comparable statistics. Specifically, for firm size S, the complementary cumulative Pareto distribution function, $F_S^C(s)$ is

$$\Pr\left(S \geq s_i\right) \equiv F_S^C\left(s;\alpha,s_0\right) = \left(\frac{s_0}{s}\right)^{\alpha}, s \geq s_0, \alpha > 0. \tag{11}$$

where s_0 is the minimum size, unity for size measured by employees. The U.S. data are well fit by $\alpha \approx$ -1.06 (Axtell 2001), the line in figure 11a. The Pareto is a power law, and for $\alpha = 1$ is known as Zipf's law. A variety of explanations for power laws have been put forward.[27] Common to these theories is the idea that systems

[26] Movies areavailable at www.css.gmu.edu/rob/research.html.

[27] For instance, Bak (1996: 62-64), Marsili and Zhang (1998), Gabaix, (1999), Reed (2001), and Saichev *et al*. (2010).

described by (11) are far from (static) equilibrium at the microscopic level. Our model is non-equilibrium at the agent level with agents regularly changing jobs. Note that power laws well fit the *entire distribution* of firm sizes. Simon (1977) argued that such highly skew distributions are so odd as to constitute *extreme* hypotheses. That our simple model reproduces this peculiar distribution is strong evidence it captures some essence of basic firm dynamics.[28]

Labor Productivity

Firm output per employee is productivity. Figure 12 is a plot of average firm output as a function of firm size. Fitting a line by several distinct methods indicates that output scales linearly with size, implying constant returns to scale.

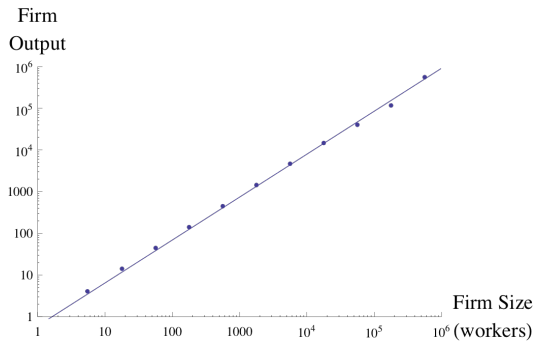


Figure 12: Essentially constant returns at the aggregate level, despite increasing returns at the micro-level

Approximately constant returns is also a feature of the U.S. output data; see Basu and Fernald (1997). That *constant returns* occur at the *aggregate* level occurs despite *increasing returns* at the *micro*-level suggests the difficulties of making any inferences across levels. An explanation of why this occurs is apparent. As the increasing returns-induced advantages that accrue to a firm with size are consumed by free riding behavior, agents migrate to more productive firms. Each agent who changes jobs acts to 'arbitrage' the returns across firms. Since output per worker represents wages in our simple model it is clear there is no wage-size effect here (Brown and Medoff 1989), a phenomenon that seems to be fading in the real world (Even and Macpherson 2012).

[28] At the very least it is preferable to models of identical firms (e.g., Robin 2011) or unit size firms (e.g., Shimer 2005).

While average labor productivity is constant across firms, there is substantial variation in productivity, as given by the distribution in figure 13.

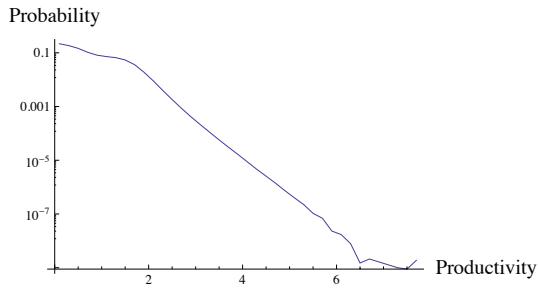


Figure 13: Labor productivity distribution

Average productivity is about 0.7 with a standard deviation of 0.6, but clearly there are some firms with extreme productivities. In the semilog coordinates of figure 13 labor productivities are approximately exponentially-distributed, at least the larger ones, not Pareto-distributed as has become a fashionable specification among theorists (Helpman 2006). Interestingly, small and large firms have about the same productivity distribution.

Firm Ages

Using data from the BLS Business Employment Dynamics program, figure 14 gives the age distribution (*pmf*) of U.S. firms, in semi-log coordinates, with each colored line representing the distribution in a recent year. Model output is overlaid on the raw data as points and agrees reasonably well. While the exponential distribution (Coad 2010) is a rough approximation, the curvature (i.e., the departure from exponential) is important, indicating that failure probability is a function of age.

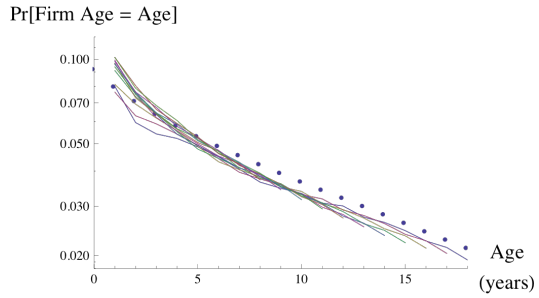


Figure 14: Firm age distributions (*pmf*s), U.S. data 2000-2011 (lines) and model output (points); source: BLS (www.bls.gov/bdm/us_age_naics_00_table5.txt) and author calculations

In these figures average firm lifetime ranges from 16-18 years, which is also the

approximate standard deviation.

Joint Distribution of Firms by Size and Age

With unconditional size and age distributions now analyzed, their joint distribution is shown in figure 15, a normalized histogram in *log* probabilities.

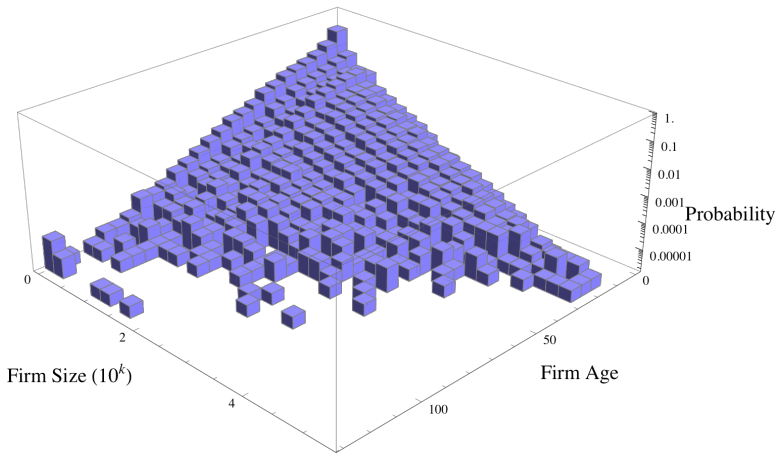
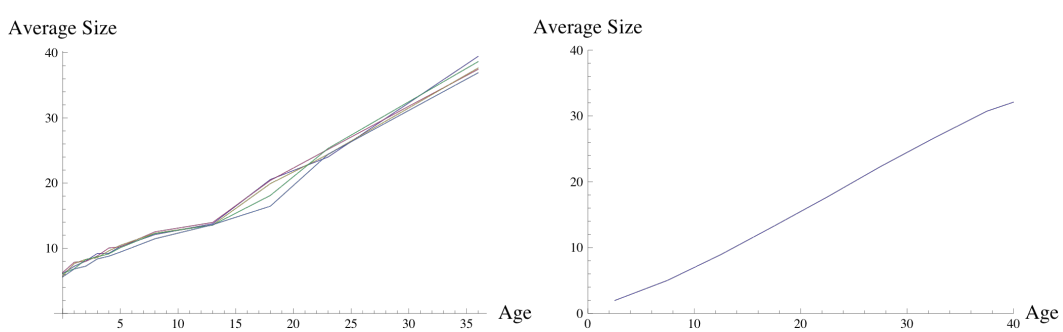


Figure 15: Histogram of the steady-state distribution of firms by *log*(size) and age

Note that log probabilities decline approximately linearly as a function of age and *log* firm size. From the BLS data one can determine average firm size conditional on firm age. In figure 16a these data are plotted for five recent years, starting with 2005, each year its own line. To first order there is a linear relation between firm size and age: firms that are 10 years old have slightly more than 10 employees on average, firms 20 years old have 20 employees, 30 year old firms have roughly 30 employees, and so on. There must be a cut-off beyond some age but the data are censored for large ages. From the model we get approximately the same linear effect but a slightly different intercept, figure 16b.



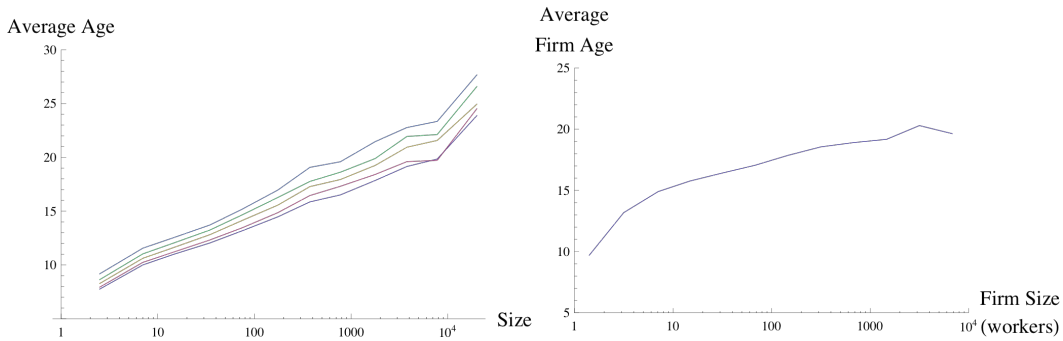


Figure 16: (a) Average firm size by age bins in (a) the U.S. for 2005-2009 and (b) the model; average firm age by size bins in (c) the U.S. and (e) the model; source: BLS and author calculations

The conditional in the other direction—the dependence of average age on firm size—is shown in figure 16c in semilog coordinates. To first order, average age increases linearly with log size: firms with 10 employees are on average 10 years old, firms with 100 employees average nearly 15 years of age, and firms with 1000 employees are roughly 20 years old, on average. The model yields a similar result, figure 16d: linearly increasing age with *log* size.

Firm Survival Rates

If firm ages were exactly exponentially distributed then the survival probability would be constant, and independent of age (Barlow and Proschan 1965). The departures from exponential in figure 14 indicate that survival probability does depend on age. Empirically it is well-known that survival probability *increases* with age (Evans 1987; Hall 1987). In figure 17 firm survival probabilities over recent years are shown for U.S. companies (colored lines) with points representing model output.

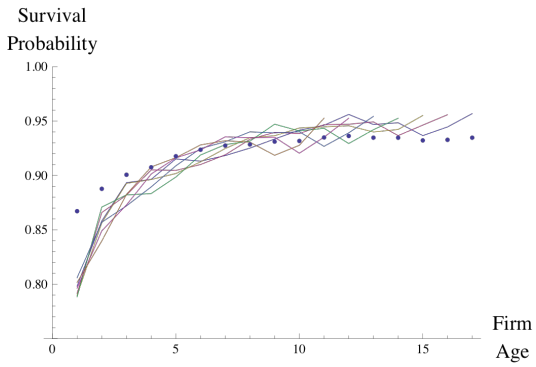


Figure 17: Firm survival probability increases with firm age, U.S. data 1994-2000 (lines) and model (points), and firm size; source: BLS and author calculations

Firm survival rates also rise with firm size in both the U.S. data and the model.

Firm Growth Rates

Calling a firm's size at time t, S_t, a common specification of firm growth rate is $G_{t+1} \equiv S_{t+1}/S_t$. This raw growth rate has support on $\boldsymbol{R}_+$ and is right skew, since there is no upper limit to how much a firm can grow yet it cannot shrink by more than its current size. The quantity $g_{t+1} \equiv ln(G_{t+1})$ has support on $\boldsymbol{R}$ and tends to be roughly symmetric. Gibrat's (1931) proportional growth model—all firms have the same growth rate distribution—implies that G_t is lognormally distributed (e.g., Sutton 1997), meaning g_t is Gaussian. In the basic proportional growth model these distributions are not stationary as their variance grows with time. Adding firm birth and death processes can lead to stationary firm size distributions (see de Wit (2005) for a review).

Gaussian specifications for g were common in IO for many years (e.g., Hart and Prais 1956; Hymer and Pashigian 1962), often based on small samples of firms. Stanley *et al*. (1996) reported that data on g for all publicly-traded U.S. manufacturing firms (Compustat) were well-fit by the Laplace (double exponential) distribution, which is heavy-tailed in comparison to the Gaussian.[29] Subsequently, growth rate data for European pharmaceuticals (Bottazzi et al. 2001), Italian and French manufacturers (Bottazzi et al. 2007; Bottazzi et al. 2011), and all U.S. establishments (Teitelbaum and Axtell 2005) were shown to be Laplacian. Schwarzkopf (2011) argues that g is stable.

Theoretical models of Laplace and stably-distributed firm growth rates are based on departures from the standard central limit theorem (Bottazzi and Secchi 2006). When the number of summands is geometrically distributed the Laplace distribution results (Kotz et al. 2001) while heavier-tails yield stable laws (Schwarzkopf 2010).

Empirically, the so-called Subbotin or exponential power distribution is useful as it embeds both the Laplace and Gaussian distributions. Its *pdf* has the form

[29] For g Laplace-distributed, G follows the log-Laplace distribution, a kind of double-sided Pareto distribution (Reed 2001), technically a combination of the power function distribution on (0, 1) and the Pareto on (1, ∞).

$$\frac{\eta}{2\sigma_g \Gamma(1/\eta)} \exp\left[-\left(\frac{g - \bar{g}}{\sigma_g} \right)^{\eta} \right], \tag{12}$$

where $\bar{g}$ is the average log growth rate, σ_g is proportional to the standard deviation, and η is a parameter; $\eta = 2$ is the normal distribution, $\eta = 1$ the Laplace. Semilog plots of (12) vs g yield distinctive 'tent-shaped' figures for $\eta \approx 1$, parabolas for $\eta = 2$. Empirical estimates often yield $\eta \leq 1$ (Perline et al. 2006; Bottazzi et al. 2011), thus even more non-Gaussian than the Laplace.[30] Overall, g has several empirical characteristics:

1. Typically, there is more variance for negative g, i.e., firm decline, corresponding to more variability in job destruction than job creation (Davis et al. 1996), requiring an asymmetric Subbotin distribution (Perline et al. 2006).
2. While Mansfield (1962), Birch (1981), Evans (1987) and Hall (1987) all demonstrate that average growth declines with firm size, or at least is positive for small firms and negative for large firms, there is evidence this an artifact of the specification of g (Haltiwanger et al. 2011; Dixon and Rollin 2012).
3. Mansfield (1962), Evans (1987), Hall (1987) and Stanley *et al*. (1996) all show that growth rate variance declines with firm size, on average in the first three cases, for the full distribution in the latter. This is significant insofar as it vitiates Gibrat's simple growth rate specification: all firms are *not* subject to the same growth rate distribution, as large firms face significantly less variable growth.
4. Average growth falls with age (Haltiwanger et al. 2008; Haltiwanger et al. 2011).
5. Over longer time periods g tends to become more normal (Perline et al. 2006), i.e., η increases with the duration over which firm growth is

[30] An alternative definition of G is $2(S_{t+1} - S_t)/(S_t + S_{t+1})$, making $G \in [-2, 2]$ (Davis et al. 1996). Although advantageous because it keeps exiting and entering firms in datasets for one additional period, it is objectionable on the grounds that it muddies the water in distinguishing Laplace from normally-distributed growth rates.

measured.

With this as background, figure 18 shows distributions of g emanating from the model for seven classes of firm sizes, with larger firms nested inside the more numerous small ones.

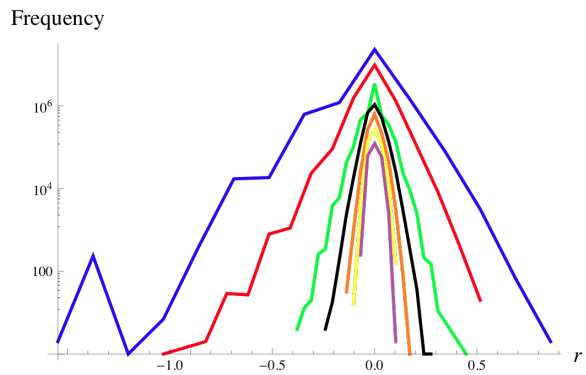


Figure 18: Distributions of g annually, as a function of firm size, from the model; sizes 8-15 (blue), 16-31 (red), 32-63 (green), 64-127 (black), 128-255 (orange), 256-511 (yellow), and 512-1023 (purple)

Overall, $\overline{g}$ is very close to 0.0 (no growth) and figure 19a shows its dependence on size (blue). The red line is an alternative definition of G (see footnote 30).

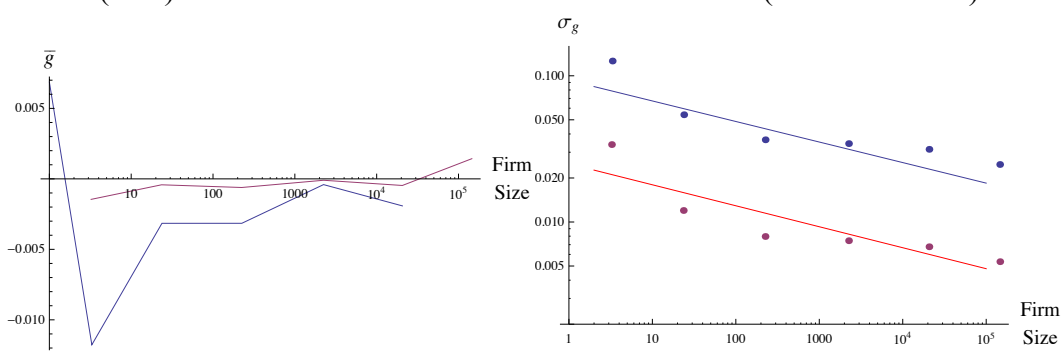


Figure 19: Dependence of the (a) mean and (b) standard deviation of g on firm size, in agreement with Dixon and Rollin [2012] for (a) and Stanley *et al*. [1996] for (b)

The variability of g clearly declines with firm size in figure 18, and figure 19b shows how, with the colors corresponding to those in figure 19a. Stanley *et al*. (1996) find that the standard deviation in g decreases with size like $s^{-\tau}$, and estimate $\tau = 0.16 \pm 0.03$ for size based on employees (data from Compustat manufacturing firms) while we get $\tau = 0.14 \pm 0.02$, the blue and red lines. A value of $\tau = 0.5$ would mean the central limit theorem applies but clearly this is not the case. If $\tau = 0$ all firms would be perfectly correlated and variability would not be a function of size. Several explanations for this dependence have been proposed

(Buldyrev et al. 1997; Amaral et al. 1998; Sutton 2002; Wyart and Bouchaud 2002; Klette and Kortum 2004; Fu et al. 2005; Luttmer 2007; Riccaboni et al. 2008), none particularly relevant to the set-up of the present model.

Firm growth rates decline with age, as mentioned above. Figure 20 shows model output as a smoothed histogram. The insets depict $\overline{g}$ and standard deviation of g vs. age.

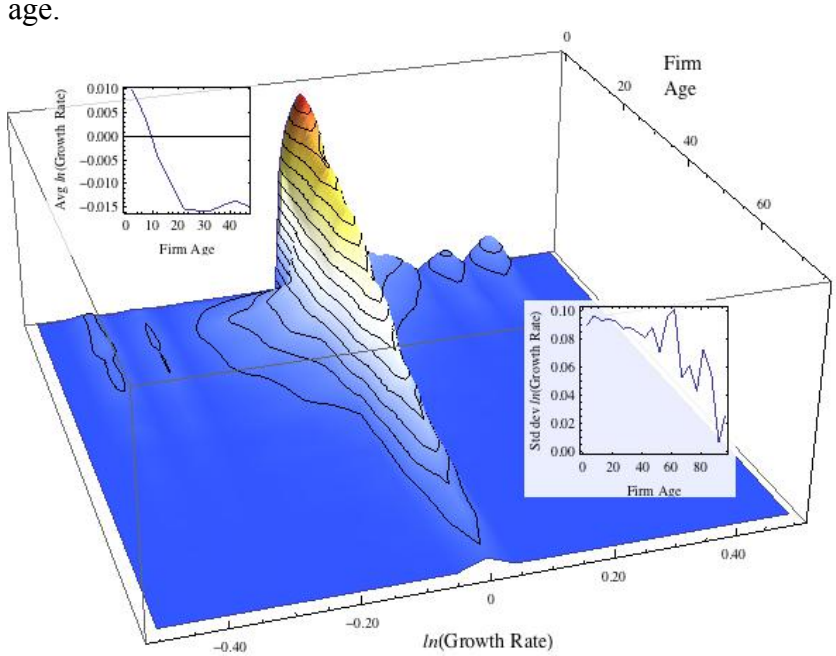


Figure 20: Smoothed histogram of firm growth rates as a function of firm age; the dependence of the mean and standard deviation of g on firm age are shown in the two insets

Having explored firms cross-sectionally, we next turn to agents.

3.4 Agents in Cross-Section: Income, Job Tenure, and Employment

In this section agent behavior in the aggregate steady-state is quantified.

Income Distribution

While income and wealth are famously heavy-tailed (Pareto 1971 [1927]) *wages* are less so. A recent empirical examination of U.S. adjusted gross incomes—primarily salaries, wages and tips—argues that below about $125K the data are well-described by an exponential distribution, while a power law better fits the upper tail (Yakovenko and Rosser 2009). In figure 21a the income distribution from the model is shown.

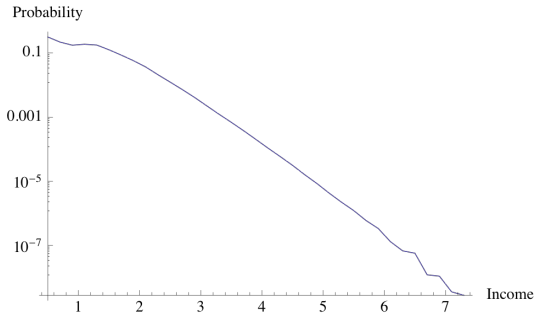


Figure 21: Income distribution (arbitrary units)

Since incomes are nearly linear in semi-log coordinates, they are approximately exponentially-distributed. Although there is not room to analyze these data further, it is the case that incomes increase rapidly with endowments, ω, slowly with preferences for income, θ, and are independent of firm size and age.

Job Tenure Distribution

Job tenure in the U.S. has a median of just over 4 years and a mean of about 8.5 years (BLS Job Tenure 2010). The counter-cumulative distribution for 2010 is shown in figure 22a (points) with the straight line being the estimated exponential distribution. The model-generated job tenure counter-cumulative distribution is shown in figure 22b.

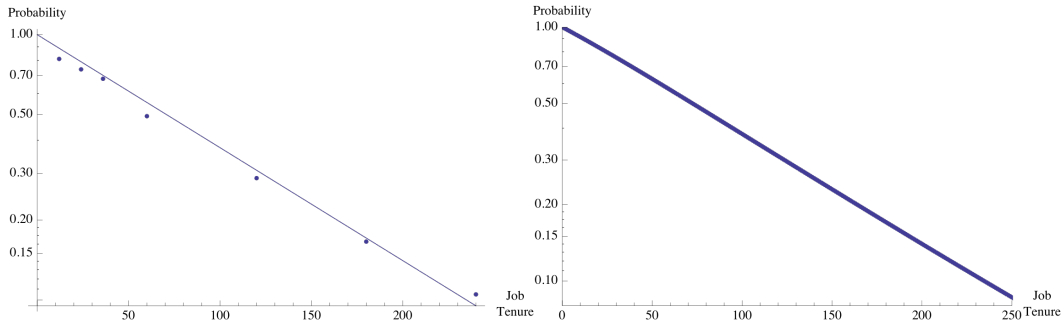


Figure 22: Job tenure (months) is exponentially-distributed (a) in the U.S. and (b) in the model; source: BLS and author calculations

The base case of the model is calibrated to make these distributions coincide. That is, the number of agent activations per period is specified in order to bring these two figures into agreement, thus defining the meaning of one unit of time in the model, here a month. The many other dimensions of the model having to do with time—firm growth rates, ages, and so on—derive from this basic calibration.

Employment as a Function of Firm Size and Age

Because the model's firm size distribution by employees is approximately right (figure 11a), it is also the case that employment as a function of firm size also comes out about right. But the dependence of employment on firm age is not directly available from analytical manipulations without making certain distributional assumptions. In figure 23 we count the number of employees in firms as a function of age. About half of American private sector workers are in firms younger than 28 years of age. The first panel are the U.S. data, available online via BLS BDM, shown as a counter-cumulative distribution of employment by firm age, while the second is the same plot using output from the model.

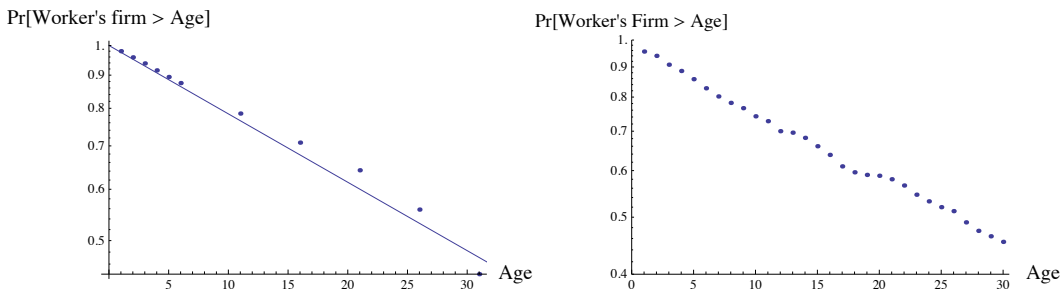


Figure 23: Employment by firm age in years: (a) U.S. data and (b) model output; source: BLS

These two panels show broad agreement between the model and the data on this issue.

3.5 Agent Welfare in Endogenous Firms

Each time an agent is activated it seeks higher utility, which is bounded from below by the singleton utility. Therefore, it must be the case that all agents prefer the non-equilibrium state to one in which each is working alone—the state of all firms being size one is Pareto-dominated by the dynamical configurations studied above.

To analyze welfare of agents, consider homogeneous groups of maximum stable size. Associated with such groups are the utility levels shown in figure 4b above. Figure 24 starts out as a recapitulation of figure 4b: a plot of the optimal utility for both singleton firms as well as optimal size homogeneous ones, as a function of θ. Overlaid on these smooth curves is the cross-section of utilities in realized groups.

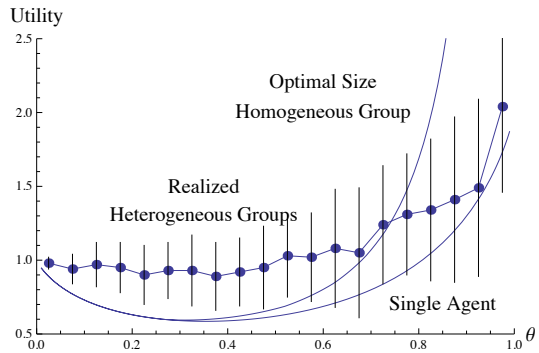


Figure 24: Utility in single agent firms, in optimal homogeneous firms, and realized firms, by θ

The main result here is that most agents prefer the non-equilibrium world to the equilibrium outcome with homogeneous groups.

4 Robustness of the Results

In this section the base model of table 2 is varied and the effects described. One specification found to have no effect on the model in the long run is the initial condition. Starting the agents in groups seems to modify only the duration of the initial transient. The main lesson of this section is that, while certain behavioral and other features can be added to this model and the empirical character of the results preserved, relaxation of any of the basic structural specifications of the model, individually, is sufficient to break its connection to data.

4.1 The Importance of Purposive Behavior

Against this simple model it is possible to mount the following critique. Since certain stochastic growth processes are known to yield power law distributions, perhaps the model described above is simply a complicated way to generate stochasticity. That is, although the agents are behaving purposively, this may be just noise at the macro level. What if agent behavior were truly random, would this too yield power law firm sizes? We have investigated this in two ways. First, imagine that agents randomly select whether to stay in their current firm, leave for another firm, or start-up a new firm, while still picking an optimal effort where they end up. It turns out that this specification yields only small firms, under size 10. Second, if agents select the best firm to work in but then choose an effort level

at random, again nothing like skew size distributions arise. These results suggest that any systematic departure from (locally) purposive behavior is unrealistic.

4.2 Effect of Population Size

While the base case of the model has been realized for 120 million agents, it has often been run with fewer agents. Figure 25 gives the largest firm realized vs. population size.

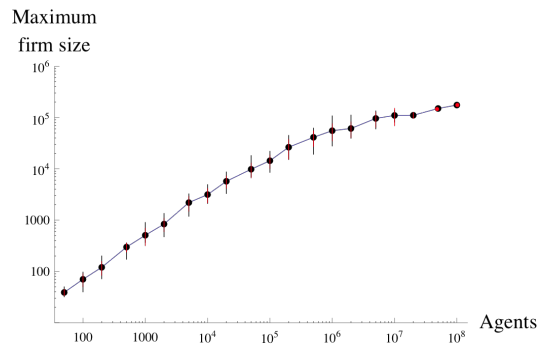


Figure 25: Largest firm size realized as a function of the number of agents

The maximum firm size rises sub-linearly with the size of the population.

4.3 Effect of the Agent Activation Rule and Rate

While it is well-known that *synchronous* activation can produce anomalous output (Huberman and Glance 1993), for the *asynchronous* activation model there can be subtle effects based on whether agents are activated randomly or uniformly (Axtell et al. 1996). The same effect has been found here but it primarily affects firm growth rates (Axtell 2001).

4.4 Effect of the Production Parameters

Of the three parameters that specify the production function, a, b and β, as increasing returns are made stronger, larger firms are realized and average firm size increases. For $\beta > 2$, very large firms arise; these are 'too big' empirically.[31]

4.5 Alternative Specifications of Agent Characteristics

Preferences are distributed uniformly on (0,1) in the base case. This yields a certain number of agents having extreme preferences: those with $\theta \approx 0$ are leisure lovers and those with $\theta \approx 1$ love income. Other distributions (e.g., beta, triangular)

[31] The model can occasionally 'run away' to a single large firm for β in this range.

were investigated and found to change the results very little. Removing agents with extreme preferences from the population can modify the main findings quantitatively. If agent prferences are too homogeneous the model output is qualitatively different from the empirical data. Finally, CES preferences do not alter the general character of the results. Overall, the model is insensitive to preferences as long as they are sufficiently heterogeneous.

4.6 Effect of the Extent and Composition of Social Networks

In the base case each agent has 2 to 4 friends. This number is a measure of the size of an agent's search or information space, since the agent queries these other agents when active to assess the feasibility of joining their firms. The main qualitative impact of increasing the number of friends is to slow model execution.

However, when agents query *firms* for jobs something new happens. Picking an agent to talk to may lead to working at a big firm. But picking a firm at random almost always leads to small firms and empirically-irrelevant model output.

4.7 Bounded Rationality: Groping for Better Effort Levels

So far, agents have adjusted their effort levels to anywhere within the feasible range [0, ω]. A different behavioral model involves agents making only small changes from their current effort level each time they are activated. Think of this as a kind of prevailing *work ethic* within the group or *individual habit* that constrains the agents to keep doing what they have been, with small changes.

Experiments have been conducted for each agent searching over a range of 0.10 around its current effort level: an agent working with effort e_i picks its new effort from the range $[e_L, e_H]$, where $e_L = \max(0, e_i - 0.05)$ and $e_H = \min(e_i + 0.05, 1)$. This slows down the dynamics somewhat, yielding larger firms. This is because as large firms tend toward non-cooperation, sticky effort adjustment dampens the downhill spiral to free riding. I have also experimented with agents who 'grope' for welfare gains by randomly perturbing current effort levels.

4.8 Effect of Agent Loyalty to Its Firm

In the basic model an agent moves immediately to a new firm when doing so

makes it better off. Behaviorally, this seems implausible. The idea of agent loyalty involves agents not changing jobs right away even when it is ostensibly better to do so.[32] Imagine an agent counting how many times it should have moved but did not. Only when its count exceeds a parameter, μ, does it move to a new firm and reset its counter. Setting $\mu = 0$ corresponds to the base model. Increasing μ produces larger, longer-lived firms. That is, loyalty is a stabilizing factor, even when μ is heterogeneous in the population.

4.9 Hiring

One aspect of the base model is very unrealistic: that agents can join whatever firms they want, as if there is no barrier to getting hired by any firm. The model can be made more realistic by instituting local hiring policies. It turns out that such policies have little effect at the aggregate level.

Let us say that one agent in each firm does all hiring, say the agent who founded the firm, or the agent with the most seniority. We will call this agent the 'boss' or 'residual claimant'. A simple hiring policy has the boss compare current productivity to what would be generated by the addition of a new worker, *assuming that no agents adjust their effort levels*. The boss computes the minimum effort, $\phi E/n$, for a new hire to raise productivity as a function of a, b, β, E and n, where ϕ is a fraction of average effort:

$$\frac{aE + bE^{\beta}}{n} < \frac{a\left(E + \phi\frac{E}{n}\right) + b\left(E + \phi\frac{E}{n}\right)^{\beta}}{n+1} = \frac{aE\left(1 + \frac{\phi}{n}\right) + bE^{\beta}\left(1 + \frac{\phi}{n}\right)^{\beta}}{n+1}. \qquad (13)$$

For $\beta = 2$ this can be solved explicitly for the minimum ϕ necessary

$$\phi_* = \frac{-n(a + 2bE) + \sqrt{n^2(a + 2bE)^2 + 4bEn(a + bE)}}{2bE}.$$

For all values of ϕ_* exceeding this level it makes sense to hire the prospective worker. For the case of $a = 0$, (13) can be solved for any value of β: $\phi_* = n\left(\frac{n+1}{n}\right)^{1/\beta} - n$; this is independent of b and E. Numerical values for ϕ_* as a

[32] Loyalty is a prominent feature in Tesfatsion's (1998) models of labor markets, as well as in Kirman and Vriend (2000; 2001), where loyalty between buyers and sellers emerges in bilateral exchange markets.

function of β and n are show in Table 3.

$n\backslash\beta$	1.0	1.5	2.0	2.5
1	1.0	0.59	0.41	0.32
2	1.0	0.62	0.45	0.35
5	1.0	0.65	0.48	0.38
10	1.0	0.66	0.49	0.39
100	1.0	0.67	0.50	0.40

Table 3: Dependence of the minimum fraction of average effort on firm size and increasing returns

As n increases for a given β, ϕ_* increases. In the limit of large n, ϕ_* equals $1/\beta$. So with sufficient increasing returns the boss will hire just about any agent who wants a job! These results can be generalized to hiring multiple workers at a time.

Adding this functionality to the computational model changes the behavior of individual firms and the life trajectories of individual agents but does not substantially alter the overall macrostatistics of the artificial economy.

4.10 Effort Monitoring, Job Termination and Unemployment

In the base model, shirking goes completely undetected and unpunished. Effort level monitoring is important in real firms, and a large literature has grown up studying it; see Olson (1965), the models of mutual monitoring of Varian (1990), Bowles and Gintis (1998), and Dong and Dow (1993), the effect of free exit (Dong and Dow 1993), and endowment effects (Legros and Newman 1996); Ostrom (1990) describes mutual monitoring in institutions of self-governance.

It is possible to *perfectly* monitor workers in our model and fire the shirkers, but this breaks the model by pushing it toward static equilibrium. All real firms suffer from imperfect monitoring. Indeed, many real-world compensation systems can be interpreted as ways to manage incentive problems by substituting reward for supervision, from efficiency wages to profit-sharing (Bowles and Gintis 1996). Indeed, if incentive problems in team production were perfectly handled by monitoring there would be no need for corporate law (Blair and Stout 1999).

At any instant of time, some firms are growing and others are declining. However, growing firms shed workers and declining firms do some hiring. In figure 26 the left panel represents empirical data on the U.S. economy (Davis et al. 2006), and shows that growing firms have to hire in excess of the separations they

suffer, while declining firms keep hiring even when separations are the norm.

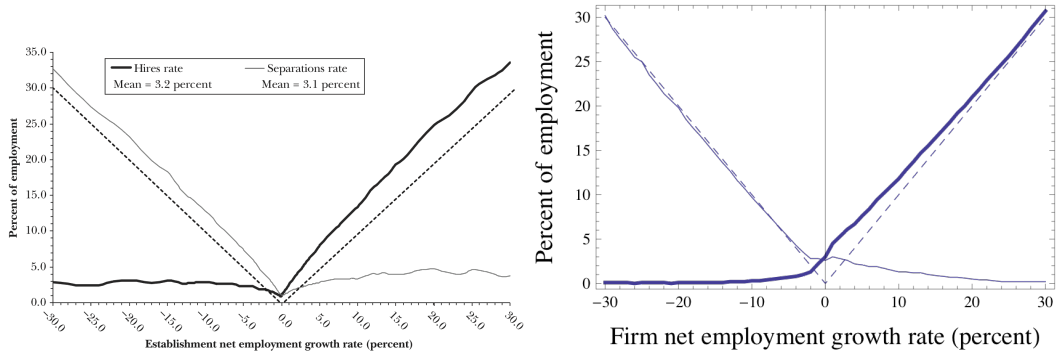


Figure 26: Labor transitions as a function of firm growth rate

In the right panel are data from my model, and clearly firms can both gain and lose workers. Note that the 'hiring' line in the two figures looks comparable, but the 'separations' line is different, with too few separations in the model.

To introduce involuntary separations, imagine the residual claimant knows the effort of each agent and can thus determine if the firm would be better off if the least hard working one were let go. Opposite calculations of those for hiring yield:

$$\frac{aE+bE^{\beta}}{n} < \frac{a\left(E-\phi\frac{E}{n}\right)+b\left(E-\phi\frac{E}{n}\right)^{\beta}}{n-1} = \frac{aE\left(1-\frac{\phi}{n}\right)+bE^{\beta}\left(1-\frac{\phi}{n}\right)^{\beta}}{n-1}$$

Introducing this logic into the code there results unemployment: agents are terminated and do not immediately find another firm to join. Computational experiments with terminations and unemployment have been undertaken and many new issues are raised, so we leave full investigation of this for future work.

4.11 Alternative Compensation Schemes

Agents in a group have so far shared equally. Here alternative compensation rules are investigated involving pay in proportion to effort:[33]

$$U_i^p\left(e_i;\theta_i,E_{\sim i}\right)=\left(\frac{e_i}{E_{\sim i}+e_i}O\left(e_i,E_{\sim i}\right)\right)^{\theta_i}\left(\omega_i-e_i\right)^{1-\theta_i}$$

Interestingly, this change, when implemented globally across the entire economy, leads to a breakdown in the basic model results, with one giant firm forming. The

[33] Encinosa *et al*. (1997) studied compensation systems empirically for team production environments in medical practices. They find that "group norms" are important in determining pay practices. Garen (1998) empirically links pay systems to monitoring costs. More recent work is Shaw and Lazear (2008).

reason for this is that there are great advantages from the increasing returns to being in a large firm and if everyone is compensated in proportion to their effort level no one can do better away from the one large firm. Thus, while there is a certain 'perfection' in the microeconomics of this compensation, it completely breaks all connections of the model to empirical data.

Next consider a mixture of compensation schemes, with workers paid partially in proportion to how hard they work and partially based on total output.

$$U_i(e_i) = fU_i^e(e_i) + (1-f)U_i^p(e_i) = \left(\frac{f}{n^{\theta_i}} + \frac{(1-f)e_i^{\theta_i}}{(E_{\sim i} + e_i)^{\theta_i}}\right)\left[O(e_i, E_{\sim i})\right]^{\theta_i}(\omega_i - e_i)^{1-\theta_i}.$$

Parameter f determines whether compensation is more 'equal' or 'proportional'. This can be solved analytically for $\beta = 2$, but is long and messy. Experiments with $f \in [½, 1]$ reveal that the qualitative character of the model is not sensitive to f.

4.12 Finite Lifetimes and Demographics (Population Growth)

Experiments adding agents progressively to the model over time produces a growing economy. Finite lifetimes are a further source of endogenous dynamics in the model as retirement and death force firms to seek new workers.

5 Summary, Discussion and Conclusions

A microeconomic model of firm formation has been analyzed mathematically, studied computationally, and tested empirically. Stable equilibrium configurations of firms *do not exist* in this model. Rather, agents constantly adapt to their economic circumstances, changing firms when it is in their self-interest to do so. This simple model, consisting of locally optimizing agents in a world of increasing returns, is sufficient to generate macro-statistics on firm size, growth rates, ages, job tenure, and so on, that closely resemble U.S. data. Overall, firms serve as vehicles through which agents realize greater utility than they would otherwise achieve. The general character of these results is robust to variations in the model specification. However, it is possible to sever connections to empirical data with agents who are too homogeneous, too random, or too rational.

5.1 The Emergence of Firms, Out of Equilibrium

The main result of this research is to connect an explicit microeconomic model of team formation to emerging micro-data on the population of U.S. business firms. Agent behavior is specified at the micro-level with firms emerging at a meso-level, and the population of firms becoming a well-defined statistical entity at the aggregate level. This micro-to-macro picture has been created with agent-based computing, realized at full-scale with the U.S. private sector workforce.[34] However, despite the vast scale of the model, its specification is actually very *minimal*, so spare as to seem quite unrealistic[35]—no product markets are modeled, no prices computed, no consumption represented. How is it that such a stripped-down model could ever resemble empirical data?

This model works because its *dynamics* capture elements of the real world more closely than the *static equilibrium* models conventional in the theory of the firm. This is so despite our agents being myopic and incapable of figuring out anything remotely resembling optimal multi-period strategies. Two defenses of such simple agents are clear. First, the environments in which the agents find themselves are *combinatorially too complex* for even highly capable agents to compute rational behaviors. There are just too many possible coalition structures, so each agent finds itself in perpetually novel circumstances.[36] Second, the strategic environment is *dynamically too complex* for agents to make accurate forecasts, even in the short run:[37] agents are constantly moving between firms, new firms are forming while others exit, and although the macro-level is stationary there is constant flux and adaptation in every agent's local economy.

More generally, equating social equilibrium with agent-level equilibrium, common throughout the social sciences, is problematical (Foley 1994). While the goal of social science is to explain *aggregate* regularities, agent-level equilibria are

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[34] It is folk wisdom that agent models are 'macroscopes,' illuminating macro patterns from the micro rules.
[35] In this it is reminiscent of Gode and Sunder and zero-intelligence traders (Gode and Sunder 1993).
[36] Anderlini and Felli (1994) assert the impossibility of complete contracts due to the complexity of nature.
[37] Anderlini (1998) describes the kinds of forecasting errors that are intrinsic in such environments.

commonly treated as *necessary* when in fact they are only *sufficient*—that is, the micro- and macro-worlds are commonly viewed as *homogeneous* with respect to equilibrium. But macroscopic regularities that have the character of statistical equilibria—stationary distributions, for instance—may have two conceptually distinct origins. When equilibrium at the agent level is achieved, perhaps as stochastic fluctuations about one or more deterministic equilibria (e.g., Young 1993), then there is a definite sense in which macro-stationarity is a direct consequence of micro-equilibrium. But when there do not exist stable agent-level equilibria, the assumption of homogeneity across levels is invalid, yet it may nonetheless be the case that regularities and patterns will appear at the macro-level. Furthermore, when stable equilibria exist but require an amount of time to be realized that is long in comparison to the economic process under consideration, one may be better off looking for regularities in the long-lived transients. This is particularly relevant to coalition formation games in large populations, where the number of coalitions is given by the unimaginably vast Bell numbers, making it unlikely that anything like optimal coalitions could ever be realized. Perpetual flux in the composition of groups must result, leading to the conclusion that microeconomic equilibria have little explanatory power.

5.2 Theories of the Firm Versus a Theory of Firms

Extant theories of the firm are steeped in this kind of micro-to-macro homogeneity. They begin innocuously enough, with firms conceived as being composed of a few actors. They then go on to derive firm performance in response to strategic rivals, uncertainty, information processing constraints, and so on. But these derivations interpret the overall performance of many-agent groups and organizations in terms of a few agents in equilibrium.[38] I suggest that preoccupation with equilibrium notions is largely responsible for the neglect of the gross empirical regularities of industrial organization in textbooks.[39] There do not

[38] Least guilty of this charge is the evolutionary paradigm.

[39] For example, neither Shy (1995) nor Cabral (2000) make no mention of size and growth rate distributions!

exist microeconomic explanations for most of these regularities, from firm age to job tenure. Indeed, it has been conjectured that power laws are generically *not* the result of perturbations about static configurations. If so then it may be that no equilibrium theory can ever reproduce heavy-tailed firm size and growth rate data.

There are two senses in which our model is a theory of firms. First, from a purely descriptive point of view, the model reproduces many facts. Theories of the firm able to explain more than a few of these facts do not exist.[40] Nor are most theories sufficiently explicit to be operationalized in software—although stated at the microeconomic level, the focus on equilibrium leaves behavior away from equilibrium unspecified.[41] In the language of Simon (1976), these theories are substantively rational, not procedurally so. Or, if micro-mechanisms are given, the model is not quantitatively related to data (e.g., Kremer 1993; Rajan and Zingales 2001), or else the model generates the wrong patterns (e.g., Cooley and Quadrini (2001) get *exponential* firm sizes). The second sense in which my model is a theory of firms is that agent models always are *explanations* of the phenomena they reproduce.[42] In the philosophy of science an explanation is defined with respect to a theory.[43] A theory has to be general enough to permit many instantiations—to provide explanations of whole classes of phenomena—while not being so vague that it can rationalize all phenomena. Each parameterization of an agent-based model is an instantiation of a more general agent 'theory'. Executing an instance yields patterns and regularities that can be compared to data, thus making the instance, the model and the theory all falsifiable.[44]

My 'explanation' for firms is simply this: purposive agents in increasing returns environments form transient coalitions; freedom of movement between

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[40] A variety of models target one of these desiderata, often the firm size distribution (e.g., Kwasnicki 1998).

[41] I began this work with the expectation of drawing heavily on extant theory. While I did not expect to be able to turn Coase's elegant prose into software line-for-line, I did expect to find significant guidance on the micro-mechanisms of firm formation. These hopes were soon dashed on the shoals of equilibrium theorizing.

[42] According to Simon (Ijiri and Simon 1977: 118): "To 'explain' an empirical regularity is to discover a set of simple mechanisms that would produce the former in any system governed by the latter."

[43] This is the so-called deductive-nomological (D-N) view of explanation; see Hempel (1966).

[44] In models that are intrinsically stochastic, multiple realizations must be made to find robust regularities.

such coalitions 'arbitrages' away super linear returns and induces firms to compete for talent, Suitably parameterized, empirically-salient firms result. Someday a mathematical derivation from the micro (agent) level to the macro (firm population) level—through the meso (firm) level—may appear, but for now we must be content with the *discovery* that the latter result from the former.

Real-world organizations provided economics with the important concept of bounded rationality.[45] Now micro-data on organizations are re-shaping our ideas about firms. It is hoped that the present work will provide impetus for the development of heterogeneous agent, non-equilibrium theories in economics. If so, computational methods will be as useful as analytical ones, at least if other branches of science are any guide. It is sometimes said that physics owes more to the steam engine than the steam engine owes to physics. Perhaps someday the same will be said of economic theory and firms.

5.3 Agent-Based Economics

This model is a first step toward a more realistic, dynamical theory of the firm, one with explicit micro-foundations. Clearly this approach yields empirically rich results. These results are produced computationally. Typical uses of computers by economists today are to numerically solve equations (Judd 1998) or mathematical programs, to run regressions (Sala-i-Martin 1997), or to simulate stochastic processes—all complementary to conventional theorizing. The way computer power is being harnessed here is different. Agent computing facilitates heterogeneity, so representative agents are not needed (Kirman 1992). It encourages use of behavioral specifications featuring direct (local) interactions, so networks are natural (Kirman 1997). Agents possess a limited amount of information and are of necessity boundedly rational, since full rationality is computationally intractable (Papadimitriou and Yannakakis 1994). Aggregation happens, just like in the real world, by summing up numerical quantities, without concern for functional forms (of utility and production functions). Macro-

[45] Simon's early work on organizations (1947) led him to question optimizing models of human behavior..

relationships *emerge* and are not limited *a priori* by what the 'armchair economist' (Simon 1986) can first imagine and then write down mathematically. There is no need to postulate the attainment of equilibrium since one merely interrogates a model's noiseless output for patterns, which may or may not include stable equilibria. Indeed, agent computing is a natural technique for studying economic processes that are far from (agent-level) equilibrium. The present work has just scratched the surface of the pregnant interface between agent computing and the theory of the firm. Much work remains.

Appendix A: Existence and Instability of Nash Equilibria

Relaxing the functional forms of §2, each agent has preferences for income, I, and leisure, Λ, with more of each preferred to less. Agent i's income is monotone non-decreasing in its effort level e_i as well as that of the other agents in the group, $E_{\sim i}$. Its leisure is a non-decreasing function of $\omega_i - e_i$. The agent's utility is thus $U_i(e_i; E_i) = U_i(I(e_i; E_{\sim i}), \Lambda(\omega_i - e_i))$, with $\partial U_i/\partial I > 0$, $\partial U_i/\partial \Lambda > 0$, and $\partial I(e_i; E_{\sim i})/\partial e_i > 0$, $\partial \Lambda(e_i)/\partial e_i < 0$. Furthermore, assuming $U_i(I = 0, \cdot) = U_i(\cdot, \Lambda = 0) = 0$, U is single-peaked. Each agent selects the effort that maximizes its utility. The first-order condition is straightforward. From the inverse function theorem there exists a solution to this equation of the form $e_i^* = \max\,[0, \zeta(E_{\sim i})]$. From the implicit function theorem both ζ and e_i^* are continuous, non-increasing functions of $E_{\sim i}$.

Team effort equilibrium corresponds to each agent contributing its optimal effort, e_i^*, assuming that the other agents are doing so as well, i.e., substituting $E_{\sim i}^*$ for $E_{\sim i}$. Since each e_i^* is a continuous function of $E_{\sim i}$ so is the vector of optimal efforts, $e^* \in [0, \omega]^N$, a compact, convex set. By the Leray-Schauder-Tychonoff theorem an effort level fixed point exists. Furthermore, such a solution constitutes a Nash equilibrium, which is Pareto-dominated by effort vectors having larger amounts of effort for all agents.

An upper bound on size exists for effort adjustments $e_i(t+1) = h_i(E_{\sim i}(t))$, s.t.

$$\frac{dh_i\left(E_{\sim i}\right)}{dE_{\sim i}} = \frac{\partial h_i\left(E_{\sim i}\right)}{\partial e_j} \leq 0\,, \qquad \text{(A.1)}$$

for all $j \neq i$. Under these circumstances the Jacobian matrix retains the structure

described in § 2.3, where each row contains N-1 identical entries and a 0 on the diagonal. The bounds on the dominant eigenvalue derived in section 2.3 guarantee that there exists an upper bound on the stable group size, as long as (A.1) is a strict inequality, thus establishing the onset of instability above some critical size.

Appendix B: Empirical Data

Table 4 summarizes the firm data to which the model outputs are compared. Data that are conceptually similar are colored similarly. Note that because many of the moments do not exist for several of the distributions considered, modal and median quantities are sometimes used as bases for comparison.

	Datum or data compared	**Source**	**In text**
1	Size of the U.S. workforce: 120 million	U.S. Census	Table 2
2	Number of firms with employees: 6 million	U.S. Census	Figure 7
3	Number of new firms monthly: 100 thousand	Kauffman Foundation	Figure 7
4	Number of exiting firms monthly: 100 thousand	Kauffman Foundation	Figure 7
5	Variance higher for exiting firms to new firms	Davis, Haltiwanger and Schuh	Figure 7
6	Average firm size: 20 employees/firm	U.S. Census	Figure 8
7	Maximum firm size: 1 million employees	Forbes 500	Figure 8
8	Number of job-to-job changes monthly: 3-4 million	Fallick and Fleischman	Figure 10
9	Number of jobs created monthly: 2 million	Davis, Haltiwanger and Schuh	Figure 10
10	Number of jobs destroyed monthly: 2 million	Davis, Haltiwanger and Schuh	Figure 10
11	Variance higher for jobs destroyed than jobs created	Davis, Haltiwanger and Schuh	Figure 10
12	Modal firm size (employees): 1	U.S. Census	Figure 11a
13	Median firm size (employees): 3-4	U.S. Census	Figure 11a
14	Firm size distribution (employees): Pareto	U.S. Census	Figure 11a
15	Pareto exponent: near 1 (so-called Zipf distribution)	Axtell	Figure 11a
16	Firm size distribution (output): Pareto	Axtell	Figure 11b
17	Pareto exponent of output distribution: near 1	Axtell	Figure 11b
18	Aggregate returns to scale: constant	Basu and Fernald	Figure 12
19	Productivity distribution: exponential	various	Figure 13
20	Firm age distribution: exponential with mean 18 years	Bureau of Labor Statistics	Figure 14
21	Joint dist. of firms, size and age: linear in age, log size	various	Figure 15
22	Average firm size vs age: increasing linearly in age	Bureau of Labor Statistics	Figure 16ab
23	Avg. firm age vs size: increasing linearly in log size	Bureau of Labor Statistics	Figure 16cd
24	Firm survival probability: increasing with age	Bureau of Labor Statistics	Figure 17
25	Log firm growth rate distribution: heavy-tailed	Stanley *et al.* [1996]	Figure 18
26	Mean log firm growth rate: 0.0	Stanley *et al.* [1996]	Figure 18
27	Mean log firm growth rate vs size: sensitive to def'n	Dixon and Rollin	Figure 19a
28	Std. dev. log firm growth rate vs firm size: exp = 0.14	Stanley *et al.*	Figure 19b
29	Mean log firm growth rate vs firm age: decreasing	Dixon and Rollin	Figure 20
30	Std. dev. log firm growth rate vs firm age: decreasing	Dixon and Rollin	Figure 20
31	Income distribution: exponential	Yakovenko	Figure 21
32	Job tenure dist.: exponential with mean 80 months	Bureau of Labor Statistics	Figure 22
33	Employment vs age: exp. with mean 1000 employees	Bureau of Labor Statistics	Figure 23
34	Florence median: 500 employees	U.S. Census	
35	Large firm vs workforce size: increasing sublinearly	historical Forbes 500	Figure 25
36	Simultaneous hiring and separation	Davis, Faberman and Haltiwanger	Figure 26

Table 4: Empirical data to which the model is compared

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