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TWO ESSAYS ON INDIVIDUAL CHOICE

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Leigh Tesfatsion

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CONTENTS

AN EXPECTED UTILITY MODEL WITH ENDOGENOUSLY DETERMINED GOALS*

by

Leigh Tesfatsion

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ABSTRACT

An expected utility model of individual choice is formulated which allows the decision maker to specify his available actions in the form of "controls" (partial contingency plans) and to simultaneously choose goals and controls in end-mean pairs. It is shown that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard optimal control model can be viewed as special cases of this "goal-control expected utility model."

1. INTRODUCTION

The formal description of decision problems under uncertainty in terms of "states," "consequences," and "acts" (complete contingency plans), functionally mapping states into consequences, is now common in economics. Yet, as has often been noted (e.g., see J. H. Drèze [2]), the descriptive realism of this decision framework is somewhat limited. Most importantly, in actual problem contexts the specification of available actions in the form of complete contingency plans is often not feasible. Information may be incomplete; alternatively, the required calculations may be too costly. Second, the implicit requirement that the occurrence of a state should be unaffected by the decision maker's choice of act can greatly complicate the formal statement of even the simplest problems.

In this paper an alternative decision framework is presented within which these two difficulties do not arise. The decision maker is allowed to specify his available actions ("controls") in the form of partial contingency plans. His choice set is assumed to be a collection of end-mean, candidate goal-control pairs ("policies"). The "candidate-goals" are operationally interpreted as potential objectives whose realization the decision maker can attempt to achieve by appropriate choice of control; for example, profit, sales, or market share aspiration levels. States and consequences are subsumed into "state flows," over which policy-conditioned preference and probability orders are both defined. Thus, in a manner to be made precise below, state flows need not be defined independently of the decision maker's choice of policy. An expected utility representation is obtained for the decision maker's preferences among policies.

A number of previous researchers have explicitly introduced goals into their models (for example, R. M. Cyert and J. G. March [1], H. Simon [10], and J. Tinbergen [12]). One rationale for analytically distinguishing goals from controls is that goals and controls play distinct strategic roles in many decision problems. Whereas controls, by definition, can be realized at will by the decision maker, rarely will he have such power over his problem environment that he can ensure the attainment of a desired goal. For example, he may need the cooperation of other persons who are not entirely under his control. The amount of cooperation he receives, and hence also the probability of relevant future events, may vary depending on which goal he specifies. Thus goal specification may have strategic importance for extending control over future events. A second rationale for distinguishing goals from controls is that goals are often important for the feedback evaluation of chosen policies; i.e., the utility (cost) of a chosen policy may be a function of the "distance" between the realized outcome and the desired outcome (goal).

An interesting complementary relationship exists between the "goal-control expected utility model" presented in this paper and the "satisficing" approach to decision making first investigated by H. Simon (see [10]; also R. Radner and M. Rothschild [7]). The goal-control expected utility model seems to be particularly appropriate for the initial stages of a decision problem when relatively few details are available and the basic questions are: what should be our goal; what should be our overall line of attack (control). Once the general goal-control policy has been selected, satisficing search methods can be used by those individuals who are charged with implementing the policy and who therefore will need to make numerous sub-decisions concerning details not provided for in the policy. The specified goal plays the role of Simon's "aspiration level" in terms of which the satisficing policy implementers evaluate the adequacy of their sub-decisions.

The organization of this paper is as follows. The goal-control model primitives are presented and discussed in section 2. In section 3 it is shown that the goal-control primitive sets can be interpreted in terms of certain primitive sets used by L. Savage [9]. An expected utility representation for the goal-control model is presented in section 4. (An axiomatization for this representation is established in [11].) In section 5 it is shown that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard optimal control model can be viewed as special cases of the goal-control model with expected utility representation as in section 4. For example, the Savage expected utility model can be identified with a goal-control expected utility model in which the control set is Savage's set of acts and the collection of candidate goals is a trivial one-element set. Two examples illustrating the goal-control expected utility model are given in section 6.

2. PRIMITIVES FOR THE GOAL-CONTROL MODEL

Let $G = \{g, \ldots\}$ be a set of candidate goals, and for each $g \in G$ let $\Lambda_g = \{\lambda_g, \ldots\}$ be a set of controls. The primitives for the goal-control model ("gc-model") are then characterized by a vector

$$(\langle \Theta, \succcurlyeq \rangle, \{\langle \Omega_\theta, \succcurlyeq_\theta \rangle \mid \theta \in \Theta\}, \{\langle \mathcal{E}_\theta, \geq_\theta \rangle \mid \theta \in \Theta\})$$

where

$\Theta = \{\theta, \ldots\} = \bigcup_{g \in G} \{(g, \lambda_g) \mid \lambda_g \in \Lambda_g\}$ is the policy choice set consisting of candidate goal-control pairs (policies);

$\succcurlyeq$ (policy preference order) is a weak order[1] on Θ;

and for each policy $\theta \in \Theta$,

$\Omega_\theta = \{\omega_\theta, \ldots\}$ is a nonempty set of state flows associated with the policy θ;

$\succcurlyeq_\theta$ (θ-conditioned preference order) is a weak order on Ω_θ;

$\mathcal{E}_\theta = \{E_\theta, \ldots\}$ is an algebra[2] of subsets of Ω_θ whose elements E_θ will be called event flows associated with the policy θ;

$\geq_\theta$ (θ-conditioned probability order) is a weak order on $\mathcal{E}_\theta$.

Remark. State-consequence-act models generally assume the existence of only one primitive order, a preference order over acts, and hence obtain a simpler statement for their primitives. Nevertheless, subsequent axiomatizations to justify an expected utility representation for the preference order then invariably impose strong nonnecessary restrictions on the primitives (e.g., Savage's reliance on constant acts). In contrast, assuming a certain finiteness restriction on the state flow sets Ω_θ, necessary and sufficient conditions can be given which justify an expected utility representation for the gc-model policy preference order $\succcurlyeq$ (see section 4 below).

The controls may be operationally interpreted as possibly conditioned sequences of actions (i.e., partial contingency plans) entirely under the control of the decision maker at the time of his choice. The candidate goals $g \in G$ may be operationally interpreted as potential objectives (e.g., production targets) whose realization the decision maker can attempt to achieve through appropriate choice of a control. The grouping of the controls into sets $\{\Lambda_g | g \in G\}$ reflects the possibility that different sets of controls may be relevant for different goals; e.g., for a decision maker in San Francisco, the control "travel by bus" is suitable for the goal "vacation in Los Angeles" but not for the goal "vacation in Hawaii." A control $\lambda_g \in \Lambda_g$ may or may not provide for the communication of the goal g to other persons in the decision maker's problem environment.

The weak order $\succcurlyeq$ on Θ can be operationally interpreted as a preference order as follows. For all $\theta', \theta'' \in \Theta$,

$$\theta' \succcurlyeq \theta'' \Leftrightarrow \text{the choice of policy } \theta' \text{ is at least as desirable to the decision maker as the choice of policy } \theta'' .$$

The decision maker is assumed to choose a policy (candidate goal-control pair) $\theta' \in \Theta$ which is optimal in the sense that $\theta' \succcurlyeq \theta$ for all $\theta \in \Theta$. Throughout this paper we use "choose policy $\theta = (g, \lambda_g)$" and "implement control λ_g with g as the objective" interchangeably.

For each $\theta \in \Theta$, the set Ω_θ of state flows ω_θ can be interpreted as the decision maker's answer to the following question: "If I choose policy θ, what distinct situations (i.e., state flows ω_θ) might obtain?" The state flows may include references to past, present, and future happenings. In order for subsequent probability assessments to be realistically feasible, the state flow sets should include the decision maker's background information concerning the problem at hand.

The θ-conditioned preference orders $\succcurlyeq_\theta$ can be interpreted as follows. For all $\omega, \omega' \in \Omega_\theta$, $\theta \in \Theta$,

$$\omega \succcurlyeq_\theta \omega' \Leftrightarrow \text{the realization of } \omega \text{ is at least as desirable to the decision maker as the realization of } \omega' \text{, given the event "decision maker chooses } \theta \text{."}$$

Similarly, the θ-conditioned probability orders $\geq_\theta$ can be interpreted as follows. For all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$,

$E \geq_\theta E' \Leftrightarrow$ in the judgment of the decision maker, the realization of E is at least as likely as the realization of E', given the event "decision maker chooses θ."

A state flow ω may be relevant for the decision maker's problem under distinct potential policy choices; e.g., $\omega \in \Omega_\theta \cap \Omega_{\theta'}$ for some $\theta, \theta' \in \Theta$. Similarly, the algebras $\{\mathcal{E}_\theta\}$ may overlap. Given state flows $\omega, \omega' \in \Omega_\theta \cap \Omega_{\theta'}$ for some $\theta, \theta' \in \Theta$, it may hold that $\omega \succcurlyeq_\theta \omega'$ whereas $\omega' \succcurlyeq_{\theta'} \omega$. Verbally, the relative utility of the state flows ω and ω' may depend on which conditioning event the decision maker is considering, "decision maker chooses θ" or "decision maker chooses θ'." Similarly for the relative likelihood of event flows $E, E' \in \mathcal{E}_\theta \cap \mathcal{E}_{\theta'}$, $\theta, \theta' \in \Theta$.

Examples illustrating these interpretations are given in section 6.

3. INTERPRETATION OF PRIMITIVE SETS IN TERMS OF A MODEL USED BY SAVAGE

Although best known for his expected utility axiomatization of a complete contingency plan model in the tradition of Ramsey and Von Neumann-Morgenstern, Savage was also fully aware of the practical need for decision frameworks with more limited information requirements. In a rarely cited section of his famous "Foundations of Statistics" [9, page 82] he constructs an interesting limited information decision framework ("small world model") in terms of the primitive sets used in his basic, complete contingency plan model. As Savage cautions, however, the small world model seems to take this basic model "much too seriously;" e.g., consequences in the small world model are defined to be acts (complete contingency plans) from the basic model.

In contrast to Savage's small world model primitives, the gc-model primitives do not rely on the existence of a complete contingency plan model. Nevertheless, as is demonstrated below, the gc-model primitive sets can also be given an interpretation in terms of the primitive sets used in Savage's basic model. This interpretation helps to clarify the relationship between the gc-model primitive sets and the state-consequence-act primitive sets appearing in more traditional models.

Assume a decision maker is faced with a certain problem at time t^o . Following Savage [9], define

S = exhaustive set of possible descriptions ("states") of the world at time t^o, including all aspects relevant for the problem at hand;

C = set of all future life histories of the decision maker ("consequences");

F = set of all functions ("acts") $f : S \to C$.

For any set M , let "2^M" denote the set of all subsets of M . Then gc-model primitive sets for the decision maker's problem can be given in terms of $(2^C , 2^F , 2^S)$, as indicated below:

$G \subseteq 2^C$ (candidate goal set);

$\Lambda_g \subseteq 2^F$ (control set associated with g), for each $g \in G$;

$\Theta = \cup_{g \in G}\{(g, \lambda) \mid \lambda \in \Lambda_g\}$ (policy choice set);

and for each policy $\theta \in \Theta$,

$\Omega_\theta = S_\theta \times C_\theta$ (state flow set associated with θ) , where $S_\theta \subseteq 2^S$ is a partition of S , and $C_\theta \subseteq 2^C$ is a partition of C ;

$\mathcal{E}_\theta = \mathcal{F}_\theta \times \mathcal{A}_\theta$ (algebra of event flows associated with θ) , where $\mathcal{F}_\theta$ is an algebra of subsets of S_θ and $\mathcal{A}_\theta$ is an algebra of subsets C_θ .

In other words, a candidate goal can be interpreted as a subset of Savage's consequence set C, and a control can be interpreted as a subset of Savage's set F of acts. For example, a candidate goal for player one in a chess game might be g : (player one checkmates player two's king), which can be identified with the set of all Savage consequences(future life histories of player one) in which the event g obtains. A control for player one might be λ : (open by moving king's pawn two spaces), which can be identified with the set of all Savage acts (complete contingency plans) available to player one for which the opening move in the chess game at hand is specified to be λ. Similarly, state and event flows can be interpreted as subsets of $S \times C$ and $2^S \times 2^C$. A detailed example illustrating this interpretation is given in section 6.2.

The relationship between the Savage model and the gc-model will be further discussed (section 5.1) after the expected utility representation for the gc-model has been presented.

4. EXPECTED UTILITY REPRESENTATION

In a separate paper [11] conditions are given which ensure that the gc-model has an expected utility representation in the following sense: To each policy $\theta \in \Theta$ there corresponds a finitely additive probability measure $\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]$ satisfying

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \gtrsim_\theta E' , \tag{1}$$

for all $E, E' \in \mathcal{E}_\theta$, and a utility function $u(\cdot|\theta) : \Omega_\theta \to R$ satisfying

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \succcurlyeq_\theta \omega' , \tag{2}$$

for all $\omega, \omega' \in \Omega_\theta$, such that

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta') \Leftrightarrow \theta \succcurlyeq \theta' , \tag{3}$$

for all $\theta, \theta' \in \Theta$. Given that each state flow set Ω_θ is finite, with $\mathcal{E}_\theta = 2^{\Omega_\theta}$, the remaining conditions are shown to be necessary and sufficient for the desired representation (1), (2), and (3).

This expected utility representation for the policy preference order $\succcurlyeq$ can be interpreted as follows. To each state flow $\omega' \in \Omega_{\theta'}$, $\theta' \in \Theta$, the decision maker assigns a utility number $u(\omega'|\theta')$ representing the desirability of $\{\omega'\}$ obtaining, conditioned on the event "decision maker chooses θ'," and a probability number $\sigma(\{\omega'\}|\theta')$ representing the likelihood of $\{\omega'\}$ obtaining, conditioned on the event "decision maker chooses θ'." He then calculates the expected utility

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta)$$

corresponding to each choice of policy $\theta \in \Theta$, and chooses a policy which yields maximum expected utility.

<u>Definition</u>. A gc-model with numerical representation as in (1), (2), and (3) will be referred to as a <u>gc-expected utility model</u>, characterized by a vector

$$(\Theta,\ \{u(\cdot|\theta) : \Omega_\theta \to R | \theta \in \Theta\},\ \{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0,\ 1] | \theta \in \Theta\})\ ,$$

with <u>objective function</u> $U : \Theta \to R$ given by

$$U(\theta) = \int_{\Omega_\theta} u(\omega|\theta)\,\sigma(d\omega|\theta)\ , \qquad \theta \in \Theta\ .$$

For each policy $\theta \in \Theta$, the function $u(\cdot|\theta) : \Omega_\theta \to R$ will be called the <u>utility function associated with θ</u> and the probability measure $\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0,\ 1]$ will be called the <u>probability measure associated with θ</u> .

Two distinctive features of the gc-expected utility model will now be discussed: namely, utility and probability are both conditioned on the chosen policy; and utility and probability are both defined over (subsets of) state flows rather than having utility defined over a consequence set and probability defined over subsets of a state set. We begin by giving a brief historical perspective.

Nearly all expected utility models for decision making under uncertainty which do not presuppose the existence of numerical probability measures have as primitives a set S of "states" whose subsets have probability but no utility, a set C of "consequences" which have utility, and a set F of "acts," ordered in preference by the decision maker, which functionally relate the states to the consequences. The existence of a utility function over consequences and a probability measure over states are subsequently derived from axioms. In order for the (unconditional) probability measure over states to be well defined, the realization of a state cannot depend on the decision maker's choice of act. Many decision theorists have noted that in practice it is difficult to specify states in such a way that they are utility-free; and the required independence condition between states and acts is often awkward to achieve.

In order to avoid these two difficulties, Krantz, Luce <u>et. al</u>. [4, Chapter 8] replace each original act $f \in F$ by a set of conditional acts $\{f_A : A \to C | A \subseteq S\}$, where f_A, $A \subseteq S$, is interpreted as "act f conditional on A." The utility representation they subsequently axiomatize for their conditional utility model is of the form:

$$f_A \succcurlyeq g_B \Leftrightarrow u(f_A) \geq u(g_B) \ ;$$

$$\text{if} \quad A \cap B = \emptyset \ , \quad \text{then}$$

$$u(f_A \cup g_B) = u(f_A)\, \text{Prob}(A|A \cup B) + u(g_B)\, \text{Prob}(B|A \cup B) \quad .$$

In a radically different approach, partly in response to the same difficulties, Jeffrey [3] takes as his only primitive set a certain set M of propositions, and he replaces the concept of an act by the concept of a "proposition made true." His subsequent utility (desirability) representation over the disjunction $a \vee b$ of propositions a and b in M is of the form: if $\text{Prob}(a \wedge b) = 0$ and $\text{Prob}(a \vee b) > 0$, then

$$u(a \vee b) = [u(a)\ \text{Prob}(a) + u(b)\ \text{Prob}(b)] / [\text{Prob}(a) + \text{Prob}(b)] \quad .$$

The gc-expected utility model uses to some extent both the conditional approach of Krantz _et. al_. and the homogeneous approach of Jeffrey. The conditioning of the state flow sets on the policy choice of the decision maker simply reflects the realistic consideration that different state flows may be relevant for different policy choices; it is not an essential feature of the gc-model. In contrast, the conditioning of the utility functions $\{u(\cdot|\theta) : \Omega_\theta \rightarrow R | \theta \in \Theta\}$ and the probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \rightarrow [0, 1] | \theta \in \Theta\}$ on the chosen policy is essential in order to avoid the type of independence difficulties mentioned above which arise in the application of more traditional state-consequence-act models. Without such a conditioning it would not be legitimate for the decision maker to assign utility and probability to state flows prior to his choice of policy. (As will be discussed in section 6 in the context of an example, a state flow may have a different utility and a different probability depending on which policy, i.e., candidate goal-control pair, is chosen. In addition, controls and candidate goals appear to be independently significant in this conditioning.)

By subsuming states and consequences into state flows and having utility defined over state flow sets, the gc-expected utility model also avoids the second difficulty mentioned above concerning the specification in practice of utility-free states. Moreover, as in Jeffrey's model, the specification of a functional relationship between states and consequences is not required.

On the other hand, the elements of choice in the gc-model are not in conditional form, as in the Krantz et. al. model; nor are they completely subsumed into a single primitive set, as in the Jeffrey model. Consequently, whereas the Krantz et. al. and Jeffrey utility representations are strikingly different from the more traditional expected utility representation expressed in terms of states, consequences, and acts, the gc-model primitives support an expected utility representation which generalizes this more traditional representation (see section 5).

5. COMPARISON WITH OTHER MODELS

In section 3 an interpretation was given for the primitive sets of the gc-model in terms of the primitive sets used by Savage [9]. By assuming various other interpretations for the gc-model primitive sets, it will be shown below that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard, continuous time, fixed terminal time control model can be viewed as special cases of the gc-expected utility model.

The fact that these four models can be placed in one general framework reveals that they are not as disparate as their terminology and notation might indicate. In order to facilitate comparisons, the four models will be discussed in terms of their gc-expected utility representations (given below). In each of the models the decision maker is assumed to choose a control in order to maximize expected utility. In the statistical decision model the controls are "experiment"-function pairs. In each of the other models the controls are functions. Explicitly specified goals do not play an essential role in the first three models, whereas in the control model a fixed target trajectory is used as an "aspiration level" in terms of which the effectiveness of various controls is evaluated. Yet, as is clear from their gc-expected utility representations, each of the four models could explicitly handle endogenously determined goals without any drastic change in basic framework.

On the other hand, the four models differ in their use of conditioning. In both the Savage and the team models, all state flow sets $\{\Omega_\theta | \theta \in \Theta\}$ are identical and all probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ are identical. In contrast, the state flow sets in the statistical decision model are non-trivially conditioned on the control, and the state flow sets in the control model are nontrivially conditioned on both control and goal. The probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ in the statistical decision model are nontrivially conditioned on the control; the probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ in the deterministic control model are trivial. The utility functions $\{u(\cdot|\theta) : \Omega_\theta \to R | \theta \in \Theta\}$ in all four models are nontrivially conditioned on the control (those in the control model are also nontrivially conditioned on the goal).

Another characteristic distinguishing the basic frameworks of these four models is the implicit structural relationship between the control set and the state flow sets. For example, in the Savage and the team models the controls are functions on the (single) state flow set. In the control model the controls are defined independently of the state flow sets. Intuitively, the state flows in the Savage and team models are pure "states" whereas the state flows in the control model, aside from boundary conditions, are pure "consequences."

5.1 THE SAVAGE EXPECTED UTILITY MODEL AND THE GC-EXPECTED UTILITY MODEL COMPARED. It will first be shown that the Savage expected utility model can be identified with a particular gc-expected utility model. Conversely, it will then be shown that the gc-expected utility model can be identified with a particular Savage expected utility model with primitives defined in an unusual, interesting manner.[3] As in section 3, let

S = set of states;

C = set of consequences;

F = set of acts; i.e., all functions $f: S \to C$.

Under the Savage axioms, there exists a weak (preference) order $\langle F, \geq \rangle$, a utility function $\tilde{u}: C \to R$, and a probability measure $P: 2^S \to [0, 1]$ such that for all $f, g \in F$,

$$\int_S \tilde{u}(f(s))\, P(ds) \geq \int_S \tilde{u}(g(s))\, P(ds) \Leftrightarrow f \geq g \quad .$$

Set

$G' = \{C\}$, a trivial one-element candidate goal set;

$\Lambda'_C = F$, the control set associated with the candidate goal C;

$\Theta' = \{C\} \times F = \{(C, f), \ldots\}$, the policy choice set;

and for each policy $(C, f) \in \Theta'$,

$\Omega'_{(C,f)} = S$, the set of state flows associated with (C, f);

$\mathcal{E}'_{(C,f)} = 2^S$, the algebra of event flows associated with (C, f);

$u'(\cdot|(C, f)) = \tilde{u}(f(\cdot)) : \Omega'_{(C,f)} \to R$, the utility function associated with (C, f);

$\sigma'(\cdot|(C, f)) = P(\cdot) : \mathcal{E}'_{(C,f)} \to [0, 1]$ the probability measure associated with (C, f).

Then for each policy $(C, f) \in \Theta'$,

$$\int_{\Omega'_{(C,f)}} u'(\omega|(C, f))\, \sigma'(d\omega|(C, f)) = \int_S \tilde{u}(f(s))\, P(ds) \quad .$$

Hence the Savage expected utility model can be identified with the gc-expected utility model

$$(\Theta', \{u'(\cdot|\theta):\Omega'_\theta \to R | \theta \in \Theta'\}, \{\sigma'(\cdot|\theta):\mathcal{E}'_\theta \to [0, 1] | \theta \in \Theta'\})$$

with objective function $U':\Theta' \to R$ given by

$$U'(C, f) = \int_{\Omega'_{(C,f)}} u'(\omega|(C, f))\, \sigma'(d\omega|(C, f)), \ (C, f) \in \Theta' \quad .$$

Conversely, let $(\Theta, \{u(\cdot|\theta):\Omega_\theta \to R | \theta \in \Theta\}, \{\sigma(\cdot|\theta):\mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\})$ be a given gc-expected utility model. Let

$C' = \{(\Omega_\theta, \mathcal{E}_\theta, \sigma(\cdot|\theta)) | \theta \in \Theta\}$, a set of consequences;

$S' = \{S^*\}$, a one-element set of states, where S^* is the function mapping Θ into C' given by $S^*(\theta) = (\Omega_\theta, \mathcal{E}_\theta\ \sigma(\cdot|\theta))$, $\theta \in \Theta$;

$F' = \Theta$, a set of acts, where for each $\theta \in F'$, $\theta:S' \to C'$ is given by $\theta(S^*) = S^*(\theta)$.

$\tilde{u}:C' \to R$, a utility function given by $\tilde{u}(\Omega_\theta, \mathcal{E}_\theta, \sigma(\cdot|\theta)) = \int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta)$;

$P:2^{S'} \to [0, 1]$, a probability measure given trivially by $P(\{S^*\}) = 1$.

Then for each act $\theta \in F'$,

$$\int_{S'} \tilde{u}(\theta(s))\ P(ds) = \int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \quad .$$

Hence the gc-expected utility model can be identified with the Savage expected utility model (C', S', F') with objective function $U':F' \to R$ given by

$$U'(\theta) = \int_{S'} \tilde{u}(\theta(s))\ P(ds),\ \theta \in F' \quad .$$

5.2 THE TEAM MODEL AND THE GC-EXPECTED UTILITY MODEL COMPARED. Intuitively, a team is an organization in which there is a single payoff function reflecting the common preferences of the members. The formal team model presented below is taken from Radner [8]. It will be shown that this team model can be identified with a particular gc-expected utility model. The components of the team model are as follows.

$S = \{\omega, \ldots\}$ = set of alternative <u>states</u> of the environment;

$C = \{c, \ldots\}$ = set of alternative <u>consequences</u>;

$A = \{a, \ldots\}$ = set of alternative <u>acts</u> available to the team, where every act a in A is a function from S to C;

$N = \{1, \ldots, M\}$, the set of <u>team members</u>;

$Y = \Pi_{i=1}^{M} Y_i$, where Y_i = set of alternative <u>signals</u> that $i \in N$ can receive as information;

$\mathcal{Y} = \{\eta, \eta', \ldots\}$ = set of available <u>information structures</u>, where each $\eta \in \mathcal{Y}$ is a vector $(\eta_1, \ldots, \eta_M)$ with $\eta_i : S \to Y_i$ an <u>information function</u> for $i \in N$;

$D = \Pi_{i=1}^{M} D_i$, where D_i is a set of alternative <u>decisions</u> that $i \in N$ can take;

$\mathcal{D} = \{\delta, \delta', \ldots\}$ the set of <u>decision functions</u> available to the team, where each $\delta \in \mathcal{D}$ is a vector $(\delta_1, \ldots, \delta_M)$, with $\delta_i : Y_i \to D_i$ the <u>decision function for i</u>;

$\rho: S \times D \to C$, a given team outcome function;

and for each $\eta \in \mathfrak{U}$ and $\delta \in \mathfrak{D}$,

$a_{\delta \circ \eta}: S \to C$ given by $a_{\delta \circ \eta}(\omega) = \rho(\omega, \delta \circ \eta(\omega))$, the act determined by η and δ (given ρ) .

Also it is assumed there exists

$\tilde{u}: C \to R$, a unique utility function reflecting the (identical) preferences of the team members;

$\phi: 2^S \to [0, 1]$ a probability function;

$\langle A, \geq \rangle$ a weak preference ordering expressing the preference ordering of the team (as a unit) over the set of acts.

Finally, it is assumed that $\forall\ a, a' \in A$,

$$a \geq a' \Leftrightarrow \int_S \tilde{u}(a(s))\, d\phi(s) \geq \int_S \tilde{u}(a'(s))\, d\phi(s) \quad .$$

Set

$G'' = \{C\}$, a trivial one-element candidate goal set;

$\Lambda''_C = \{\delta \circ \eta: S \to D \mid \delta \in \mathfrak{D}, \eta \in \mathfrak{U}\} = \{\lambda, \ldots\}$, the control set associated with the candidate goal C ;

$\Theta'' = \{C\} \times \Lambda''_C = \{(C, \lambda), \ldots\}$, the policy choice set;

and for each policy $(C, \lambda) \in \Theta''$,

$\Omega''_{(C,\lambda)} = S$, the set of state flows associated with (C, λ);

$\mathcal{E}''_{(C,\lambda)} = 2^S$, the algebra of event flows associated with (C, λ);

$u''(\cdot|(C, \lambda)) = \tilde{u}\circ\rho(\circ, \lambda(\circ)) : \Omega''_{(C,\lambda)} \to R$ the utility function associated with (C, λ);

$\sigma''(\cdot|(C, \lambda)) = \phi(\cdot) : \mathcal{E}''_{\theta} \to [0, 1]$ the probability measure associated with (C, λ).

Then for all policies $(C, \lambda) \in \Theta''$,

$$\int_{\Omega''_{(C,\lambda)}} u''(\omega|(C,\lambda))\ \sigma''(d\omega|(C, \lambda)) = \int_S \tilde{u}(a_\lambda(\omega))\ \phi(d\omega),$$

where $a_\lambda(\cdot) \equiv \rho(\cdot, \lambda(\cdot))$ is the act determined by λ (given ρ).

The team model can therefore be identified with the gc-expected utility model

$$(\Theta'', \{u''(\cdot|\theta) : \Omega''_\theta \to R | \theta \in \Theta''\}, \{\sigma''(\cdot|\theta) : \mathcal{E}''_\theta \to [0, 1] | \theta \in \Theta''\})$$

with objective function $U'' : \Theta'' \to R$ given by

$$U''(C, \lambda) = \int_{\Omega''_{(C,\lambda)}} u''(\omega|(C, \lambda))\ \sigma''(d\omega|(C, \lambda)),\ (C, \lambda) \in \Theta''\ .$$

5.3 THE BAYESIAN STATISTICAL DECISION MODEL AND THE GC-EXPECTED UTILITY MODEL COMPARED. The Bayesian statistical decision model presented below is taken from D. V. Lindley [5, pages 1 - 20]. It will be shown that this model can be identified with a particular gc-expected utility model.

Let $E \equiv \{(X_e, \Phi_e, \{p(\cdot|\psi, e) : \mathcal{T}_e \to [0, 1] | \psi \in \Phi_e\}) | e \in E\}$ be a collection of experiments, where for each $e \in E$

X_e is a sample space;

$\mathcal{T}_e$ is a σ-algebra of X_e ;

Φ_e is a parameter space;

$\{p(\cdot| \psi, e) : \mathcal{T}_e \to [0, 1] | \psi \in \Phi_e\}$ is a set of probability measures.

Let D be a decision space, and for each $e \in E$ let

$\mathcal{L}_e$ be a σ-algebra of Φ_e ;

$p(\cdot|e):\mathcal{L}_e \to [0, 1]$ be a probability measure (prior distribution);

$\Delta_e = \{\delta_e, \ldots\}$ be the set of all functions (decision functions) taking X_e into D ;

$\tilde{u} : (\{e\} \times \Delta_e \times X_e \times \Phi_e) \to R$ a utility function.

Then the Bayesian statistical decision problem is to choose $(e^*, \delta^*) \in \bigcup_{e \in E} (\{e\} \times \Delta_e)$ so that

$$\int_{X_{e^*} \times \Phi_{e^*}} \tilde{u}(e^*, \delta^*(x), x, \psi)\, P(dx, d\psi|e^*) \tag{4}$$

$$= \max_{e \in E} \int_{X_e} \left[\max_{\delta \in \Delta_e} \int_{\Phi_e} \tilde{u}(e, \delta(x), x, \psi)\, p(d\psi|x, e) \right] p(dx|e) ,$$

where $P(dx, d\psi|e) \equiv p(d\psi|x, e)\, p(dx|e) = p(dx|\psi, e)\, p(d\psi|e)$,

and $p(x|e) \equiv \int_{\Phi_e} p(x|\psi, e)\, p(\psi|e)\, d\psi$.

Assume that for each $e \in E$, $\tilde{u}(e, \cdot, \cdot, \cdot):(\Delta_e \times X_e \times \Phi_e) \to R$ is dominated by a function in $L^1(P(\cdot|e))$ (e.g., $\tilde{u}(e, \cdot, \cdot, \cdot)$ is bounded). Then by Lebesgue's Dominated Convergence Theorem, for every $e \in E$,

$$\int_{X_e} \left[\max_{\delta \in \Delta_e} \int_{\Phi_e} \tilde{u}(e, \delta(x), x, \psi)\, p(d\psi|x, e)\right] p(dx|e) \tag{5}$$

$$= \max_{\delta \in \Delta_e} \int_{X_e \times \Phi_e} \tilde{u}(e, \delta(x), x, \psi)\, P(dx, d\psi|e) .$$

Set

$G^* = \{R\}$, a trivial one-element candidate goal set;

$\Lambda^*_R = \cup_{e \in E} (\{e\} \times \Delta_e) = \{(e, \delta), \ldots\}$, the control set associated with the candidate goal R ;

$\Theta^* = \{R\} \times \Lambda^*_R = \{(R, e, \delta) | \delta \in \Delta_e, e \in E\} = \{\theta, \ldots\}$, the policy choice set;

and for each policy $(R, e, \delta) \in \Theta^*$,

$\Omega^*_{(R,e,\delta)} = X_e \times \Phi_e = \{(x_e, \psi_e), \ldots\}$, the state flow set associated with (R, e, δ) ;

$\mathcal{E}^*_{(R,e,\delta)} = \mathcal{T}_e \times \mathcal{L}_e$, the algebra of event flows associated with (R, e, δ) ;

$u^*(\cdot|(R, e, \delta)):\Omega^*_{(R,e,\delta)} \to R$ the utility function associated with (R, e, δ), given by

$$u^*((x, \psi)|(R, e, \delta)) = \tilde{u}(e, \delta(x), x, \psi), \ (x, \psi) \in \Omega^*_{(R,e,\delta)} ;$$

$\sigma^*(\cdot|(R, e, \delta)) = P(\cdot|e):\mathcal{E}^*_{(R,e,\delta)} \to [0, 1]$ the probability measure associated with (R, e, δ).

Then, using (5),

$$\max_{\theta \in \Theta^*} \int_{\Omega^*_\theta} u^*(\omega|\theta)\ \sigma^*(d\omega|\theta) \equiv \max_{(e,\delta)\in\Lambda^*_R} \int_{X_e \times \Phi_e} \tilde{u}(e, \delta(x), x, \psi)\, P(dx, d\psi|e)$$

$$= \max_{e\in E} \int_{X_e} \left[\max_{\delta\in\Delta_e} \int_{\Phi_e} \tilde{u}(e, \delta(x), x, \psi)\ p(d\psi|x, e)\right] p(dx|e) \ . \qquad (6)$$

Comparing (4) and (6), the Bayesian statistical decision model can be identified with the gc-expected utility model

$$(\Theta^*, \{u^*(\cdot|\theta):\Omega^*_\theta \to R|\theta \in \Theta^*\}, \{\sigma^*(\cdot|\theta):\mathcal{E}^*_\theta \to [0, 1]|\theta \in \Theta^*\})$$

with objective function $U^*:\Theta^* \to R$ given by

$$U^*(R, e, \delta) = \int_{\Omega^*_{(R,e,\delta)}} u^*(\omega|(R, e, \delta))\ \sigma^*(d\omega|(R, e, \delta)), \ (R, e, \delta) \in \Theta^* \ .$$

5.4 THE STANDARD OPTIMAL CONTROL MODEL AND THE GC-EXPECTED UTILITY MODEL COMPARED. It will be shown that the standard, continuous time, fixed terminal time optimal control model can be identified with a particular gc-expected utility model. (Although conceptually and notationally more difficult to present, the stochastic optimal control model could be similarly treated.)

Let a dynamical system be described by a system of ordinary differential equations

$$d\omega(t)/dt = h(\omega(t), \lambda(t)), \ t \in [0, T]; \tag{7}$$

$$\omega(0) = \omega_o ;$$

$$g(\omega(T)) = 0 \in E^m ,$$

where

$\lambda(t) \in E^p$ is the control input at time t ;

$\omega(t) \in E^n$ is the state of the system at time t ;

$\omega_o \in E^n$ is a fixed initial state;

$g: E^n \to E^m$, $m \leq n$, is a terminal constraint function;

T is a fixed terminal time.

The state function $\omega: [0, T] \to E^n$ is often interpreted as a difference $\omega' - \bar{\omega}$, where $\bar{\omega}: [0, T] \to E^n$ is some exogenously given target trajectory. Without loss of generality it is usually assumed that $\bar{\omega}$ is the zero function $\bar{0}:[0, T] \to \{0\} \subseteq E^n$.

Let $B \subseteq E^p$, and let $F:E^n \times E^p \to R$ be a <u>cost function</u>. Then the standard, continuous time, fixed terminal time optimal control problem in terms of the system (7) and the cost function f is as follows (cf L. S. Pontryagin [6]):

> Find a piecewise continuous control $\lambda^o:[0, T] \to B \subseteq E^p$ for the system (7) which minimizes
>
> $$J(\omega_{(\cdot)}, \cdot, \bar{0}) \equiv \int_0^T f((\omega_{(\cdot)} - \bar{0})(t), (\cdot)(t))\, dt \quad .$$

The control model presented above can be identified with a gc-expected utility model as follows. Set

$G^{**} = \{\bar{0}:[0, T] \to \{0\} \subseteq E^n\}$, a one-element candidate goal set consisting of the zero function;

$\Lambda^{**}_{\bar{0}} = \{\lambda:[0, T] \to B | \lambda$ piecewise continuous$\}$, the control set associated with the candidate goal $\bar{0}$;

$\Theta^{**} = (\{\bar{0}\} \times \Lambda^{**}_{\bar{0}}) = \{(\bar{0}, \lambda), \ldots\}$, the policy choice set;

and for each policy $(\bar{0}, \lambda') \in \Theta^{**}$,

$\Omega^{**}_{(\bar{0},\lambda')} = \{\omega_{\lambda'}:[0, T] \to E^n | \omega_{\lambda'}$ satisfies (7) with control $\lambda'\}$, a one-element state flow set;

$\mathcal{E}^{**}_{(\bar{0},\lambda')} = \{\emptyset, \Omega^{**}_{(\bar{0},\lambda')}\}$, the algebra of event flows associated with $(\bar{0}, \lambda')$;

$u^{**}(\cdot | (\bar{0}, \lambda')) : \Omega^{**}_{(\bar{0},\lambda')} \to R$, the utility function associated with $(\bar{0}, \lambda')$, given by $u^{**}(\omega_{\lambda'} | (\bar{0}, \lambda')) = - J(\omega_{\lambda'}, \lambda', \bar{0})$;

$\sigma^{**}(\cdot | (\bar{0}, \lambda')) : \mathcal{E}^{**}_{(\bar{0},\lambda')} \to [0, 1]$, the (trivial) probability measure associated with $(\bar{0}, \lambda')$, given by $\sigma^{**}(\Omega^{**}_{(\bar{0},\lambda')} | (\bar{0}, \lambda')) = 1$.

Then for each policy $(\bar{0}, \lambda) \in \Theta^{**}$,

$$\int_{\Omega^{**}_{(\bar{0},\lambda)}} u^{**}(\omega | (\bar{0}, \lambda)) \, \sigma^{**}(d\omega | (\bar{0}, \lambda)) = - J(\omega_{\lambda}, \lambda, \bar{0}) \quad .$$

Hence the control model presented above can be identified with the gc-expected utility model

$$(\Theta^{**}, \{u^{**}(\cdot | \theta) : \Omega^{**}_{\theta} \to R \,|\, \theta \in \Theta^{**}\}, \{\sigma^{**}(\cdot | \theta) : \mathcal{E}^{**}_{\theta} \to [0, 1] \,|\, \theta \in \Theta^{**}\})$$

with objective function $U^{**} : \Theta^{**} \to R$ given by

$$U^{**}(\bar{0}, \lambda) = \int_{\Omega^{**}_{(\bar{0},\lambda)}} u^{**}(\omega | (\bar{0}, \lambda)) \, \sigma^{**}(d\omega | (\bar{0}, \lambda)) \, , \; (\bar{0}, \lambda) \in \Theta^{**} \quad .$$

6. EXAMPLES

The Construction Firm Example and the Egg Example given below are both formulated in terms of the goal-control expected utility model (see section 4).

The Construction Firm Example (adapted from a case study) illustrates the following three points. First, certain decisions can be given an expected utility rationalization even though the decision maker specifies his available acts in the form of partial rather than complete contingency plans. Second, the choice of an end-mean goal-control pair arises naturally in many decision problems. Third, both choice of goal and choice of control are operationally significant in that each can affect the decision maker's probability and utility judgments concerning future events.

The principal purpose of the Egg Example (a modified version of an egg example by Savage) is to illustrate how the goal-control model primitive sets might be interpreted in terms of certain primitive sets used by Savage (see section 3). The Egg Example also illustrates the three points mentioned above.

6.1 CONSTRUCTION FIRM EXAMPLE (adapted from a case study; see Cyert and March [1, 4.2.2, pages 54 - 60]). The market share of the Home Specialties Department (HSD) for a medium-sized construction firm had been steadily declining for two years, primarily because the department's facilities in the main office building were inadequate in size and badly equipped. The top executives of the construction firm had divided into several factions over what should be done in the long run and in the short run to ameliorate the situation.

Several executives supported moving the HSD to a new location in order to increase its chances of improving its market position. Others supported such a move because they felt it would be easier to eventually ease the department out of the firm. On the other hand, certain executives who believed in the efficiency of centralization argued that the facilities of the HSD should either be improved at the current site or phased out. Still others, heads of expanding departments, were simply anxious to acquire the space used by the HSD one way or another.

A previous attempt to directly phase out the HSD had failed when the HSD head, a powerful senior executive, had opposed it. He had furthermore announced that he would not cooperate in any search for a new location for the HSD without a prior public commitment from the president in support of the long run goal of improving the market position of the HSD. His current attempts to secure backing for his position were seriously disrupting the normal operations of the firm.

In view of this background information, denoted hereafter by "ω^0," the president of the construction firm decided that the time had come to settle the problem. He considered the following candidate (long run) goals, (short run) controls, and derived policies.

Candidate Goal Set

$$G = \left\{ g' : \begin{pmatrix} \text{market share} \\ \text{of HSD} \\ \text{increased} \end{pmatrix} \; ; \; g'' : \begin{pmatrix} \text{HSD phased out} \end{pmatrix} \right\} .$$

Control Sets (one control set Λ_g associated with each candidate goal g)

$$\Lambda_{g'} = \left\{ \lambda_1' : \begin{pmatrix} \text{buy new equipment} \\ \text{for HSD at} \\ \text{current site} \end{pmatrix} \; ; \; \lambda_2' : \begin{pmatrix} \lambda_1', \text{ and} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right. ;$$

$$\left. \lambda_3' : \begin{pmatrix} \text{appoint committee} \\ \text{to search for} \\ \text{new HSD site} \end{pmatrix} ; \; \lambda_4' : \begin{pmatrix} \lambda_3', \text{ and} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right\} ;$$

$$\Lambda_{g''} = \left\{ \lambda_1'' : \begin{pmatrix} \text{attempt to force} \\ \text{HSD head out} \\ \text{of firm} \end{pmatrix} ; \; \lambda_2'' : \begin{pmatrix} \lambda_1'', \text{ and} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right. ;$$

$$\left. \lambda_3'' : \begin{pmatrix} \text{appoint committee} \\ \text{to search for} \\ \text{new HSD site} \end{pmatrix} ; \; \lambda_4'' : \begin{pmatrix} \lambda_3'', \text{ and} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right\} .$$

Policy Choice Set (set of all candidate goal-control pairs)

$$\Theta \equiv \bigcup_{g \in G} (\{g\} \times \Lambda_g)$$

$$= \{(g', \lambda_1'), (g', \lambda_2'), (g', \lambda_3'), (g', \lambda_4')\}$$

$$\cup \{(g'', \lambda_1''), (g'', \lambda_2''), (g'', \lambda_3''), (g'', \lambda_4'')\}$$

$$= \{\theta, \ldots\} .$$

For each policy θ' in Θ the president asked himself the following question: "What distinct situations (i.e., state flows $\omega_{\theta'}$) might obtain if I implement policy θ'?"

<u>State Flow Sets</u> (one set Ω_θ of state flows associated with each policy $\theta \in \Theta$)

$$\text{For policies } \left(g' : \begin{pmatrix} \text{market share} \\ \text{of HSD} \\ \text{increased} \end{pmatrix} ; \quad \lambda_1' : \begin{pmatrix} \text{buy new equipment} \\ \text{for HSD at} \\ \text{current site} \end{pmatrix} \right)$$

$$\text{and } \left(g', \ \lambda_2' : \begin{pmatrix} \lambda_1', \text{ and} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right) :$$

$$\Omega_{(g',\lambda_1')} = \Omega_{(g',\lambda_2'')}$$

$$= \left\{ \omega^o \right\} \times \left\{ m_1 : \begin{pmatrix} \text{HSD market} \\ \text{share declines} \\ \text{further} \end{pmatrix}, \ m_2 : \begin{pmatrix} \text{HSD market} \\ \text{share} \\ \text{stabilizes} \end{pmatrix}, \ m_3 : \begin{pmatrix} \text{HSD market} \\ \text{share} \\ \text{increases} \end{pmatrix} \right\}$$

$$\times \left\{ n_1 : \begin{pmatrix} \text{negative disruptive} \\ \text{attitude of other} \\ \text{departments towards} \\ \text{HSD continues} \end{pmatrix}, \ n_2 : \begin{pmatrix} \text{negative disruptive} \\ \text{attitude of other} \\ \text{departments towards} \\ \text{HSD dissipates} \end{pmatrix} \right\},$$

where ω^o is the "information state" of the president (see above).

For policies $\left(g': \begin{pmatrix}\text{market share}\\ \text{of HSD}\\ \text{increased}\end{pmatrix},\ \lambda_3': \begin{pmatrix}\text{appoint committee}\\ \text{to search for}\\ \text{new HSD site}\end{pmatrix}\right)$

and $\left(g', \lambda_4': \begin{pmatrix}\lambda_3', \text{ and}\\ \text{announce}\\ \text{goal}\end{pmatrix}\right)$:

$$\Omega_{(g',\lambda_3')} = \Omega_{(g',\lambda_4')}$$

$$= \left\{\omega^o\right\} \times \left\{S_1 : \begin{pmatrix}\text{search}\\ \text{fails}\end{pmatrix},\ S_2 : \begin{pmatrix}\text{search}\\ \text{succeeds}\end{pmatrix}\right\} \times \left\{m_1, m_2, m_3\right\} \times \left\{n_1, n_2\right\} .$$

For policies $\left(g'': \begin{pmatrix}\text{HSD}\\ \text{phased}\\ \text{out}\end{pmatrix},\ \lambda_1'': \begin{pmatrix}\text{attempt to}\\ \text{force HSD head}\\ \text{out of firm}\end{pmatrix}\right)$

and $\left(g'', \lambda_2'': \begin{pmatrix}\lambda_1'', \text{ and}\\ \text{announce}\\ \text{goal}\end{pmatrix}\right)$:

$$\Omega_{(g'', \lambda_1'')} = \Omega_{(g'', \lambda_2'')}$$

$$= \left\{\omega^o\right\} \times \left\{f_1 : \begin{pmatrix}\text{force out}\\ \text{attempt}\\ \text{fails}\end{pmatrix},\ f_2 : \begin{pmatrix}\text{force out}\\ \text{attempt}\\ \text{succeeds}\end{pmatrix}\right\} \times \left\{n_1, n_2\right\}$$

$$\times \left\{P_1 : \begin{pmatrix}\text{phasing out}\\ \text{of HSD made}\\ \text{more difficult}\end{pmatrix};\ P_2 : \begin{pmatrix}\text{way opened for}\\ \text{smooth phasing}\\ \text{out of HSD}\end{pmatrix}\right\} .$$

For policies $\left(g'' : \begin{pmatrix}\text{HSD}\\ \text{phased}\\ \text{out}\end{pmatrix},\ \lambda_3'' : \begin{pmatrix}\text{appoint committee}\\ \text{to search for a}\\ \text{new site for HSD}\end{pmatrix}\right)$

and $\left(g'',\ \lambda_4'' : \begin{pmatrix}\lambda_3'',\ \text{and}\\ \text{announce}\\ \text{goal}\end{pmatrix}\right)$:

$$\Omega_{(g'',\lambda_3'')} = \Omega_{(g'',\lambda_4'')}$$
$$= \left\{\omega^o\right\} \times \left\{h_1 : \begin{pmatrix}\text{HSD head}\\ \text{fights}\\ \text{move}\end{pmatrix},\ h_2 : \begin{pmatrix}\text{HSD head}\\ \text{cooperates}\\ \text{in move}\end{pmatrix}\right\}$$
$$\times \left\{S_1, S_2\right\} \times \left\{n_1, n_2\right\} \times \left\{P_1, P_2\right\} .$$

<u>Event Flow Algebras</u>

For each $\theta \in \Theta$, let $\mathcal{E}_\theta = 2^{\Omega_\theta}$.

The policy chosen by the president was

$$\left(g' : \begin{pmatrix}\text{market share}\\ \text{of HSD}\\ \text{increased}\end{pmatrix},\ \lambda_4' : \begin{pmatrix}\text{appoint committee}\\ \text{to search for new}\\ \text{HSD site, and}\\ \text{announce goal}\end{pmatrix}\right) .$$

This choice might have been rationalized as follows (see section 4). To each state flow $\omega' \in \Omega_{\theta'}$, $\theta' \in \Theta$, the president assigned a utility number $u(\omega'|\theta')$ representing the desirability of $\{\omega'\}$ obtaining, conditioned on the event "decision maker chooses θ'," and a probability number $\sigma(\{\omega'\}|\theta')$ representing the likelihood of $\{\omega'\}$ obtaining, conditioned on the event "decision maker chooses θ'." He then calculated the expected utility

$$\int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta')$$

corresponding to each policy $\theta' \in \Theta$, and chose the policy yielding the maximum expected utility.

Discussion. The president's probability and utility judgments concerning various event and state flows can reasonably be assumed to have depended on both his contemplated goal and his contemplated control. In support of this claim with respect to probabilities, consider the controls

$$\lambda_4' : \begin{pmatrix} \text{appoint committee} \\ \text{to search for new} \\ \text{HSD site;} \\ \text{announce goal} \end{pmatrix} \in \Lambda_{g'} ; \quad \lambda_4'' : \begin{pmatrix} \text{appoint committee} \\ \text{to search for new} \\ \text{HSD site;} \\ \text{announce goal} \end{pmatrix} \in \Lambda_{g''} ;$$

and the event flows

$$E' : (\{\omega^o\} \times \{S_1\} \times \{m_1, m_2, m_3\} \times \{n_1, n_2\}) \in \mathcal{E}_{(g', \lambda_4')} \quad ;$$

$$E'' : (\{\omega^o\} \times \{h_1, h_2\} \times \{S_1\} \times \{n_1, n_2\} \times \{P_1, P_2\} \in \mathcal{E}_{(g'', \lambda_4'')} \quad .$$

Since the two controls λ_4', λ_4'' and the two event flows E', E'' are logically identical in content, without loss of generality let

$$\lambda^* : \begin{pmatrix} \text{appoint committee} \\ \text{to search for new} \\ \text{HSD site;} \\ \text{announce goal} \end{pmatrix} \equiv \lambda_4' \equiv \lambda_4'' \quad ;$$

$$E^* : \left(\{\omega^o\} \times \{S_1 : \begin{pmatrix} \text{search} \\ \text{successful} \end{pmatrix}\}\right) \equiv E' \equiv E'' \quad .$$

By assumption $\{\omega^o\}$ is the president's current information state with respect to the HSD problem; thus presumably

$$\sigma(\{\omega^o\} | (g', \lambda^*)) = \sigma(\{\omega^o\} | (g'', \lambda^*)) = 1 \quad ,$$

and

$$\sigma(E^* | (g', \lambda^*)) = \text{Prob}(\{S_1\} | \omega^o, (g', \lambda^*)) \cdot \sigma(\{\omega^o\} | (g', \lambda^*))$$

$$= \text{Prob}(\{S_1\} | \omega^o, (g', \lambda^*)) \quad ; \tag{8}$$

$$\sigma(E^* | (g'', \lambda^*)) = \text{Prob}(\{S_1\} | \omega^o, (g'', \lambda^*)) \cdot \sigma(\{\omega^o\} | (g'', \lambda^*))$$

$$= \text{Prob}(\{S_1\} | \omega^o, (g'', \lambda^*)) \quad .$$

Since ω^o contains the information that the powerful HSD head will not cooperate in the search for a new HSD site unless g' : (market share of HSD increased) is announced by the president as his chosen goal, it seems plausible to assume that

$$\text{Prob}(\{S_1 : \begin{pmatrix} \text{search} \\ \text{successful} \end{pmatrix}\} | \omega^o, (g', \lambda^*)) \tag{9}$$

$$> \text{Prob}(\{S_1 : \begin{pmatrix} \text{search} \\ \text{successful} \end{pmatrix}\} | \omega^o, (g'', \lambda^*)) \quad .$$

In combination with (8), inequality (9) implies that

$$\sigma(E^*|(g', \lambda^*)) > \sigma(E^*|(g'', \lambda^*)) \quad ;$$

i.e., for fixed choice of control $\lambda^* \equiv \lambda_4' \equiv \lambda_4''$, the president assigns greater probability to $E^*: (\{\omega^o\} \times \{S_1 : \text{(search successful)}\}) \in \mathcal{E}_{(g', \lambda_4')} \cap \mathcal{E}_{(g'', \lambda_4'')}$ when contemplating the goal g' : (market share of HSD increased) than when contemplating the goal g'' : (HSD phased out) .

Similarly, it seems plausible to assume that

$$\sigma(E^*|(g', \lambda_4')) > \sigma(E^*|(g', \lambda_3')) \quad ;$$

i.e., that the president assigns more probability to E^* : $(\{\omega^o\} \times \{S_1$: (search successful)) under the policy choice (g', λ_4') than under the policy choice (g', λ_3') . For λ_3' : (appoint committee to search for new HSD site) and λ_4' : (λ_3', and announce goal) differ only in the announcement of the goal. Yet, as the president knows from $\{\omega^o\}$, the HSD head has stated that he will not support the search for a new site unless the president publicly commits himself to g' .

In support of the contention that control specification affects utility judgments concerning future events, suppose that in place of the control λ_1' : (buy new equipment for HSD at current site) the president had specified the more detailed controls

$$\lambda'_{11}:\begin{pmatrix}\text{buy expensive brand}\\ \text{X equipment for}\\ \text{HSD at current site}\end{pmatrix}\;;\quad \lambda'_{12}:\begin{pmatrix}\text{buy cheaper brand}\\ \text{Y equipment for}\\ \text{HSD at current site}\end{pmatrix}\;,$$

with identical associated state flow sets $\Omega_{(g',\lambda'_{11})} = \Omega_{(g',\lambda'_{12})} = \Omega_{(g',\lambda'_1)}$. Consider the state flow

$$\omega^*:\left(\{\omega^o\},m_3:\begin{pmatrix}\text{HSD market}\\ \text{share increases}\end{pmatrix},\ n_2:\begin{pmatrix}\text{negative disruptive}\\ \text{attitude of other}\\ \text{departments towards}\\ \text{HSD dissipates}\end{pmatrix}\right),$$

an element of $\Omega_{(g',\ \lambda'_{11})} \cap \Omega_{(g',\ \lambda'_{12})}$. It seems plausible to assume that

$$u(\omega^*|(g',\ \lambda'_{12})) > u(\omega^*|(g',\ \lambda'_{11})) \quad ;$$

i.e., that for "fixed output" ω^* , the president prefers to minimize costs.

Finally, in support of the contention that goal specification affects utility judgments concerning future events, it should first be noted that satisficing search models, control models, and econometric policy models accept this as commonplace. The specified goals (target trajectories) play the role of "aspiration levels" in terms of which the effectiveness of alternative controls is evaluated. Utility (cost) is a function of the "distance" between the state flow which obtains and the desired goal. (In the gc-expected utility representation for the control model established above in 5.4, the target trajectory is an "ideal" state flow.)

Returning to the Construction Firm Example, consider the goal $g'' : (\text{HSD phased out})$, the control

$$\lambda_2'' : \begin{pmatrix} \text{attempt to force} \\ \text{HSD head out} \\ \text{of firm, and} \\ \text{announce goal} \end{pmatrix} \in \Lambda_{g''} \quad ,$$

and the state flow

$$\omega'' : \left(\{\omega^o\},\ f_1 : \begin{pmatrix} \text{force} \\ \text{out} \\ \text{attempt} \\ \text{fails} \end{pmatrix};\ n_1 : \begin{pmatrix} \text{negative dis-} \\ \text{ruptive attitude} \\ \text{of other depart-} \\ \text{ments towards} \\ \text{HSD continues} \end{pmatrix},\ P_1 : \begin{pmatrix} \text{phasing} \\ \text{out of HSD} \\ \text{made more} \\ \text{difficult} \end{pmatrix} \right),$$

an element of $\Omega_{(g'',\lambda_2'')}$. Once the president publicly commits himself to a goal, it seems reasonable to assume that he views the attainment of that goal as a measure of his effectiveness as a top executive. Suppose in addition to g'' the president had also considered the candidate goal

$$g^o : \begin{pmatrix} \text{HSD head phased out if six} \\ \text{month trial run with new HSD} \\ \text{head doesn't improve HSD} \\ \text{market share} \end{pmatrix} \quad ,$$

with control set $\Lambda_{g^o} = \Lambda_{g''}$ and state flow set $\Omega_{(g^o,\lambda_2'')} = \Omega_{(g'',\lambda_2'')}$. Then, under ω'', the event "new HSD head" does not obtain, and the goal g^o simply becomes irrelevant. In contrast, under ω'' the goal g'' is directly blocked. It thus seems plausible that

$$u(\omega'' | (g^o, \lambda_2'')) > u(\omega'' | (g'', \lambda_2'')) \quad .$$

In order to give an alternative, state-consequence-act formulation for the HSD problem, it would be necessary to specify the choice set in terms of functions mapping states into consequences. For example, it might be assumed that the "state" s:(no suitable new location for HSD available), the "act" λ:(appoint committee to search for new HSD site, and announce goal), and the "consequence" c:(search fails and disruption in firm continues) satisfy the functional relationship

$$\lambda(s) = c \quad .$$

In the actual case study the control λ was implemented by the president; but, even if s had obtained, a unique consequence would not have been determined by s and λ . An unforeseen event s':(market share position of HSD stabilized due to external market factors) dissipated the disruption in the firm before the search was even concluded. Since relevant but unforeseen events such as s' commonly arise in real world decision problems, the specification of real world actions in the form of functions taking states into consequences would generally involve some immeasurable amount of approximation.

Secondly, the specification of such functions in effect requires the decision maker to behave as if he believed that certain conditioned events had probability one (e.g., Prob(c given s and λ) = 1) . For many decision problems (e.g., the HSD problem) this requirement seems to entail a significant distortion of the actual decision making process.

As the gc-formulation of the HSD problem demonstrates, certain decisions can be given an expected utility rationalization even though the decision maker specifies his available actions in the form of partial rather than complete contingency plans.

6.2 EGG EXAMPLE. The principal purpose of the following simple example, a modified version of an "egg problem" by Savage [9, pages 13 - 15], is to illustrate how the goal-control model primitive sets might be interpreted in terms of certain primitive sets used by Savage (see section 3). Comparison of the two egg problems may clarify a major distinction between the goal-control expected utility model and the Savage expected utility model. In Savage's egg problem, relevant but "unobservable" sets such as S_1, C_1, and F_1 (see below) cannot exist; for otherwise the elements available for choice (e.g., the elements of F_o below) do not functionally map the observable set of states into the observable set of consequences.

The egg problem also illustrates the three points listed in the introduction to section 6.

A decision maker breaks five good eggs into a pan on the kitchen stove, and then decides to instruct his assistant to complete the omelet. A sixth egg, which for some reason must be used for the omelet or discarded, lies unbroken in the kitchen icebox located next to a wastebasket and across the kitchen from the stove.

Assume the full problem can be represented in terms of Savage's primitive sets (S, C, F) as follows:

<u>Set of States</u>

$$S = \left\{ s' : \begin{pmatrix}\text{sixth}\\ \text{egg}\\ \text{rotten}\end{pmatrix},\ s'' : \begin{pmatrix}\text{sixth}\\ \text{egg}\\ \text{good}\end{pmatrix} \right\}$$

$$\times \left\{ s''' : \begin{pmatrix}\text{center of}\\ \text{kitchen}\\ \text{floor}\\ \text{slippery}\end{pmatrix},\ s'''' : \begin{pmatrix}\text{center of}\\ \text{kitchen}\\ \text{floor not}\\ \text{slippery}\end{pmatrix} \right\}$$

$$\equiv S_0 \times S_1 \quad .$$

<u>Set of Consequences</u>

$$C = \left\{ g' : \begin{pmatrix}\text{ruined}\\ \text{six egg}\\ \text{omelet}\end{pmatrix},\ g'' : \begin{pmatrix}\text{tasty}\\ \text{six egg}\\ \text{omelet}\end{pmatrix},\right.$$

$$\left. g''' : \begin{pmatrix}\text{tasty five}\\ \text{egg omelet}\\ \text{and good}\\ \text{sixth egg}\\ \text{destroyed}\end{pmatrix},\ g'''' : \begin{pmatrix}\text{tasty five}\\ \text{egg omelet}\\ \text{and bad}\\ \text{sixth egg}\\ \text{destroyed}\end{pmatrix} \right\}$$

$$\times \left\{ \begin{pmatrix}\text{one egg}\\ \text{mess on}\\ \text{the floor}\end{pmatrix},\ \begin{pmatrix}\text{no one}\\ \text{egg mess}\\ \text{on floor}\end{pmatrix} \right\}$$

$$\equiv C_0 \times C_1 \quad .$$

Set of Acts

$$F = \left\{ f' : \begin{pmatrix} \text{tell assistant} \\ \text{to make a sixth} \\ \text{egg omelet} \end{pmatrix}, \quad f'' : \begin{pmatrix} \text{tell assistant} \\ \text{to throw sixth} \\ \text{egg away and} \\ \text{make a five} \\ \text{egg omelet} \end{pmatrix}, \right.$$

$$\left. f''' : \begin{pmatrix} f', \text{ plus} \\ \text{announce} \\ \text{goal} \end{pmatrix}, \quad f'''' : \begin{pmatrix} f'', \text{ plus} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right\}$$

$$\times \left\{ \begin{pmatrix} \text{tell assistant} \\ \text{to take central} \\ \text{route from} \\ \text{icebox to} \\ \text{stove} \end{pmatrix}, \begin{pmatrix} \text{tell assistant} \\ \text{to take non-} \\ \text{central route} \\ \text{from icebox} \\ \text{to stove} \end{pmatrix} \right\}$$

$$\equiv F_0 \times F_1 \quad .$$

Now assume that the decision maker is aware of the possibilities listed in the sets S_0, C_0, and F_0, but either through ignorance or considerations of time and cost he does not consider the possibilities listed in the sets S_1, C_1, and F_1. Moreover, assume that he realizes that his description of his problem in terms of S_0, C_0, and F_0 is partial; in particular, the elements in F_0 are not functions mapping each element in S_0 uniquely into an element of C_0. Hence, by telling the assistant his goal, the decision maker might be able to increase the likelihood that the assistant will act in conformity with the decision maker's wishes in the face of unforeseen circumstances.

Assume that the decision maker has decided to model his problem in terms of a goal-control expected utility model. In particular, let the candidate goals, controls and derived policies considered by the decision maker be as follows.

Set of Candidate Goals

$$G = \left\{ g'' : \begin{pmatrix} \text{tasty} \\ \text{six egg} \\ \text{omelet} \end{pmatrix}, \quad g'''' : \begin{pmatrix} \text{tasty five} \\ \text{egg omelet} \\ \text{and bad sixth} \\ \text{egg destroyed} \end{pmatrix} \right\} \subseteq C_0 \ .$$

Control Sets (one control set associated with each candidate goal)

$$\Lambda_{g''} = \left\{ f' : \begin{pmatrix} \text{tell assistant} \\ \text{to make a six} \\ \text{egg omelet} \end{pmatrix}, \ f''' : \begin{pmatrix} f', \text{ plus} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right\} \subseteq F_0 \ ;$$

$$\Lambda_{g''''} = \left\{ f'' : \begin{pmatrix} \text{tell assistant} \\ \text{to throw sixth} \\ \text{egg away and} \\ \text{make a five} \\ \text{egg omelet} \end{pmatrix}, \ f'''' : \begin{pmatrix} f'', \text{ plus} \\ \text{announce} \\ \text{goal} \end{pmatrix} \right\} \subseteq F_0 \ .$$

Policy Choice Set (set of all candidate goal-control pairs)

$$\Theta \equiv \bigcup_{g \in G} (\{g\} \times \Lambda_g)$$

$$= \{(g'', f'), (g'', f''')\} \cup \{(g'''', f''), (g'''', f'''')\}$$

$$= \{\theta, . . .\} \ .$$

For each policy $\theta \in \Theta$, the decision maker asks himself the following question: "What distinct situations (i.e., state flows) might obtain if I choose θ?"

<u>State Flow Sets</u> (one set Ω_θ of state flows for each policy $\theta \in \Theta$)

$$\text{For policies } \left(g'' : \begin{pmatrix}\text{tasty}\\ \text{six egg}\\ \text{omelet}\end{pmatrix},\ f' : \begin{pmatrix}\text{tell assistant}\\ \text{to make a six}\\ \text{egg omelet}\end{pmatrix}\right)$$

$$\text{and } \left(g'',\ f''' : \begin{pmatrix}f',\ \text{plus}\\ \text{announce}\\ \text{goal}\end{pmatrix}\right):$$

$$\Omega_{(g'',f')} = \Omega_{(g'',f''')}$$

$$= \left\{\begin{pmatrix}\text{sixth}\\ \text{egg}\\ \text{rotten}\end{pmatrix}, \begin{pmatrix}\text{sixth}\\ \text{egg}\\ \text{good}\end{pmatrix}\right\} \times \left\{\begin{pmatrix}\text{ruined six}\\ \text{egg omelet}\\ \text{obtains}\end{pmatrix}, \begin{pmatrix}\text{tasty six}\\ \text{egg omelet}\\ \text{obtains}\end{pmatrix}, \begin{pmatrix}\text{no six}\\ \text{egg omelet}\\ \text{obtains}\end{pmatrix}\right\};$$

$$\text{and for policies } \left(g'''' : \begin{pmatrix}\text{tasty five}\\ \text{egg omelet}\\ \text{and bad}\\ \text{sixth egg}\\ \text{destroyed}\end{pmatrix},\ f'' : \begin{pmatrix}\text{tell assistant}\\ \text{to throw sixth}\\ \text{egg away and}\\ \text{make a five}\\ \text{egg omelet}\end{pmatrix}\right)$$

$$\text{and } \left(g'''',\ f'''' : \begin{pmatrix}f'',\ \text{plus}\\ \text{announce}\\ \text{goal}\end{pmatrix}\right):$$

$$\Omega_{(g'''',f'')} = \Omega_{(g'''',f'''')}$$

$$= \left\{\begin{pmatrix}\text{sixth}\\ \text{egg}\\ \text{rotten}\end{pmatrix}, \begin{pmatrix}\text{sixth}\\ \text{egg}\\ \text{good}\end{pmatrix}\right\} \times \left\{\begin{pmatrix}\text{tasty five}\\ \text{egg omelet}\\ \text{and good}\\ \text{sixth egg}\\ \text{destroyed}\end{pmatrix}, \begin{pmatrix}\text{tasty five}\\ \text{egg omelet}\\ \text{and bad}\\ \text{sixth egg}\\ \text{destroyed}\end{pmatrix}, \begin{pmatrix}\text{no tasty}\\ \text{five egg}\\ \text{omelet}\\ \text{obtains}\end{pmatrix}\right\}.$$

Event Flow Algebras

For each $\theta \in \Theta$, let $\mathcal{E}_\theta = 2^{\Omega_\theta}$.

To each state flow $\omega' \in \Omega_{\theta'}$, $\theta' \in \Theta$, the decision maker assigns a utility number $u(\omega'|\theta')$ representing the desirability of $\{\omega'\}$ obtaining, conditioned on the event "decision maker chooses θ'," and a probability number $\sigma(\{\omega'\}|\theta')$ representing the likelihood of $\{\omega'\}$ obtaining, conditioned on the event "decision maker chooses θ'." He then calculates his expected utility

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta$$

corresponding to each choice of policy $\theta \in \Theta$, and chooses a policy which yields maximum expected utility; e.g.,

$$\left(g'': \begin{pmatrix}\text{tasty six}\\ \text{egg}\\ \text{omelet}\end{pmatrix},\quad f''': \begin{pmatrix}\text{tell assistant}\\ \text{to make a six egg}\\ \text{omelet, plus}\\ \text{announce goal}\end{pmatrix}\right).$$

In keeping with this choice, the decision maker tells the assistant to make a six egg omelet; and in addition he informs him that he would like the omelet to be tasty.

When the assistant later enters the kitchen, he notices an aspect of the true world state whose possible realization the decision maker has overlooked or ignored; namely,

$$s''': \begin{pmatrix}\text{center of}\\ \text{kitchen floor}\\ \text{slippery}\end{pmatrix} \in S_1 .$$

The assistant is momentarily unsure whether to take the central, slippery, timesaving route or the noncentral, nonslippery time-consuming route from the kitchen icebox to the kitchen stove, with the sixth egg in hand. Nevertheless, upon consulting his instructions, he observes that the decision maker's goal is to have a tasty six egg omelet, not a fast six egg omelet. Hence there is no reason to attempt the central, slippery, time-consuming route and risk ending up with a five instead of six egg omelet by way of a one-egg mess on the floor.

Remarks. The gc-model primitive sets (i.e., control, goal, state and event flow sets) are constructed from the observable components S_0, C_0 and F_0 of the Savage primitive sets $S \equiv S_0 \times S_1$, $C \equiv C_0 \times C_1$, and $F \equiv F_0 \times F_1$. Specification of the unobservable (but relevant) sets S_1, C_1, and F_1 is not required.

The elements of S_0, C_0, and F_0 are in natural correspondence with the subsets of S, C, and F. For example, the element s':(sixth egg rotten) in S_0 can be identified with the element

$$\{s'\} \times \{s''': \begin{pmatrix}\text{center of} \\ \text{floor slippery}\end{pmatrix}, \; s'''': \begin{pmatrix}\text{center of floor} \\ \text{not slippery}\end{pmatrix}\}$$

in 2^S. Under this correspondence each state flow $\omega_\theta \in \Omega_\theta$, $\theta \in \Theta$, is an element of $2^S \times 2^C$ as in section 3.

FOOTNOTES

[1]A binary relation $\succcurlyeq$ on a set D is a <u>weak order</u> if for all a, b, c $\in$ D

(1) $a \succcurlyeq b$ or $b \succcurlyeq c$ (i.e., $\succcurlyeq$ is connected);

(2) $a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$
(i.e., $\succcurlyeq$ is transitive).

Weak orders have also been referred to as "complete preorderings."

[2]A collection $\mathcal{F}$ of subsets of a nonempty set X is said to be an <u>algebra</u> in X if $\mathcal{F}$ has the following three properties:

(1) $X \in \mathcal{F}$;

(2) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where A^c is the complement of A relative to X;

(3) If $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

[3]This observation is due to C. Hildreth.

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AXIOMATIZATION FOR AN EXPECTED UTILITY MODEL WITH ENDOGENOUSLY DETERMINED GOALS*

by

Leigh Tesfatsion

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ABSTRACT

In [7] a "goal-control expected utility model" was formulated which allows the decision maker to specify his acts in the form of "controls" (partial contingency plans) and to simultaneously choose goals and controls in end-mean pairs. It was shown that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard optimal control model can be viewed as special cases of this model.

In this paper the goal-control expected utility representation for the goal-control model primitives is axiomatized.

1. INTRODUCTION

In [7] a "goal-control expected utility model" was formulated which allows the decision maker to specify his acts in the form of "controls" (partial contingency plans) and to simultaneously choose goals and controls in end-mean pairs. It was shown that the Savage expected utility model, the Marschak-Radner team model, the Bayesian statistical decision model, and the standard optimal control model can be viewed as special cases of this model.

In this paper the expected utility representation for the goal-control model primitives is axiomatized. The primitives are reviewed in order to make this paper reasonably self-contained. However, for a detailed discussion of the goal-control model together with examples illustrating the expected utility representation, the reader is referred to [7].

2. PRIMITIVES FOR THE GOAL-CONTROL MODEL

Let $G = \{g, \ldots\}$ be a set of *candidate goals*, and for each $g \in G$ let $\Lambda_g = \{\lambda_g, \ldots\}$ be a set of *controls*. The primitives for the *goal-control model* ("gc-model") are then characterized by a vector

$$(<\Theta, \succcurlyeq>, \{<\Omega_\theta, \succcurlyeq_\theta> \mid \theta \in \Theta\}, \{<\mathcal{E}_\theta, \geq_\theta> \mid \theta \in \Theta\})$$

where

$\Theta = \{\theta, \ldots\} = \cup_{g \in G} \; \{(g, \lambda_g) \mid \lambda_g \in \Lambda_g\}$ is the *policy choice set* consisting of candidate goal-control pairs (*policies*);

$\succcurlyeq$ (*policy preference order*) is a weak order[1] on Θ;

and for each policy $\theta \in \Theta$,

$\Omega_\theta = \{\omega_\theta, \ldots\}$ is a nonempty set of *state flows* associated with the policy θ;

$\succcurlyeq_\theta$ (*θ-conditioned preference order*) is a weak order on Ω_θ;

$\mathcal{E}_\theta = \{E_\theta, \ldots\}$ is an algebra[2] of subsets of Ω_θ whose elements E_θ will be called *event flows* associated with the policy θ;

$\geq_\theta$ (*θ-conditioned probability order*) is a weak order on $\mathcal{E}_\theta$.

The controls may be operationally interpreted as possibly conditioned sequences of actions (i.e., partial contingency plans) entirely under the control of the decision maker at the time of his choice. The candidate goals $g \in G$ may be operationally interpreted as potential objectives (e.g., production targets) whose realization the decision maker can attempt to achieve through appropriate choice of a control. The grouping of the controls into sets $\{\Lambda_g \mid g \in G\}$ reflects the possibility that different sets of controls may be relevant for different goals; e.g., for a decision maker in San Francisco, the control "travel by bus" is suitable for the goal "vacation in Los Angeles" but not for the goal "vacation in Hawaii." A control $\lambda_g \in \Lambda_g$ may or may not provide for the communication of the goal g to other persons in the decision maker's problem environment.

The weak order $\succcurlyeq$ on Θ can be operationally interpreted as a preference order as follows. For all $\theta', \theta'' \in \Theta$,

$$\theta' \succcurlyeq \theta'' \Leftrightarrow \text{the choice of policy } \theta' \text{ is at least as desirable to the decision maker as the choice of policy } \theta''.$$

The decision maker is assumed to choose a policy (candidate goal-control pair) $\theta' \in \Theta$ which is optimal in the sense that $\theta' \succcurlyeq \theta$ for all $\theta \in \Theta$. Throughout this paper we use "choose policy $\theta = (g, \lambda_g)$" and "implement control λ_g with g as the objective" interchangeably.

For each $\theta \in \Theta$, the set Ω_θ of state flows ω_θ can be interpreted as the decision maker's answer to the following question: "If I choose policy θ, what distinct situations (i.e., state flows ω_θ) might obtain?" The state flows may include references to past, present, and future happenings. In order for subsequent probability assessments to be realistically feasible, the state flow sets should include the decision maker's background information concerning the problem at hand.

The θ-conditioned preference orders $\succcurlyeq_\theta$ can be interpreted as follows. For all $\omega, \omega' \in \Omega_\theta$, $\theta \in \Theta$,

$\omega \succcurlyeq_\theta \omega' \Leftrightarrow$ the realization of ω is at least as desirable to the decision maker as the realization of ω', given the event "decision maker chooses θ."

Similarly, the θ-conditioned probability orders $\geq_\theta$ can be interpreted as follows. For all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$,

$E \geq_\theta E' \Leftrightarrow$ in the judgment of the decision maker, the realization of E is as likely as the realization of E', given the event "decision maker chooses θ."

A state flow ω may be relevant for the decision maker's problem under distinct potential policy choices; e.g., $\omega \in \Omega_\theta \cap \Omega_{\theta'}$, for some $\theta, \theta' \in \Theta$. Similarly, the algebras

$\{\mathcal{E}_\theta\}$ may overlap. Given state flows $\omega, \omega' \in \Omega_\theta \cap \Omega_{\theta'}$ for some $\theta, \theta' \in \Theta$, it may hold that $\omega \succcurlyeq_\theta \omega'$ whereas $\omega' \succcurlyeq_{\theta'} \omega$. Verbally, the relative utility of the state flows ω and ω' may depend on which conditioning event the decision maker is considering, "decision maker chooses θ" or "decision maker chooses θ'." Similarly for the relative likelihood of event flows $E, E' \in \mathcal{E}_\theta \cap \mathcal{E}_{\theta'}$, $\theta, \theta' \in \Theta$.

3. AXIOMATIZATION: INTRODUCTION

In sections 4 and 5 axioms will be given which ensure that the gc-model has an expected utility representation in the following sense: To each policy $\theta \in \Theta$ there corresponds a finitely additive probability measure $\sigma(\cdot \mid \theta) : \mathcal{E}_\theta \to [0, 1]$ satisfying

$$\sigma(E \mid \theta) \geq \sigma(E' \mid \theta) \Leftrightarrow E \geq_\theta E' , \qquad (1)$$

for all $E, E' \in \mathcal{E}_\theta$, and a utility function $u(\cdot \mid \theta) : \Omega_\theta \to R$ satisfying

$$u(\omega \mid \theta \geq u(\omega' \mid \theta) \Leftrightarrow \omega \succcurlyeq_\theta \omega' , \qquad (2)$$

for all $\omega, \omega' \in \Omega_\theta$, such that

$$\int_{\Omega_\theta} u(\omega \mid \theta)\, \sigma(d\omega \mid \theta) \geq \int_{\Omega_{\theta'}} u(\omega \mid \theta')\, \sigma(d\omega \mid \theta') \Leftrightarrow \theta \succcurlyeq \theta' , \qquad (3)$$

for all $\theta, \theta' \in \Theta$.

This expected utility representation for the policy preference order $\succcurlyeq$ can be interpreted as follows. To each state flow $\omega \in \Omega_\theta$, $\theta \in \Theta$, the decision maker assigns a utility number $u(\omega|\theta)$ representing the desirability of $\{\omega\}$ obtaining, conditioned on the event "decision maker chooses θ," and a probability number $\sigma(\{\omega\}|\theta)$ representing the likelihood of $\{\omega\}$ obtaining, conditioned on the event "decision maker chooses θ." He then calculates the expected utility

$$\int_{\Omega_\theta} u(\omega \mid \theta)\ \sigma(d\omega \mid \theta)$$

corresponding to each choice of policy $\theta \in \Theta$, and chooses a policy which yields maximum expected utility.

Before beginning the statement of axioms, it might be helpful to briefly discuss the relationship of the gc-expected utility model axiomatization to previously established axiomatizations.

Ideally, an expected utility axiomatization should be calculationally feasible and all the primitives should be relevant for the decision maker's problem. In actuality, most expected utility axiomatizations extend the "basic primitives" (i.e., the primitives essential for the decision maker's problem) for mathematical reasons, and this extension often implies an impossible calculational ability on the part of the decision maker. (See Fishburn [1] and Krantz, Luce <u>et.al.</u> [3] for reviews of the expected utility literature.)

In certain axiomatizations the basic primitive sets are explicitly extended by introducing over these sets a collection of extraneous gambles, usually infinite in number, which the decision maker is required to order in preference. In other axiomatizations the basic primitive sets are implicitly extended. For example, in the expected utility model of L. Savage [4] the primitive sets consist of a set S of "states of the world," a set C of "consequences," and a set F containing <u>all</u> "acts" (functions) taking S into C . For most decision problems the presence of constant functions in F represents an extension of the basic primitive set of acts available to the decision maker. In addition, Savage's axioms require S to be uncountably infinite. Since the decision maker is assumed to order in preference all functions in F , the uncountability of S introduces a calculational infeasibility.

Reliance on a single primitive preference order seems to be the principal reason why extraneous elements are introduced into the primitives of most individual choice models. In order for a preference order over consequences to be derived from a primitive preference order over a set B of acts or gambles, the consequence set must somehow be imbedded into B ; e.g., through constant acts or degenerate gambles. Similarly, in order for probability judgments to be assessed from a primitive preference order over B , the acts or gambles in B must be suitably varied.

In contrast to most individual choice models, the gc-model primitives include three different types of orders whose existence is implied by the desired expected utility representation: a policy preference order $\succcurlyeq$, θ-conditioned preference orders $\succcurlyeq_\theta$, and θ-conditioned probability orders $\geq_\theta$. Consequently, the expected utility representation is obtained under minimal restrictions on the basic primitives.

Specifically, in section 4 the representations (2) and (3) are established under three assumptions (Axioms I - III) which include the temporary assumption (Axiom I) that probability representations satisfying (1) have been obtained. Axiom II is a finiteness restriction on the state flow sets. Axiom III requires the decision maker's primitive preference and probability orders to be compatible with the existence of a certain weak order over a mixture set constructed from primitive elements. As will be discussed in 4.2 below, this weak order can (but need not) be interpreted as a preference order over extraneous gambles. Given Axioms I and II, Axiom III will be shown (4.4) to be necessary and sufficient for the existence of the desired representations (2) and (3).

In section 5 two different axiomatizations for probability representations satisfying (1) are presented. The first axiomatization, due essentially to C. Kraft, J. Pratt, and A. Seidenberg, establishes necessary and sufficient conditions for the existence of the desired probability representations. The second axiomatization, due to Krantz, Luce *et. al*., establishes only sufficient conditions for the existence of the desired representations, but uniqueness is guaranteed. Uniqueness is of interest in relation to the preference order interpretation offered for Axiom III in section 4 (see 4.2). On the other hand, when the state flow sets Ω_θ are assumed to be finite as in Axiom II, the Krantz *et. al*. nonnecessary condition which ensures uniqueness is strong. These points will be further discussed in section 5.

Under both axiomatizations the resulting probability representations are finitely rather than countably additive. In view of Axiom II, this is all that is needed for the expected utility representation. If Axiom II were to be eventually weakened to allow for σ-algebras, the extension to countably additive probability representations would present no problems. A simple necessary and sufficient condition for a finitely additive probability representation on a σ-algebra to be countably additive has been obtained by C. Villegas (see [3, pages 215 - 216]).

4. AXIOMATIZATION: UTILITY

Let the primitives $(\langle \Theta, \succcurlyeq \rangle, \{\langle \Omega_\theta, \succcurlyeq_\theta \rangle \mid \theta \in \Theta\}, \{\langle \mathcal{E}_\theta, \geq_\theta \rangle \mid \theta \in \Theta\})$ for a gc-model be given (see section 2). The first axiom presented below will be replaced in section 5 by conditions on the primitives.

AXIOM I (TEMPORARY). To each policy $\theta \in \Theta$ there corresponds a finitely additive probability measure $\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]$ satisfying

$$\sigma(E|\theta \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E' \quad ,$$

for all $E, E' \in \mathcal{E}_\theta$.

In the next axiom finiteness of the state flow sets $\{\Omega_\theta\}$ will be assumed in order to use 4.3 below. Although finiteness of the state flow sets is realistic, it is often convenient to work with connected sets, e.g., intervals of the real line. Moreover, as will be seen in section 5, this finiteness restriction is not essential for establishing the existence of the desired probability representations as in Axiom I. Thus it would be desirable to weaken Axiom II to allow for infinite state flow sets.

AXIOM II. For every policy $\theta \in \Theta$, the associated set Ω_θ of state flows is a finite set $\{\omega'_\theta, \ldots, \omega_\theta^{n_\theta}\}$, and the associated algebra $\mathcal{E}_\theta$ of event flows is given by $\mathcal{E}_\theta = 2^{\Omega_\theta}$ (i.e., the set of all subsets of Ω_θ) .

In the next axiom the decision maker's primitive preference and probability orders will be required to be compatible with the existence of a certain extraneous weak order. A preference order interpretation for the axiom will be discussed after it is stated. The following definitions and notation will be used.

4.1 DEFINITIONS AND NOTATION. A set M is the mixture set for a set K if

1) $K \subseteq M$;
2) For all $t \in [0, 1]$ and $B, D \in M$, there exists an element $tB + [1-t]D \in M$;
3) For all $t, r \in [0, 1]$ and $B, D \in M$,
   (a) $1B + 0D = B$;
   (b) $tB + [1-t]D = [1-t]D + tB$;
   (c) $t[rB + [1-r]D] + [1-t]D = trB + [1-tr]D$;
4) M is the minimal set with properties 1), 2), and 3).

For each policy $\theta \in \Theta$ let $M\Omega_\theta = \{\psi_\theta, \ldots\}$ denote the mixture set for $\Omega_\theta = \{\omega_\theta^1, \ldots, \omega_\theta^{n_\theta}\}$ (see Axiom II); and let $T(\theta) \equiv \sum_{i=1}^{n_\theta} \omega_\theta^i \, \sigma(\{\omega_\theta^i\} \mid \theta) \in M\Omega_\theta$, where $\sigma(\cdot \mid \theta) : \mathcal{E}_\theta \to [0, 1]$ is the finitely additive probability representation for the θ-conditioned probability order $\langle \mathcal{E}_\theta, \geq_\theta \rangle$ whose existence is guaranteed by Axiom I.

Let $Q \equiv \{(\psi_\theta \mid \theta) \mid \psi_\theta \in M\Omega_\theta, \theta \in \Theta\}$, and let $MQ = \{b, c, d, \ldots\}$ denote the mixture set for Q.

AXIOM III. There exists a weak order $\succcurlyeq^*$ over MQ which satisfies the following five conditions: For all $\theta, \theta' \in \Theta$, $\omega, \omega' \in \Omega_\theta$, $\psi, \psi' \in M\Omega_\theta$, and $b, c, d \in MQ$,

1) $(\omega|\theta) \succcurlyeq^* (\omega'|\theta) \Leftrightarrow \omega \succcurlyeq_\theta \omega'$;
2) $(T(\theta)|\theta) \succcurlyeq^* (T(\theta')|\theta') \Leftrightarrow \theta \succcurlyeq \theta'$;
3) $c \succ^* b,\ 0 < t < 1 \Rightarrow tc + [1 - t]d \succ^* tb + [1 - t]d$;
4) $d \succ^* c \succ^* b \Rightarrow$ there exist $t, s \in (0, 1)$ such that $tb + [1 - t]d \succ^* c \succ^* sb + [1 - s]d$;
5) $t(\psi|\theta) + [1 - t](\psi'|\theta) \sim^* (t\psi + [1 - t]\psi'|\theta)$ for all $t \in [0, 1]$,

where $\succ^*$ is defined on MQ by $[b \succ^* d] \equiv [b \succcurlyeq^* d$ and not $(d \succcurlyeq^* b)]$; and $\sim^*$ is defined on MQ by $[b \sim^* d] \equiv [b \succcurlyeq^* d$ and $d \succcurlyeq^* b]$.

<u>Remarks</u>. Axiom III - 2) is well defined only if Axiom I holds. Assuming conditions 1) and 2) are compatible, a weak order on MQ satisfying conditions 1) and 2) always exists. (By assumed connectedness and transitivity of the orders $\langle \Theta, \succcurlyeq \rangle$ and $\{\langle \Omega_\theta, \succcurlyeq_\theta \rangle | \theta \in \Theta\}$, the partial order $\succcurlyeq^o$ induced on MQ by the compatible conditions 1) and 2) is transitive and reflexive. Hence $\succcurlyeq^o$ can be extended to a weak order over MQ (see [6]).)

4.2 EXTRANEOUS GAMBLE - PREFERENCE ORDER INTERPRETATION FOR $\langle MQ, \succcurlyeq^* \rangle$. The mixture sets $M\Omega_\theta$, $\theta \in \Theta$, may be interpreted as sets of extraneous gambles as follows. For each set $\{t_1, \ldots t_{n_\theta}\}$ of nonnegative coefficients satisfying $\sum_i t_i = 1$, let the corresponding element $\psi = \sum_i \omega_\theta^i t_i \in M\Omega_\theta$ be interpreted as the gamble which awards "prize" ω_θ^i with "probability" t_i. Under this interpretation, if the probability representation $\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]$ for the probability order $\langle \mathcal{E}_\theta, \geq_\theta \rangle$ guaranteed by Axiom I is unique, then

$$T(\theta) \equiv \sum_{i=1}^{n_\theta} \omega_\theta^i \, \sigma(\{\omega_\theta^i\} \mid \theta) \in M\Omega_\theta$$

is the gamble which the decision maker will participate in if he chooses policy θ, according to his own judgments. If $\sigma(\cdot|\theta)$ is not unique, then $T(\theta)$ approximates this gamble.

Similarly, the mixture set MQ for $Q \equiv \{(\psi_\theta|\theta) | \psi_\theta \in M\Omega_\theta, \theta \in \Theta\}$ may be interpreted as a set of extraneous gambles as follows. Let each element $(\psi|\theta) \in Q$ be interpreted as the event "decision maker participates in gamble ψ" conditioned on the event "decision maker chooses policy θ." Then for each set $\{r_1, \ldots, r_m\}$ of nonnegative coefficients with $\sum_j r_j = 1$, and each set of elements $\{(\psi_{\theta_j}|\theta_j) \in Q | j = 1, \ldots, m\}$, the element $b \equiv \sum_j (\psi_{\theta_j}| \theta_j) r_j \in MQ$ can be interpreted as the gamble which awards "prize" $(\psi_{\theta_j}|\theta_j)$ with "probability" r_j. To "participate" in the gamble b, the decision maker imagines that with probability r_j he must participate in the gamble ψ_{θ_j}, with θ_j as his policy choice.

The weak order $\succcurlyeq^*$ can then be interpreted as a preference order over the gambles in MQ as follows.

$b \succcurlyeq^{*} c \Leftrightarrow$ Participation in the gamble b is at least as desirable to the decision maker as participation in the gamble c .

Under this gamble-preference order interpretation for $\langle MQ, \succcurlyeq^{*}\rangle$, conditions 1) - 5) in Axiom III can be given straightforward interpretations. Condition 1) is tautological, and condition 2) is essentially tautological if the probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ are unique. Verbally, condition 2) reads: The desirability of participating in the gamble $T(\theta)$, given the event "decision maker chooses policy θ ," is at least as great for the decision maker as the desirability of participating in the gamble $T(\theta')$, given the event "decision maker chooses policy θ' ," if and only if the choice of policy θ is at least as desirable to the decision maker as the choice of policy θ'. (Intuitively, the "tighter" the probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1]\}$, the closer the gambles $\{T(\theta)\}$ approximate the gambles the decision maker believes he would participate in for each choice of θ ; hence the more plausible condition 2) becomes.)

Finally, conditions 3) - 5) can be compared to standard axioms in the von Neumann-Morgenstern tradition. Condition 3) resembles Savage's "sure thing principle" (see [4, page 21 and page 114]) and can be given a similar defense. Condition 4) is a typical Archimedean constraint. Condition 5) states that the decision maker is indifferent between a one stage and a two stage gamble as long as both offer him the same expected return.

Although conditions 1) - 5) in Axiom III become intuitively plausible under this gamble-preference order interpretation for $\langle MQ, \succcurlyeq^{*} \rangle$, Axiom III does not impose this interpretation for two reasons: it is not necessary; and more importantly, the underlying assumption that the decision maker can order in preference all the hypothetical, nonrealizable "gambles" in MQ is clearly strong.

4.3 LEMMA [1, 8.4, page 112]. Let M be the mixture set for a set K. Let $\geq'$ be a weak order on M, and let $>'$ be defined on M by $[B >' D] \equiv [B \geq' D$ and not $(D >' B)]$.[3] Then for all $B, D, R \in M$, the following two conditions

(a) $D >' B,\ 0 < t < 1 \Rightarrow tD + [1 - t]R >' tB + [1 - t]R$;

(b) $R >' D >' B \Rightarrow tB + [1 - t]R >' D >' rB + [1 - r]R$ for some $t, r \in (0, 1)$;

are necessary and sufficient for the existence of a function $W : M \to R$, unique up to positive linear transformation, satisfying

$$W(B) \geq' W(D) \Leftrightarrow B \geq' D ;$$

$$W(tB + [1 - t]D) = tW(B) + [1 - t]W(D) ,$$

for all $B, D \in M$ and $t \in [0, 1]$.

4.4 THEOREM. Let Axioms I and II hold. Then for each policy $\theta \in \Theta$ there exists a utility function $u(\cdot|\theta)\colon \Omega_\theta \to R$ satisfying

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \succcurlyeq_\theta \omega', \tag{4}$$

for all $\omega, \omega' \in \Omega_\theta$, such that

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta') \Leftrightarrow \theta \succcurlyeq \theta', \tag{5}$$

for all $\theta, \theta' \in \Theta$, if and only if Axiom III holds.

Proof. Assume Axioms I, II, and III hold. Then by Axiom III - 3), 4) and 4.3 there exists a function $U^*\colon MQ \to R$ satisfying

$$U^*(td + [1-t]b) = tU^*(d) + [1-t]U^*(b)\ ; \tag{6}$$

$$U^*(d) \geq U^*(b) \Leftrightarrow d \succcurlyeq^* b\ , \tag{7}$$

for all $d, b \in MQ$ and for all $t \in [0,1]$. By Axiom III - 2) and (7), for all $\theta', \theta'' \in \Theta$,

$$U^*(T(\theta')|\theta') \geq U^*(T(\theta'')|\theta'') \Leftrightarrow \theta' \succcurlyeq \theta'', \tag{8}$$

where $T(\theta)$, $\theta \in \Theta$, is as defined in 4.1 By Axiom III - 5), (7), and repeated use of (6),

$$U^*(T(\theta)|\theta) = \sum_{i=1}^{n_\theta} U^*(\omega_\theta^i|\theta)\ \sigma(\{\omega_\theta^i\}|\theta),\ \theta \in \Theta\ . \tag{9}$$

For each $\theta \in \Theta$, define a function $u(\cdot|\theta) : \Omega_\theta \to R$ by

$$u(\omega|\theta) = U^*(\omega|\theta) , \quad \omega \in \Omega_\theta . \tag{10}$$

By Axiom III - 1), (7), and (8),

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \succcurlyeq_\theta \omega' ,$$

for all $\omega, \omega' \in \Omega_\theta$, $\theta \in \Theta$. By (8), (9), (10) and Axiom II,

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta') \Leftrightarrow \theta \succcurlyeq \theta' ,$$

for all $\theta, \theta' \in \Theta$.

Conversely, assume Axioms I and II hold, and functions $\{u(\cdot|\theta) : \Omega_\theta \to R | \theta \in \Theta\}$ exist satisfying (4) and (5). Define $U^o : MQ \to R$ by

$$U^o(\Sigma_i\ (\Sigma_{j=1}^{n_i}\ \omega_{\theta_i}^j\ r_i^j|\theta_i) \cdot t_i) = \Sigma_i\ (\Sigma_{j=1}^{n_i}\ u(\omega_{\theta_i}^j|\theta_i) \cdot r_i^j) \cdot t_i \quad .$$

Clearly U^o is a well-defined function. Define a weak order $\succcurlyeq^*$ on MQ by

$$a \succcurlyeq^* b \Leftrightarrow U^o(a) \geq U^o(b) , \quad a, b \in MQ \quad .$$

By (4), $\langle MQ , \succcurlyeq^* \rangle$ satisfies Axiom III - 1); and by (5) and Axiom II, $\langle MQ , \succcurlyeq^* \rangle$ satisfies Axiom III - 2). Finally, conditions 3), 4), and 5) in Axiom III can be verified for $\langle MQ , \succcurlyeq^* \rangle$ by straightforward calculation.

Q.E.D.

5. AXIOMATIZATION: PROBABILITY

Two alternative sets of conditions for the weak orders $\{\langle \Omega_\theta, \mathcal{E}_\theta, \geq_\theta \rangle \mid \theta \in \Theta\}$ will be presented which guarantee the existence of finitely additive probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] \mid \theta \in \Theta\}$ as in Axiom I, in a manner consistent with Axioms II and III (see section 4). The first set of conditions, although necessary and sufficient for the desired probability representations, will not guarantee their uniqueness. As discussed in 4.2, if the weak order $\langle MQ, \succcurlyeq^* \rangle$ appearing in Axiom III is interpreted as a preference order over extraneous gambles, then the plausibility of the consistency requirement 2) in Axiom III varies directly with the "tightness" of the obtained representations. For this reason a second, sufficient set of conditions is presented which ensures the uniqueness of the probability representations. Since uniqueness for a probability representation over a finite set is unusual, it is not surprising that the representations obtained under the second set of conditions are somewhat rigid.

The first set of conditions will be obtained as a corollary of the following representation theorem, a reformulation by D. Scott of a result established by C. Kraft, J. Pratt, and A. Seidenberg [2]. Scott's proof (not given) involves passing by means of "indicator functions" from an algebra of subsets to a finite dimensional vector space representation for which a separating hyperplane theorem (a variant of the Hahn-Banach

Theorem) becomes applicable; hence the somewhat strange appearance of condition (iv) in the statement of the theorem.

Given an algebra $\mathcal{E}$ of subsets of a set Ω, $1_E : \Omega \to \{0, 1\}$ will denote the indicator function for E, defined by

$$1_E(\omega) = \begin{cases} 1, & \omega \in E ; \\ 0, & \omega \notin E . \end{cases}$$

A function $P : \mathcal{E} \to [0, 1]$ will be said to represent a binary relation $\succcurlyeq$ on $\mathcal{E}$ if $[E \succcurlyeq E'] \Leftrightarrow [P(E) \geq P(E')]$, for all $E, E' \in \mathcal{E}$.

5.1 THEOREM [5, Theorem 4.1, page 246]. Let $\mathcal{E}$ be an algebra of subsets of a finite set Ω, and let $\succcurlyeq$ be a binary relation on $\mathcal{E}$. Then for $\succcurlyeq$ to be representable by a finitely additive probability function P on $\mathcal{E}$ it is necessary and sufficient that the conditions

(i) $\Omega \succ \emptyset$;

(ii) $E \succcurlyeq \emptyset$;

(iii) $E \succcurlyeq E'$ or $E' \succcurlyeq E$;

(iv) $1_{E^o} + \ldots 1_{E^{n-1}} = 1_{D^o} + \ldots 1_{D^{n-1}}$ implies $D^o \succcurlyeq E^o$,

hold for all $E, E', E^i, D^i \in \mathcal{E}$, $i=0, \ldots, n-1$, where $E^i \succcurlyeq D^i$ for $0 < i < n$.

<u>Remark</u>. As Scott notes, condition (iv) is an "unpleasant feature" since the sum $1_A + 1_B$ of two indicator functions cannot be identified with an element of $\mathcal{E}$ except when

$A \cap B = \phi$. Hence the theorem establishes the representation by placing restrictions on objects outside of the proper domain of events $\mathcal{E}$. Nevertheless, the interpretation of the equation in (iv) is straightforward: every element of Ω belongs to exactly the same number of the E^i as the D^i.

A second objection which might be raised to condition (iv) is its testability. (Although Ω is finite, condition (iv) entails an infinite set of restrictions; for repetition of the indicator functions is allowed.) However, the proof of 5.1 presented in Reference [2] includes an algorithm for checking in a finite number of steps whether condition (iv) holds.

According to Scott, 5.1 can be extended to infinite Ω by appropriate use of the Hahn-Banach Theorem.

5.2 COROLLARY. Assume each state flow set Ω_θ, $\theta \in \Theta$, is finite. Then the following three conditions are necessary and sufficient for the existence of finitely additive probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ satisfying

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \gtrsim_\theta E',$$

for all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$:

1) $\Omega_\theta >_\theta \phi$, $\theta \in \Theta$;

2) $E \gtrsim_\theta \phi$ for all $E \in \mathcal{E}_\theta$, $\theta \in \Theta$;

3) $1_{E^o} + \dots 1_{E^{n-1}} = 1_{D^o} + \dots 1_{D^{n-1}} \Rightarrow D^o \gtrsim_\theta E^o$, for all $E^i, D^i \in \mathcal{E}_\theta$, $i = 0, \dots, n-1$, with $E^i \gtrsim_\theta D^i$, $0 < i < n$, for all $\theta \in \Theta$.

A second, alternative set of conditions sufficient for the existence of probability representations $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ as in Axiom I will be obtained as a corollary of the following theorem, due to Krantz et. al. We distinguish between necessary conditions which are implied by the existence of the desired representation, and structural conditions which are sufficient but not necessary for the existence of the desired representation.

5.3 THEOREM [3, Theorem 2, page 208]. Let $\mathcal{E}$ be an algebra of sets on a set Ω, and let $\geq^*$ be a relation on $\mathcal{E}$ such that for every $A, B, C, D \in \mathcal{E}$:

1. (Necessary) $\langle \mathcal{E}, \geq^* \rangle$ is a weak order;
2. (Necessary) $\Omega >^* \phi$ and $A \geq^* \phi$;
3. (Necessary) If $A \cap B = A \cap C = \phi$, then $B \geq^* C$ if and only if $A \cup B \geq^* A \cup C$;
4. (Structural) Ω is finite;
5. (Structural) If $A \cap B = \phi$, $A \geq^* C$ and $B \geq^* D$, then there exist $C', D', E \in \mathcal{E}$ such that:
   (i) $E \sim^* A \cup B$;
   (ii) $C' \cap D' = \phi$;
   (iii) $E \supseteq C' \cup D'$;
   (iv) $C' \sim^* C$ and $D' \sim^* D$,

where $[A \sim^* B] \equiv [A \succcurlyeq^* B$ and $B \succcurlyeq^* A]$ and $[A \succ^* B] \equiv [A \succcurlyeq^* B$ and not $(B \succcurlyeq^* A)]$, $A, B \in \mathcal{E}$. Then there exists a unique order-preserving measure P on $\mathcal{E}$ such that $(\Omega, \mathcal{E}, P)$ is a finitely additive probability space.

Discussion. In place of condition 4, the original Krantz et. al. representation theorem imposes a weaker, necessary Archimedean condition which is compatible with infinite algebras $(\Omega, \mathcal{E})$.

In 1949 Bruno de Finetti questioned whether conditions 5.3 - 1, 2, and 3 were sufficient as well as necessary for the existence of a finitely additive probability representation over a finite algebra $(\Omega, \mathcal{E})$. A counterexample to this conjecture, involving a Boolean algebra generated by five elements, is established in Reference [2]. The nonsufficiency of conditions 5.3 - 1, 2, and 3 for infinite algebras $(\Omega, \mathcal{E})$ is discussed by L. Savage [4, Chapter III, especially page 40]).

As Krantz et. al. note, it is difficult to give a simple interpretation for structural condition 5. Yet, in the presence of conditions 1, 2, and 3, condition 5 is strictly weaker than Savage's postulate P6′, which states: If $B, C \in \mathcal{E}$, and $C \succ^* B$, then there exists a partition $\{D_1, \ldots D_n\}$ of Ω such that $C \succ^* B \cup D_i$ for each i (see [4, pages 38 - 39] and [3, pages 206 - 207]). For example, Savage's P6′ forces Ω to be infinite, whereas conditions 5.3 - 1, 2, 3, and 5 are compatible with certain finite Ω (e.g., $\Omega = \{a, b, c, d\}$, with Prob(a) = Prob(b) = Prob(c) = .2 , and Prob(d) = .4) .

Since uniqueness is an extremely strong condition for probability representations over finite algebras, some rigidity in the Krantz et. al. representing function P is to be expected. Specifically, the probabilities assigned by P are integer multiples of a certain minimal fraction 1/n . (To verify that this restriction holds, see the constructive proof for Theorem 4 [3, pages 44 - 52] on which the proof of 5.3 is based.) The rigidity of this restriction could be somewhat alleviated if the algebra $\mathcal{E}$ were assumed to contain an event such as "N tosses of a fair coin results in N heads," for some arbitrarily large N .

5.4 COROLLARY TO 5.3. Let conditions 2 - 5 in 5.3 hold for each weak order $\langle \Omega_\theta, \mathcal{E}_\theta, \geq_\theta \rangle$, $\theta \in \Theta$. Then there exist unique finitely additive probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ satisfying

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E' ,$$

for all $E, E' \in \mathcal{E}_\theta$, $\theta \in \Theta$.

6. THE MAIN REPRESENTATION THEOREM

By combining 5.2 with 4.4, the following representation theorem is obtained.

6.1 THEOREM. Let a gc-model $(\langle\Theta, \succcurlyeq\rangle, \{\langle\Omega_\theta, \succcurlyeq_\theta\rangle | \theta \in \Theta\}, \{\langle\mathcal{E}_\theta, \geq_\theta\rangle | \theta \in \Theta\})$ be given, and assume each state flow set Ω_θ is finite, with $\mathcal{E}_\theta = 2^{\Omega_\theta}$ (Axiom II). Then conditions 5.2 - 1), 2), 3) and Axiom III are necessary and sufficient for the existence of finitely additive probability measures $\{\sigma(\cdot|\theta) : \mathcal{E}_\theta \to [0, 1] | \theta \in \Theta\}$ and utility functions $\{u(\cdot|\theta) : \Omega_\theta \to R | \theta \in \Theta\}$ satisfying for all policies $\theta, \theta' \in \Theta$:

$$\sigma(E|\theta) \geq \sigma(E'|\theta) \Leftrightarrow E \geq_\theta E', \quad \text{for all} \quad E, E' \in \mathcal{E}_\theta ;$$

$$u(\omega|\theta) \geq u(\omega'|\theta) \Leftrightarrow \omega \geq \omega', \quad \text{for all} \quad \omega, \omega' \in \Omega_\theta ;$$

$$\int_{\Omega_\theta} u(\omega|\theta)\ \sigma(d\omega|\theta) \geq \int_{\Omega_{\theta'}} u(\omega|\theta')\ \sigma(d\omega|\theta') \Leftrightarrow \theta \succcurlyeq \theta' .$$

<u>Remark</u>. In the presence of Axiom II, conditions 5.2 - 1), 2), 3) are equivalent to Axiom I (this is the content of Theorem 5.2). Hence Axiom III - 2) is well defined.

FOOTNOTES

[1]A binary relation $\succcurlyeq$ on a set D is a <u>weak order</u> if for all a, b, c $\in$ D

(i) $a \succcurlyeq b$ or $b \succcurlyeq c$
(i.e., $\succcurlyeq$ is connected);

(ii) $a \succcurlyeq b$ and $b \succcurlyeq c$ implies $a \succcurlyeq c$
(i.e., $\succcurlyeq$ is transitive).

Weak orders have also been referred to as "complete preorderings."

[2]A collection F of subsets of a nonempty set X is said to be an <u>algebra</u> in X if F has the following three properties:

(1) $X \in F$;

(2) If $A \in F$, then $A^c \in F$, where A^c is the complement of A relative to X;

(3) If $A, B \in F$, then $A \cup B \in F$.

[3]Fishburn's original proposition is stated in terms of a binary relation R which he requires to be a "weak order" in the sense that R is asymmetric and negatively transitive [1, Definition 2.1, page 11]. As is easily verified, the assumption that $\geq'$ is a weak order over M in the sense used in this paper (see Footnote 1) implies that $>'$ is a "weak order" over M in the sense of Fishburn.

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