Parameter-Sensitivity Study for a Linear-Quadratic Control Problem with Random State Coefficients

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ABSTRACT

Optimal feedback-control laws generally cannot be obtained in closed form for stochastic control problems. The characterizations which have been obtained for several simplified problems have provided valuable insight into the properties of optimal feedback-control laws as well as providing guidance for the construction of suboptimal control laws. In this paper an analytical and computer simulated parameter-sensitivity study is presented for the optimal feedback-control law and dynamic-programming optimality equations associated with a discrete-time, finite-horizon, linear-quadratic control problem with random state coefficients. One interesting characteristic revealed by analysis is the existence of simple linear relationships between parameter sensitivities for the optimal feedback-control selections and for the corresponding cost-to-go expressions.

1. INTRODUCTION

For most stochastic control problems the optimal feedback-control law cannot be obtained in closed form. Nevertheless, the optimal feedback-control law has been characterized for several simple problems, and these characterizations have provided valuable insight into the properties of optimal feedback-control laws as well as providing guidance for the construction of suboptimal control laws. (See Refs. [1]–[8].)

The purpose of this paper is to present a parameter-sensitivity study for the optimal feedback-control law and dynamic-programming optimality equations associated with a discrete-time linear-quadratic control problem with random state coefficients. Previous parameter sensitivity studies for linear-quadratic control problems (e.g., [9–11]) have generally focused either on infinite-horizon problems, with special emphasis given to optimal stationary control policies, or on essentially one- or two-period problems. The latter category includes studies such as [11] which use an N-period model but analyze only the control

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selections for the final one or two periods. A distinctive feature of the present parameter-sensitivity study is its focus on the N-period case, with arbitrary finite N. Both analytical and computer-simulation studies are carried out to test the sensitivity of state, control, single-period cost, and cost-to-go trajectories to changes in time horizon, state-equation parameters, cost-function parameters, and mean and variance values for the random state coefficients. These parameter sensitivity results are used in Ref. [12] to clarify the optimality properties of a recently proposed adaptive control technique, direct criterion-function updating.

One interesting characteristic revealed by analysis is the existence of simple linear relationships between parameter sensitivities for the optimal feedback-control selections $\theta_n^{\text{opt}}(x)$ and also for the corresponding cost-to-go terms $T_n(x)$. [See (39) and (40) in Sec. 3.] A second analytical finding is the sign determinateness of all parameter sensitivities on the basis of the signs of the parameters and current state, with one exception: changes in control intensity with respect to changes in the gain. (See Table 1, Sec. 3.) Finally, it was previously observed by Athans et al. [13] that limiting optimal controllers can fail to exist if parameter uncertainties exceed certain threshholds. Since the discussion by Athans et al. is somewhat heuristic, a simple proof is provided for the model at hand (a special case of the Athans et al. model) demonstrating the nonexistence of the limiting optimal controller when the state-coefficient standard deviation takes values greater or equal to 1.0 (Theorem 6, Sec. 3).

An important characteristic illustrated by simultions is the extreme skewness of the sample cost distribution when the state-coefficient standard deviation takes values greater than 1.0. This finding supports the contention of several previous researchers [14–16] that exclusive reliance on the standard quadratic (expected mean squared error) measure for cost associated with state and control trajectory deviations can sometimes mask large trajectory fluctuations which occur with small but significant probability. (Their recommendation is to examine more closely the entire probability distribution governing costs rather than focusing on the single distribution parameter, expectation.)

The organization of the paper is as follows. The basic control problem is outlined in Sec. 2. Analytical parameter sensitivity results are established in Sec. 3. Computer-simulation results are presented and discussed in Sec. 4. Concluding comments are given in Sec. 5.

2. THE BASIC CONTROL PROBLEM

Consider a first-order dynamical system described by the equations

$$x_1 = x_1^* \qquad \text{(initial conditions)},$$

$$x_{n+1} = \omega_n x_n + b\theta_n \equiv f(\omega_n, \theta_n, x_n), \qquad 1 \leq n \leq N, \tag{1}$$

where $x_n \in R$ is the state, $b \in R$ is a known nonzero constant, $\theta_n \in R$ is the control, and the state coefficient $\omega_n \in R$ is drawn from a probability distribution $p : \mathfrak{F} \to [0, 1]$ defined over the σ-algebra $\mathfrak{F}$ consisting of all Borel subsets of R. The set $\mathfrak{L}$ of admissable feedback control laws for the system (1) consists of all vectors $\boldsymbol{\theta} \equiv (\theta_1(\cdot), \ldots, \theta_N(\cdot))$ of functions $\theta_n : R \to R$. The control objective for the system (1) is assumed to be the minimization of expected total cost

$$E^N\left[\sum_{n=1}^{N} W(\omega_n, \theta_n(x_n), x_n)\right] = \int_{R^N}\left[\sum_{n=1}^{N} W(\omega_n, \theta_n(x_n), x_n)\right] p(d\omega_N) \cdots p(d\omega_1) \quad (2)$$

via selection of an admissable feedback control law $\boldsymbol{\theta} \in \mathfrak{L}$, where

$$W(\omega, \theta, x) \equiv c(\omega x + b\theta)^2 + q\theta^2 \quad (3)$$

for known constants $c > 0$ and $q \geqq 0$.

Letting $E[\cdot]$ denote expectation with respect to $(R, \mathfrak{F}, p)$, and letting $p^N : \mathfrak{F}^N \to [0, 1]$ denote the product probability distribution for coefficient sequences $(\omega_1, \ldots, \omega_N)$ generated in the usual way from $(R, \mathfrak{F}, p)$, a control law $(\theta_1^{\mathrm{opt}}(\cdot), \ldots, \theta_N^{\mathrm{opt}}(\cdot)) \in \mathfrak{L}$ minimizes expected total cost (2) subject to the constraints (1) if and only if the following dynamic-programming optimality equations hold p^N-a.s. (see [1, Chapter II]):

$$T_N(x_N) \equiv \min_{\theta \in R} E[W(\omega, \theta, x_N)] = E[W(\omega, \theta_N^{\mathrm{opt}}(x_N), x_N)];$$

$$T_n(x_n) \equiv \min_{\theta \in R} E[W(\omega, \theta, x_n) + T_{n+1} \circ f(\omega, \theta, x_n)]$$

$$= E[W(\omega, \theta_n^{\mathrm{opt}}(x_n), x_n) + T_{n+1} \circ f(\omega, \theta_n^{\mathrm{opt}}(x_n), x_n)], \qquad 1 \leqq n \leqq N-1. \quad (4)$$

In the following two sections a parameter-sensitivity study will be carried out for the optimal feedback control law $(\theta_1^{\mathrm{opt}}(\cdot), \ldots, \theta_N^{\mathrm{opt}}(\cdot))$ and cost-to-go functions $T_n(\cdot)$ using both analytical and computer-simulation techniques.

3. PARAMETER SENSITIVITIES: ANALYTICAL RESULTS

The first two theorems below establish the basic structural properties of the optimal feedback-control law and cost-to-go functions. The results of Theorem 1 are well known, and a proof is presented only for completeness.

THEOREM 1. *The* nth*-period optimal control* $\theta_n^{\text{opt}}(\cdot)$ *and cost-to-go function* $T_n(\cdot)$ *for the control problem described by* (1) *and* (2) *take the form*

$$\theta_n^{\text{opt}}(x) = -\left(\frac{[E\omega]cb[1+v(n+1)]}{cb^2[1+v(n+1)]+q}\right)x, \tag{5}$$

$$T_n(x) = cv(n)x^2, \qquad x \in R, \tag{6}$$

where

$$v(N+1) = 0,$$

$$v(n) = [1+v(n+1)]\left[\operatorname{Var}(\omega) + (E\omega)^2\left(\frac{q}{cb^2[1+v(n+1)]+q}\right)\right],$$

$$1 \leq n \leq N-1. \tag{7}$$

Proof. By definition,

$$E[W(\omega,\theta,x)] = (E\omega^2)cx^2 + 2(E\omega)cbx\theta + (cb^2+q)\theta^2;$$

hence

$$\theta_N^{\text{opt}}(x) = -\left(\frac{(E\omega)cb}{cb^2+q}\right)x$$

and

$$T_N(x) = (E\omega^2)cx^2 - \frac{(E\omega)^2c^2b^2x^2}{cb^2+q}$$

$$= c\left[\operatorname{Var}(\omega) + (E\omega)^2 - \frac{(E\omega)^2cb^2}{cb^2+q}\right]x^2$$

$$= c\left[\operatorname{Var}(\omega) + (E\omega)^2\left(\frac{q}{cb^2+q}\right)\right]x^2.$$

Thus Eqs. (5) and (6) hold for $n = N$.

Now suppose Eqs. (5) and (6) hold for some $n \in \{2, \ldots, N\}$. Then

$$E[W(\omega,\theta,x) + T_n \circ f(\omega,\theta,x)] = E\{W(\omega,\theta,x) + cv(n)[f(\omega,\theta,x)]^2\}$$

$$= (E\omega^2)c[1+v(n)]x^2 + 2(E\omega)cb[1+v(n)]x\theta + \{cb^2[1+v(n)]+q\}\theta^2;$$

hence

$$\theta_{n-1}^{\text{opt}}(x) = -\left(\frac{[E\omega]cb[1+v(n)]}{cb^2[1+v(n)]+q}\right)x,$$

and

$$\begin{aligned} T_{n-1}(x) &= c[1+v(n)]\left[E\omega^2 - \frac{[E\omega]^2 cb^2[1+v(n)]}{cb^2[1+v(n)]+q}\right]x^2 \\ &= c[1+v(n)]\left[\text{Var}(\omega) + (E\omega)^2\left(\frac{q}{cb^2[1+v(n)]+q}\right)\right]x^2 \\ &= cv(n-1)x^2. \end{aligned}$$

Thus Eqs. (5) and (6) hold for all $n \in \{1,\ldots,N\}$. Q.E.D.

THEOREM 2. *Let the terms* $v(1),\ldots,v(N+1)$ *be defined by* (7). *Then* $v(n) \geqq v(n+1) \geqq 0$ *for all* $n \in \{1,\ldots,N\}$.

Proof. Clearly $v(N) = \text{Var}(\omega) + (E\omega)^2 q/[cb^2+q] \geqq v(N+1) = 0$. Suppose it has been shown that $v(n) \geqq v(n+1) \geqq 0$ for some $n \in \{2,\ldots,N\}$. Then, letting $D(k) \equiv cb^2[1+v(k)]+q > 0$, $k \in \{n, n+1\}$,

$$\begin{aligned} &v(n-1) - v(n) \\ &= [1+v(n)]\left[\text{Var}(\omega) + \frac{(E\omega)^2 q}{D(n)}\right] - [1+v(n+1)]\left[\text{Var}(\omega) + \frac{(E\omega)^2 q}{D(n+1)}\right] \\ &= \text{Var}(\omega)[v(n)-v(n+1)] + \frac{\{D(n+1)[1+v(n)] - D(n)[1+v(n+1)]\}(E\omega)^2 q}{D(n)D(n+1)} \\ &= \text{Var}(\omega)[v(n)-v(n+1)] + \frac{[v(n)-v(n+1)](E\omega)^2 q^2}{D(n)D(n+1)} \\ &\geqq 0. \end{aligned}$$

Thus $v(n) \geqq v(n+1) \geqq 0$ for all $n \in \{1,\ldots,N\}$. Q.E.D.

Less obvious structural properties of the optimal feedback-control law $(\theta_1^{\text{opt}}(x)),\ldots,\theta_N^{\text{opt}}(x))$ and cost-to-go functions $T_n(x)$ are clarified by the parameter-sensitivity table (Table 1). The meaning of, e.g., the first table entry is that the sign of the partial derivative $\partial v(n)/\partial(E\omega)$ is given by the sign of $q(E\omega)$.

TABLE 1
Parameter Sensitivities: $1 \leq n \leq N$

$z=$	$E\omega$	$\mathrm{Var}(\omega)$	$q(\geq 0)$	$c(>0)$	$b(\neq 0)$
$\mathrm{sgn}\left(\frac{\partial v(n)}{\partial z}\right)$	$\mathrm{sgn}(qE\omega)$	$\mathrm{sgn}(+1)$	$\mathrm{sgn}([E\omega]^2)$	$\mathrm{sgn}(-q[E\omega]^2)$	$\mathrm{sgn}(-qb[E\omega]^2)$
$\mathrm{sgn}\left(\frac{\partial T_n(x)}{\partial z}\right)$	$\mathrm{sgn}(q[E\omega]x^2)$	$\mathrm{sgn}(x^2)$	$\mathrm{sgn}([E\omega]^2x^2)$	$\mathrm{sgn}([\mathrm{Var}(\omega)+q(E\omega)^2]x^2)$	$\mathrm{sgn}(-qb[E\omega]^2x^2)$
$\mathrm{sgn}\left(\frac{\partial \theta_n^{\mathrm{opt}}(x)}{\partial z}\right)$	$\mathrm{sgn}(-bx)$	$\mathrm{sgn}(-qb[E\omega]x)$	$\mathrm{sgn}([E\omega]bx)$	$\mathrm{sgn}(-qb[E\omega]x)$	$\mathrm{sgn}([E\omega]x)$ if $cb^2 \geq q$; ? otherwise
$\mathrm{sgn}\left(\frac{\partial b\theta_n^{\mathrm{opt}}(x)}{\partial z}\right)$	$\mathrm{sgn}(-x)$	$\mathrm{sgn}(-q[E\omega]x)$	$\mathrm{sgn}(-q[E\omega]x)$	$\mathrm{sgn}(-q[E\omega]x)$	$\mathrm{sgn}(-qb[E\omega]x)$

Derivations for the entries in Table 1 will now be given. Proof of the following lemma is obtained by straightforward calculation.

LEMMA 1. *Let* $n \in \{1, \ldots, N\}$, *and define* $B_n \equiv q/(cb^2[1+v(n+1)]+q)$, *where* $q \geq 0$, $c>0$, $b \neq 0$, *and* $v(n+1)$ *is defined by* (7). *Then*

$$B_n \begin{cases} >0 & \text{if} \quad q>0, \\ =0 & \text{if} \quad q=0; \end{cases} \tag{1.1}$$

$$\frac{\partial B_n}{\partial v(n+1)} = \begin{cases} -cb^2B_n^2/q & \text{if} \quad q>0, \\ 0 & \text{if} \quad q=0; \end{cases} \tag{1.2}$$

$$\frac{\partial B_n}{\partial q} = \begin{cases} \dfrac{B_n^2}{q^2}cb^2[1+v(n+1)] - \dfrac{B_n^2}{q}cb^2\dfrac{\partial v(n+1)}{\partial q} & \text{if} \quad q>0, \\ \dfrac{1}{cb^2[1+v(n+1)]} & \text{if} \quad q=0; \end{cases} \tag{1.3}$$

$$\frac{\partial B_n}{\partial c} = \begin{cases} -\dfrac{B_n^2}{q}b^2[1+v(n+1)] - \dfrac{B_n^2}{q}cb^2\dfrac{\partial v(n+1)}{\partial c} & \text{if} \quad q>0, \\ 0 & \text{if} \quad q=0; \end{cases} \tag{1.4}$$

$$\frac{\partial B_n}{\partial b} = \begin{cases} -\dfrac{B_n^2}{q}2cb[1+v(n+1)] - \dfrac{B_n^2}{q}cb^2\dfrac{\partial v(n+1)}{\partial b} & \text{if} \quad q>0, \\ 0 & \text{if} \quad q=0. \end{cases} \tag{1.5}$$

THEOREM 3. *Let $n \in \{1,\ldots,N\}$, and let $v(n)$ be defined by* (7). *Then*

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial E\omega}\right)=\operatorname{sgn}(qE\omega), \tag{3.1}$$

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial \operatorname{Var}(\omega)}\right)=\operatorname{sgn}(+1), \tag{3.2}$$

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial q}\right)=\operatorname{sgn}([E\omega]^2), \tag{3.3}$$

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial c}\right)=\operatorname{sgn}(-q[E\omega]^2), \tag{3.4}$$

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial b}\right)=\operatorname{sgn}(-qb[E\omega]^2), \tag{3.5}$$

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial v(n+1)}\right)=\operatorname{sgn}(\operatorname{Var}(\omega)+q(E\omega)^2). \tag{3.6}$$

Proof of (3.1). By definition of $v(n)$,

$$\begin{aligned}\frac{\partial v(n)}{\partial E\omega}&=\frac{\partial v(n+1)}{\partial E\omega}\left[\operatorname{Var}(\omega)+[E\omega]^2B_n\right]+\{2[1+v(n+1)]B_n\}E\omega\\&\quad+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial v(n+1)}\,\frac{\partial v(n+1)}{\partial E\omega}\\&=\frac{\partial v(n+1)}{\partial E\omega}\left[\operatorname{Var}(\omega)+(E\omega)^2B_n+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial v(n+1)}\right]\\&\quad+\{2[1+v(n+1)]B_n\}E\omega.\end{aligned} \tag{8}$$

If $q=0$, then by (1.1) of Lemma 1,

$$\frac{\partial v(n)}{\partial E\omega}=\frac{\partial v(n+1)}{\partial E\omega}\operatorname{Var}(\omega).$$

Since $v(N+1)\equiv 0$, it follows by backward induction on n that

$$\frac{\partial v(n)}{\partial E\omega}=0 \qquad \text{if} \quad q=0. \tag{9}$$

Suppose $q>0$. Then, using (1.2) of Lemma 1 and (8),

$$\frac{\partial v(n)}{\partial E\omega}=\frac{\partial v(n+1)}{\partial E\omega}\left[\operatorname{Var}(\omega)+(E\omega)^2B_n-\frac{cb^2[1+v(n+1)]}{q}(E\omega)^2B_n^2\right]$$

$$+\{2[1+v(n+1)]B_n\}E\omega$$

$$=\frac{\partial v(n+1)}{\partial E\omega}\left[\operatorname{Var}(\omega)+(E\omega)^2B_n^2\right]$$

$$+\{2[1+v(n+1)]B_n\}E\omega. \tag{10}$$

Using (1.1) the coefficient of $\partial v(n+1)/\partial E\omega$ in (10) is nonnegative, and using Theorem 2 and (1.1), the coefficient of $E\omega$ in (10) is positive. Since $v(N+1)\equiv 0$,

$$\operatorname{sgn}\left(\frac{\partial v(N)}{\partial E\omega}\right)=\operatorname{sgn}(E\omega) \quad \text{if} \quad q>0. \tag{11}$$

It then follows by backward induction on n that

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial E\omega}\right)=\operatorname{sgn}(E\omega) \quad \text{if} \quad q>0. \tag{12}$$

Combining (9) and (12), (3.1) is proved.

Proof of (3.2). Suppose $q=0$. Then

$$\frac{\partial v(n)}{\partial \operatorname{Var}(\omega)}=[1+v(n+1)]+\frac{\partial v(n+1)}{\partial \operatorname{Var}(\omega)}[\operatorname{Var}(\omega)].$$

Since $v(N+1)\equiv 0$ and $v(k)\geq 0$ for $k\in\{1,\ldots,N\}$, it follows by backward inducation on n that

$$\operatorname{sgn}\left(\frac{\partial v(n)}{\partial \operatorname{Var}(\omega)}\right)=\operatorname{sgn}(+1) \quad \text{if} \quad q=0.$$

Suppose $q>0$. Then, using (1.2),

$$\frac{\partial v(n)}{\partial \operatorname{Var}(\omega)}=[1+v(n+1)]+\frac{\partial v(n+1)}{\partial \operatorname{Var}(\omega)}\left[\operatorname{Var}(\omega)+(E\omega)^2 B_n\right]$$

$$+[1+v(n+1)](E\omega)^2\left[\frac{\partial B_n}{\partial v(n+1)}\frac{\partial v(n+1)}{\partial \operatorname{Var}(\omega)}\right]$$

$$=[1+v(n+1)]+\frac{\partial v(n+1)}{\partial \operatorname{Var}(\omega)}\left[\operatorname{Var}(\omega)+(E\omega)^2 B_n\right.$$

$$\left.+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial v(n+1)}\right]$$

$$=[1+v(n+1)]+\frac{\partial v(n+1)}{\partial \operatorname{Var}(\omega)}\left[\operatorname{Var}(\omega)+(E\omega)^2 B_n^2\right]. \tag{13}$$

By (1.1) the second bracketed term in (13) is nonnegative. Since $v(N+1)\equiv 0$, $\partial v(N)/\partial \operatorname{Var}(\omega)>0$. Thus (3.2) also follows easily for $q>0$ by backward induction on n.

Proof of (3.3). By definition of $v(n)$,

$$\frac{\partial v(n)}{\partial q}=\frac{\partial v(n+1)}{\partial q}\left[\operatorname{Var}(\omega)+(E\omega)^2 B_n\right]+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial q}. \tag{14}$$

By (1.1), the coefficient of $\partial v(n+1)/\partial q$ in (14) is nonnegative. If $q=0$, then by (1.3) and Theorem 2 the sign of the last expression in (14) is given by the sign of $(E\omega)^2$. Since $v(N+1)\equiv 0$, (3.3) then follows by backward induction on n.

Suppose $q>0$. Using (14) and (1.3),

$$\frac{\partial v(n)}{\partial q}=\frac{\partial v(n+1)}{\partial q}\left[\operatorname{Var}(\omega)+(E\omega)^2 B_n^2\right]+[1+v(n+1)]^2(E\omega)^2(B_n)^2\frac{cb^2}{q^2}. \tag{15}$$

Using (1.1), the coefficient of $\partial v(n+1)/\partial q$ in (15) is nonnegative, and the sign of the last expression in (15) is given by the sign of $(E\omega)^2$. Since $v(N+1)\equiv 0$, (3.3) follows by backward induction on n.

Proof of (3.4). By definition of $v(n)$,

$$\frac{\partial v(n)}{\partial c}=\frac{\partial v(n+1)}{\partial c}\left[\operatorname{Var}(\omega)+(E\omega)^2 B_n\right]+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial c}. \tag{16}$$

If $q=0$, then by (1.4) the final expression in (16) vanishes. Since $v(N+1)\equiv 0$, it follows by backward induction on n that $\partial v(n)/\partial c=0$.

Suppose $q>0$. Then, using (1.4),

$$\frac{\partial v(n)}{\partial c}=\frac{\partial v(n+1)}{\partial c}\left[\operatorname{Var}(\omega)+(E\omega)^2B_n^2\right]-b^2[1+v(n+1)]^2(E\omega)^2\frac{B_n^2}{q}. \quad (17)$$

By (1.1) the coefficient of $\partial v(n+1)/\partial c$ in (17) is nonnegative; and the final expression in (17) has the sign of $-(E\omega)^2$. Since $v(N+1)\equiv 0$, (3.4) follows by backward induction on n.

Proof of (3.5). By definition of $v(n)$,

$$\frac{\partial v(n)}{\partial b}=\frac{\partial v(n+1)}{\partial b}\left[\operatorname{Var}(\omega)+(E\omega)^2B_n\right]+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial b}. \quad (18)$$

If $q=0$, then by (1.5) the final expression in (18) vanishes. Since $v(N+1)\equiv 0$, (3.5) follows by backward induction on n.

Suppose $q>0$. Then, using (1.5),

$$\frac{\partial v(n)}{\partial b}=\frac{\partial v(n+1)}{\partial b}\left[\operatorname{Var}(\omega)+(E\omega)^2B_n^2\right]-2[1+v(n+1)]^2(E\omega)^2cb\frac{B_n^2}{q}. \quad (19)$$

By (1.1) the coefficient of $\partial v(n+1)/\partial b$ in (19) is nonnegative; and the sign of the final term in (19) is given by the sign of $-b(E\omega)^2$. Since $v(N+1)\equiv 0$, (3.5) follows by backward induction on n.

Proof of (3.6). By definition of $v(n)$,

$$\frac{\partial v(n)}{\partial v(n+1)}=\left[\operatorname{Var}(\omega)+(E\omega)^2B_n\right]+[1+v(n+1)](E\omega)^2\frac{\partial B_n}{\partial v(n+1)}. \quad (20)$$

If $q=0$, then by (1.1) and (1.2) the sign of (20) is given by the sign of $\operatorname{Var}(\omega)$. Suppose $q>0$. Then, using (1.2),

$$\frac{\partial v(n)}{\partial v(n+1)}=\operatorname{Var}(\omega)+(E\omega)^2B_n^2. \quad (21)$$

By (1.1) the sign of (21) is given by the sign of $\operatorname{Var}(\omega)+(E\omega)^2$. Combining both cases $q=0$ and $q>0$, (3.6) is proved. Q.E.D.

THEOREM 4. *Let* $T_n(x)$ *be defined by* (6). *Then*

$$\operatorname{sgn}\left(\frac{\partial T_n(x)}{\partial E\omega}\right)=\operatorname{sgn}(q[E\omega]x^2), \tag{4.1}$$

$$\operatorname{sgn}\left(\frac{\partial T_n(x)}{\partial \operatorname{Var}(\omega)}\right)=\operatorname{sgn}(x^2), \tag{4.2}$$

$$\operatorname{sgn}\left(\frac{\partial T_n(x)}{\partial q}\right)=\operatorname{sgn}([E\omega]^2x^2), \tag{4.3}$$

$$\operatorname{sgn}\left(\frac{\partial T_n(x)}{\partial c}\right)=\operatorname{sgn}\left(\left[\operatorname{Var}(\omega)+q(E\omega)^2\right]x^2\right), \tag{4.4}$$

$$\operatorname{sgn}\left(\frac{\partial T_n(x)}{\partial b}\right)=\operatorname{sgn}(-qb[E\omega]^2x^2). \tag{4.5}$$

Proof. By definition,

$$T_n(x)=cv(n)x^2, \qquad c>0. \tag{22}$$

Thus (4.1), (4.2), (4.3), and (4.5) follow immediately from Theorem 3. To prove (4.4), it suffices to establish that

$$\frac{\partial T_n(x)}{\partial c}=\begin{cases} v(n)x^2 & \text{if} \quad q=0, \quad (23)\\ \left(\sum_{j=0}^{N-n}\left[\prod_{m=0}^{j} S_{m+n}\right]\right)x^2 & \text{if} \quad q>0, \quad (24)\end{cases}$$

where B_k is as defined in Lemma 1 and

$$S_k\equiv\operatorname{Var}(\omega)+(E\omega)^2B_k^2, \qquad 1\leqq k\leqq N, \tag{25}$$

for $\operatorname{sgn}(v(n))=\operatorname{sgn}(\operatorname{Var}(\omega))$ if $q=0$, and $\operatorname{sgn}(S_k)=\operatorname{sgn}(\operatorname{Var}(\omega)+(E\omega)^2)$ for $q>0$, $1\leqq k\leqq N$, using (1.1).

By the definition (22) for $T_n(x)$,

$$\frac{\partial T_n(x)}{\partial c}=\left[v(n)+c\frac{\partial v(n)}{\partial c}\right]x^2. \tag{26}$$

If $q=0$, then by (3.4) the expression $\partial v(n)/\partial c$ in (26) vanishes, establishing (23). Suppose $q>0$. By (17), for arbitrary $k\in\{1,\ldots,N\}$,

$$\begin{aligned} v(k)+c\frac{\partial v(k)}{\partial c} &= [1+v(k+1)]\left[\mathrm{Var}(\omega)+(E\omega)^2B_k\right] \\ &\quad + c\left[\frac{\partial v(k+1)}{\partial c}S_k - b^2[1+v(k+1)]^2(E\omega)^2\frac{B_k^2}{q}\right] \\ &= [1+v(k+1)]S_k + c\frac{\partial v(k+1)}{\partial c}S_k \\ &= \left[1+v(k+1)+c\frac{\partial v(k+1)}{\partial c}\right]S_k. \end{aligned} \tag{27}$$

Since $v(N+1)\equiv 0$, it follows by backward recursion that

$$v(n)+c\frac{\partial v(n)}{\partial c} = \sum_{j=0}^{N-n}\left[\prod_{m=0}^{j} S_{m+n}\right]. \tag{28}$$

Equations (26) and (28) establish (24). Q.E.D.

LEMMA 2. *Let $v(1),\ldots,v(N+1)$ be defined by* (7). *Then for each $n\in\{1,\ldots,N\}$,*

$$c\frac{\partial v(n)}{\partial c} = -q\frac{\partial v(n)}{\partial q} = \frac{b}{2}\frac{\partial v(n)}{\partial b}; \tag{2.1}$$

$$v(n)-q\frac{\partial v(n)}{\partial q} = \sum_{j=0}^{N-n}\left[\prod_{m=0}^{j} S_{m+n}\right] \geqq 0 \qquad \text{for} \quad q>0, \tag{2.2}$$

where the terms S_k, $1\leqq k\leqq N$, are defined by (25).

Proof. The assertion (2.1) is obvious from Theorem 3 when $q=0$. Suppose $q>0$. Using (15) and (17), with $v(N+1)\equiv 0$,

$$c\frac{\partial v(N)}{\partial c} = c\left(-\frac{(E\omega)^2B_N^2b^2}{q}\right) = -q\left(\frac{(E\omega)^2B_N^2cb^2}{q^2}\right) = -q\frac{\partial v(N)}{\partial c};$$

hence (2.1) holds for $n=N$. Suppose (2.1) holds for some $n=k+1\in\{2,\ldots,N\}$.

Then, using (15) and (17) again,

$$c\frac{\partial v(k)}{\partial c}=c\left(\frac{\partial v(k+1)}{\partial c}S_k-[1+v(k+1)]^2\frac{(E\omega)^2B_k^2b^2}{q}\right)$$

$$=-q\frac{\partial v(k+1)}{\partial q}S_k-qc[1+v(k+1)]^2\frac{[E\omega]^2B_k^2b^2}{q^2}$$

$$=-q\left(\frac{\partial v(k+1)}{\partial q}S_k+[1+v(k+1)]^2\frac{(E\omega)^2B_k^2cb^2}{q^2}\right)$$

$$=-q\frac{\partial v(k)}{\partial q}.$$

Finally, using (17) and (19) with $v(N+1)\equiv 0$,

$$\frac{b}{2}\frac{\partial v(N)}{\partial b}=-\frac{B_N^2(E\omega)^2cb^2}{q}$$

$$=c\left(-\frac{(E\omega)^2B_N^2b^2}{q}\right)$$

$$=c\frac{\partial v(N)}{\partial c}.$$

Now suppose this equality holds for some $k+1\in\{2,\ldots,N\}$. Then, using (17) and (19) again,

$$\frac{b}{2}\frac{\partial v(k)}{\partial b}=\frac{b}{2}\left(\frac{\partial v(k+1)}{\partial b}S_k-\frac{2B_k^2(E\omega)^2cb[1+v(k+1)]^2}{q}\right)$$

$$=c\frac{\partial v(k+1)}{\partial c}S_k-\frac{B_k^2(E\omega)^2cb^2[1+v(k+1)]^2}{q}$$

$$=c\left(\frac{\partial v(k+1)}{\partial c}-\frac{[1+v(k+1)]^2(E\omega)^2B_k^2b^2}{q}\right)$$

$$=c\frac{\partial v(k)}{\partial c}.$$

Thus (2.1) holds for all $n\in\{1,\ldots,N\}$. Equation (2.2) then follows from (2.1), (28), the definition of the terms S_k, and (1.1). Q.E.D.

THEOREM 5. *Let $\theta_n^{\text{opt}}(x)$ be defined by* (5). *Then*

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial E\omega}\right)=\operatorname{sgn}(-bx), \tag{5.1}$$

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial \operatorname{Var}(\omega)}\right)=\operatorname{sgn}(-qb[E\omega]x), \tag{5.2}$$

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial q}\right)=\operatorname{sgn}([E\omega]bx), \tag{5.3}$$

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial c}\right)=\operatorname{sgn}(-qb[E\omega]x), \tag{5.4}$$

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial b}\right)=\begin{cases}\operatorname{sgn}([E\omega]x) & \text{if}\quad cb^2\geqq q,\\ ? & \text{otherwise,}\end{cases} \tag{5.5}$$

$$\operatorname{sgn}\left(\frac{\partial b\theta_n^{\text{opt}}(x)}{\partial b}\right)=\operatorname{sgn}(-qb[E\omega]x). \tag{5.6}$$

Proof of (5.1). By Eq. (5),

$$\frac{\partial\theta_n^{\text{opt}}(x)}{\partial E\omega}=-\frac{x}{b}\qquad\text{if}\quad q=0, \tag{29}$$

proving (5.1) for $q=0$. Suppose $q>0$. Then

$$\begin{aligned}\frac{\partial\theta_n^{\text{opt}}(x)}{\partial E\omega}&=-\frac{cb[1+v(n+1)]B_nx}{q}\\ &=-\frac{(E\omega)cb\dfrac{\partial v(n+1)}{\partial E\omega}B_nx}{q}\\ &\quad-\frac{(E\omega)cb[1+v(n+1)]\left[\dfrac{\partial B_n}{\partial v(n+1)}\dfrac{\partial v(n+1)}{\partial E\omega}\right]x}{q}\\ &=-cb[1+v(n+1)]\frac{B_nx}{q}\\ &\quad-\frac{\partial v(n+1)}{\partial E\omega}\frac{(E\omega)cbB_n^2x}{q}.\end{aligned} \tag{30}$$

Combining (1.1), Theorem 2, and (3.1), the last signed expression in (30) has the sign of $-b(E\omega)^2x$, whereas the next to last signed expression in (30) has the sign of $-bx$. Thus

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\mathrm{opt}}(x)}{\partial E\omega}\right)=\operatorname{sgn}(-bx),$$

as asserted.

Proof of (5.2). Clearly $\partial\theta_n^{\mathrm{opt}}(x)/\partial\operatorname{Var}(\omega)=0$ if $q=0$. Suppose $q>0$. Then, defining $D\equiv cb^2[1+v(n+1)]+q>0$,

$$\begin{aligned}\frac{\partial\theta_n^{\mathrm{opt}}(x)}{\partial q}&=\left(-D(E\omega)cb\frac{\partial v(n+1)}{\partial\operatorname{Var}(\omega)}+\frac{\partial D}{\partial\operatorname{Var}(\omega)}(E\omega)cb[1+v(n+1)]\right)\frac{x}{D^2}\\&=\left(-c^2b^3[1+v(n+1)](E\omega)\frac{\partial v(n+1)}{\partial\operatorname{Var}(\omega)}\right.\\&\qquad\left.+c^2b^3[1+v(n+1)]\frac{\partial v(n+1)}{\partial\operatorname{Var}(\omega)}\right)\frac{x}{D^2}\\&\qquad-\left(q(E\omega)cb\frac{\partial v(n+1)}{\partial\operatorname{Var}(\omega)}\right)\frac{x}{D^2}\\&=-\left(q(E\omega)cb\frac{\partial v(n+1)}{\partial\operatorname{Var}(\omega)}\right)\frac{x}{D^2}.\end{aligned}\tag{31}$$

Combining (3.2) with (31),

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\mathrm{opt}}(x)}{\partial\operatorname{Var}(\omega)}\right)=\operatorname{sgn}(-q[E\omega]b),$$

as asserted.

Proof of (5.3). Defining $D\equiv cb^2[1+v(n+1)]+q$,

$$\begin{aligned}\frac{\partial\theta_n^{\mathrm{opt}}(x)}{\partial q}&=\left(-D(E\omega)cb\frac{\partial v(n+1)}{\partial q}+\frac{\partial D}{\partial q}(E\omega)cb[1+v(n+1)]\right)\frac{x}{D^2}\\&=(E\omega)cb\left[1+v(n+1)-q\frac{\partial v(n+1)}{\partial q}\right]x.\end{aligned}$$

By (2.2),

$$\operatorname{sgn}\left(1+v(n+1)-q\frac{\partial v(n+1)}{\partial q}\right)=\operatorname{sgn}(+1);$$

hence

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial q}\right)=\operatorname{sgn}([E\omega]bx),$$

as asserted.

Proof of (5.4). Clearly $\partial\theta_n^{\text{opt}}(x)/\partial c=0$ if $q=0$. Suppose $q>0$. Letting $D\equiv cb^2[1+v(n+1)]+q>0$,

$$\begin{aligned}\frac{\partial\theta_n^{\text{opt}}(x)}{\partial c}&=-D\left((E\omega)b[1+v(n+1)]+(E\omega)cb\frac{\partial v(n+1)}{\partial c}\right)\frac{x}{D^2}\\&\quad+(E\omega)cb[1+v(n+1)]\frac{\partial D}{\partial c}\frac{x}{D^2}\\&=-D(E\omega)b\left(1+v(n+1)+c\frac{\partial v(n+1)}{\partial c}\right)\frac{x}{D^2}\\&\quad+(D-q)(E\omega)b\left(1+v(n+1)+c\frac{\partial v(n+1)}{\partial c}\right)\frac{x}{D^2}.\end{aligned}\tag{32}$$

By (28), Eq. (32) can be restated as

$$\frac{\partial\theta_n^{\text{opt}}(x)}{\partial c}=-q(E\omega)b\left(1+\sum_{j=0}^{N-n}\left[\prod_{m=0}^{j}S_{m+n}\right]\right)\frac{x}{D^2},$$

where the terms S_k, $1\leqq k\leqq N$, are defined by (25). By (1.1), $S_k\geqq 0$, $1\leqq k\leqq N$. Thus

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial c}\right)=\operatorname{sgn}(-q[E\omega]bx),$$

as asserted.

Proof of (5.5). Defining $D \equiv cb^2[1+v(n+1)]+q>0$,

$$\frac{\partial\theta_n^{\text{opt}}(x)}{\partial b} = -D\left((E\omega)c[1+v(n+1)]+[E\omega]cb\frac{\partial v(n+1)}{\partial b}\right)\frac{x}{D^2}$$

$$+\frac{\partial D}{\partial b}(E\omega)cb[1+v(n+1)]\frac{x}{D^2}$$

$$= -D\frac{\partial v(n+1)}{\partial b}cb(E\omega)\frac{x}{D^2}$$

$$+(E\omega)c[1+v(n+1)]\{cb^2[1+v(n+1)]-q\}\frac{x}{D^2}.$$

Suppose $cb^2 \geqq q$. Since by assumption $c>0$, $b\neq 0$, and $q \geqq 0$, and $v(n+1)\geqq 0$ by Theorem 2, either $cb^2[1+v(n+1)]=q>0$ or $cb^2[1+v(n+1)]>q\geqq 0$. In either case, using (3.5),

$$\operatorname{sgn}\left(\frac{\partial\theta_n^{\text{opt}}(x)}{\partial b}\right)=\operatorname{sgn}([E\omega]x),$$

as asserted.

Proof of (5.6). If $q=0$, then clearly $\partial b\theta_n^{\text{opt}}(x)/\partial b=0$. Suppose $q>0$, and let

$$A_n \equiv 1+v(n+1)+\frac{b}{2}\frac{\partial v(n+1)}{\partial b}.$$

Then, by (1.5),

$$\frac{\partial b\theta_n^{\text{opt}}(x)}{\partial b}=\frac{\partial(-(E\omega)B_n cb^2[1+v(n+1)]x/q)}{\partial b}$$

$$= -(E\omega)cb\left(\frac{\partial B_n}{\partial b}b[1+v(n+1)]+2B_nA_n\right)\frac{x}{q}$$

$$= -2(E\omega)cbB_nA_n\left(1-\frac{cb^2[1+v(n+1)]B_n}{q}\right)\frac{x}{q}.$$

By (1.1) and Lemma 2,

$$B_nA_n\left(1-\frac{cb^2[1+v(n+1)]B_n}{9}\right)\equiv B_n^2A_n>0.$$

Thus

$$\operatorname{sgn}\left(\frac{\partial b\theta_n^{\text{opt}}(x)}{\partial b}\right)=\operatorname{sgn}(-qb[E\omega]x)$$

as asserted. Q.E.D.

The parameter sensitivities displayed in Table 1 indicate the local response of the optimal feedback-control law (5) and associated cost-performance measures (6) to changes in the basic parameters $E\omega$, $\operatorname{Var}(\omega)$, q, c, and b. Analysis of the global response of the closed-loop system to changes in these parameters reveals the crucial role of $\operatorname{Var}(\omega)$. Specifically, no optimal control law exists in the limit as $N \to \infty$ if $\operatorname{Var}(\omega) \geqq 1$. This phenomenon appears to have first been noted by Athans et al. [13]. Since their discussion is somewhat heuristic, a simple precise proof of this fact will be given below.

The closed-loop system takes the form

$$\begin{aligned} x_{n+1} &= \omega_n x_n + b\theta_n^{\text{opt}}(x_n) \\ &= B_n[E\omega]x_n + [\omega_n - E\omega]x_n, \end{aligned} \tag{33}$$

where

$$B_n \equiv \frac{q}{cb^2[1+v(n+1)]+q}.$$

Thus for each $n \in \{2,\ldots,N\}$ the variance $\operatorname{Var}(x_{n+1})$ of x_{n+1} satisfies

$$\begin{aligned} \operatorname{Var}(x_{n+1}) &= \left[\operatorname{Var}(\omega) + B_n^2(E\omega)^2\right]\operatorname{Var}(x_n) + \operatorname{Var}(\omega)(Ex_n)^2 \\ &\geqq [\operatorname{Var}(\omega)]^n x_1^2. \end{aligned} \tag{34}$$

If $x_1 \neq 0$ and $\operatorname{Var}(\omega) > 1$, the variability of x_{n+1} increases without bound as $n \to \infty$. The following result therefore comes as no surprise.

THEOREM 6. *Consider the control problem described by* (1) *and* (2). *If* $\operatorname{Var}(\omega) \geqq 1$, *then no optimal control law exists in the limit as* $N \to \infty$.

Proof. By Theorem 1 the expected total cost associated with the optimal feedback-control law for the N-period control problem is given by

$$T_1(x_1) = cv(1)x_1^2, \tag{35}$$

where $v(1)$ is recursively generated by the relation (7). To establish that no optimal control law exists in the limit as $N \to \infty$ if $\mathrm{Var}(\omega) \geqq 1$, it suffices to show that $v(1)$ increases without bound as $N \to \infty$ if $\mathrm{Var}(\omega) \geqq 1$.

By Lemma 2, the terms $(v(n))_{n=1}^{N}$ recursively generated by (7) satisfy

$$v(n) \geqq v(n+1) \geqq v(N+1) \equiv 0;$$

and in the course of proving Lemma 2 it was shown that for $n \in \{2,\ldots,N\}$,

$$v(n-1)-v(n)=\mathrm{Var}(\omega)\,[v(n)-v(n+1)]+\frac{[v(n)-v(n+1)](E\omega)^2 q^2}{D(n)D(n+1)}, \quad (36)$$

where $D(k) \equiv cb^2[1+v(k)]+q>0$, $k \in \{1,\ldots,N\}$. Since the final term in (36) is nonnegative, and $v(N) \geqq \mathrm{Var}(\omega)$, it follows that for any $n \in \{2,\ldots,N\}$

$$v(n-1)-v(n) \geqq [\mathrm{Var}(\omega)]^{N+2-n}.$$

If $\mathrm{Var}(\omega) \geqq 1$, the distance between the monotone decreasing terms $v(1),\ldots,v(N)$ is therefore at least one, with $v(N) \geqq 1$. Thus $v(1)$ increases without bound as $N \to \infty$. Q.E.D.

As a final remark, we note the following interesting parameter sensitivity relations for the optimal-control and cost-to-go terms $\theta_n^{\mathrm{opt}}(x)$ and $T_n(x)$. By Lemma 1 and (2.1) it is easily verified that

$$c\frac{\partial B_n}{\partial c}=-q\frac{\partial B_n}{\partial q}=\frac{b}{2}\,\frac{\partial B_n}{\partial b}; \quad (37)$$

and by (5),

$$b\theta_n^{\mathrm{opt}}(x)=(E\omega)[B_n-1]x. \quad (38)$$

Combining (37) and (38),

$$c\frac{\partial \theta_n^{\mathrm{opt}}(x)}{\partial c}=-q\frac{\partial \theta_n^{\mathrm{opt}}(x)}{\partial q}=\frac{1}{2}\,\frac{\partial b\theta_n^{\mathrm{opt}}(x)}{\partial b}. \quad (39)$$

Moreover, it directly follows from (6) and (2.1) that

$$c^2\frac{\partial c^{-1}T_n(x)}{\partial c}=-q\frac{\partial T_n(x)}{\partial q}=\frac{b}{2}\,\frac{\partial T_n(x)}{\partial b}. \quad (40)$$

4. PARAMETER SENSITIVITIES: SIMULATION RESULTS

Computer simulations were carried out on an IBM 370/158 for the control problem described in Sec. 2 in order to test the sensitivity of the state, control, single-period cost, and cost-to-go trajectories to changes in the time horizon N, cost-function coefficient q, and state-coefficient mean $E\omega$ and standard deviation $\mathrm{SD}(\omega)$.[1] The initial state x_1^*, control coefficient b, and cost-function coefficient c were held constant at value 1 throughout all simulations. State-coefficient "observations" ω_n were generated using a pseudo-random-number generator for normal deviates $N(E\omega, \mathrm{SD}(\omega))$.

Examples of two complete simulation runs are presented in Tables 2 and 3 below. The principal table entries x_n, $\theta_n^{\mathrm{opt}}(x_n)$, $W(\omega_n, \theta_n^{\mathrm{opt}}(x_n), x_n)$, and $T_n(x_n)$ represent averages over 50 trial runs. The σ-values are the corresponding sample standard deviations. The final column of each table gives the sample

TABLE 2
State, Control, Single-Period Cost, and Cost-to-Go Trajectories[a]

Period n	x_n / σ	$\theta_n^{\mathrm{opt}}(x_n)$ / σ	$W(\omega_n, \theta_n^{\mathrm{opt}}(x_n), x_n)$ / σ	$T_n(x_n)$ / σ	$\overline{E}_n\omega$ / $\overline{\mathrm{SD}}_n(\omega)$
1	1.000	−1.953	4.866	45.623	0.0
	0.0	0.000	1.597	0.000	0.0
2	−0.079	0.154	4.869	42.794	1.874
	1.022	1.990	7.602	65.025	1.022
3	−0.029	0.057	4.711	31.674	1.988
	0.940	1.823	14.329	72.279	1.008
4	−0.234	0.452	5.953	42.883	1.971
	1.154	2.225	31.658	237.050	0.986
5	0.034	−0.065	3.725	20.813	2.028
	0.893	1.708	16.436	88.091	1.009
6	−0.103	0.194	4.165	17.058	1.992
	0.890	1.683	22.533	91.145	1.006
7	−0.034	0.064	5.993	21.300	2.018
	1.137	2.108	33.820	118.512	0.996
8	−0.092	0.165	5.499	18.185	1.987
	1.240	2.210	31.988	106.846	0.996
9	0.108	−0.173	2.135	3.867	1.981
	0.725	1.160	12.318	22.555	0.997
10	0.075	−0.075	1.993	2.282	2.003
	0.869	0.869	9.579	12.979	0.986
11	−0.056				
	1.109				

Total realized costs: 43.859.
[a]Parameter values: $N = 10$, $q = 1$, $E\omega = 2$, $\mathrm{SD}(\omega) = 1.0$.

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[1]Programs are available upon request.

TABLE 3

State, Control, Single-Period Cost, and Cost-toGo Trajectories[a]

Period n	x_n σ	$\theta_n^{opt}(x_n)$ σ	$W(\omega_n, \theta_n^{opt}(x_n), x_n)$ σ	$T_n(x_n)$ σ	$\bar{E}_n\omega$ $\overline{SD}_n(\omega)$
1	1.000	−2.000	6.385	13,161.813	0.0
	0.0	0.000	3.646	0.000	0.0
2	−0.188	0.376	13.984	13,952.324	1.811
	1.533	3.065	22.935	21,318.672	1.533
3	−0.068	0.135	32.225	11,541.711	1.982
	2.108	4.211	112.598	26,326.754	1.512
4	−0.799	1.595	77.739	16,655.516	1.957
	3.719	7.424	404.923	91,114.375	1.479
5	0.172	−0.342	137.094	10,216.230	2.042
	4.478	8.916	672.724	46,699.887	1.514
6	−0.876	1.735	408.129	12,838.313	1.988
	7.531	14.909	2247.859	70,068.688	1.509
7	−0.466	0.910	1109.455	17,645.297	2.028
	13.514	26.387	6222.000	98,319.375	1.494
8	−0.401	2.639	1912.439	16,558.527	1.981
	20.258	38.157	1070.383	93,933.375	1.494
9	3.042	−5.111	2230.196	6,820.063	1.972
	20.982	35.250	3856.977	42,153.148	1.495
10	2.799	−2.799	2274.022	4,086.502	2.005
	30.882	30.882	967.527	25,619.145	1.479
11	−2.697				
	36.128				

Total realized costs: 8201.664.

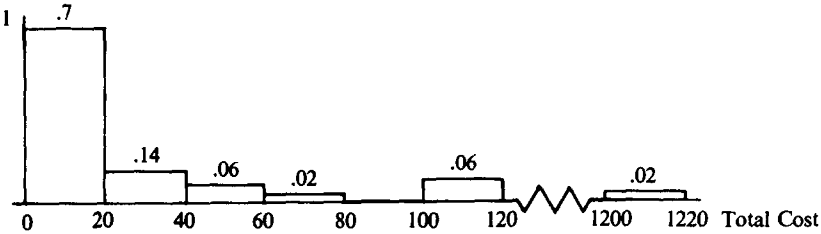
[a]Parameter values: $N=10$, $q=1$, $E\omega=2$, $\mathrm{SD}(\omega)=1.5$.

means $\bar{E}_n\omega$ and sample standard deviations $\overline{\mathrm{SD}}_n(\omega)$ for the generated pseudo random numbers $\omega_1,\ldots,\omega_{n-1}$, $2 \le n \le N$, averaged over fifty trial runs.

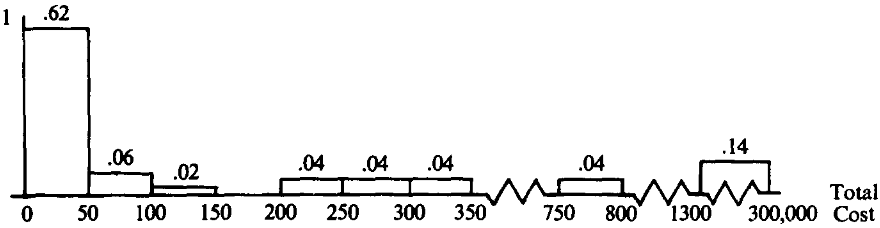
Since the standard deviation $\mathrm{SD}(\omega)$ for the state coefficient ω is deliberately being set at values for which the closed-loop system is unstable, it is not surprising that sample standard deviations are large. The sample distributions become increasingly thick-tailed as $\mathrm{SD}(\omega)$ is increased above 1.0. To illustrate this phenomenon, sample distributions for the (averaged) total realized costs appearing in Tables 2 and 3 are displayed in Fig. 1.

A second characteristic illustrated in Tables 2 and 3 is the tremendous sensitivity of the single-period cost and cost-to-go trajectories to changes in the standard deviation $\mathrm{SD}(\omega)$. As $\mathrm{SD}(\omega)$ varies from 1.0 to 1.5, total realized costs increase by a factor on the order of 10^2. Moreover, single-period costs are monotone increasing for $\mathrm{SD}(\omega)=1.5$, whereas they are obviously damping to zero for $\mathrm{SD}(\omega)=1.0$.

Since presumably a controller views total realized costs as a key aspect of the control process, this summary characteristic is displayed below in Table 4



(a)



(b)

Fig. 1. Sample distributions for total realized costs over fifty trial runs. (a) Parameter values: $N=10$, $q=1$, $E\omega=2$, $SD(\omega)=1.0$; sample mean: 43.859. (b) Parameter values: $N=10$, $q=1$, $E\omega=2$, $SD(\omega)=1.5$; sample mean: 8201.664.

TABLE 4
Total Realized Costs[a]

		$q=0$	$q=1$		
N	SD(ω)	$E\omega=0, 1, 2$	$E\omega=0$	$E\omega=1$	$E\omega=2$
10	1	9.116	9.116	16.966	43.859
	1.5	2,366.137	2,366.137	3,849.684	8,201.664
50	1	6.038	6.038	13.166	34.429
	1.5	2,351,999.000	2,351,999.000	4,700,150.000	11,745,118.000
100	1	2.996	2.996	6.911	18.833
	1.5	83,622.125	83,622.125	167,244.875	418,116.688

[a]All table entries represent averages over 50 trial runs.

for simulations using the following parameter values:

N (time horizon): 10, 50, 100
q (cost coefficient): 0, 1
$E\omega$ (state-coefficient mean): 0, 1, 2
SD(ω) (state-coefficient standard deviation): 1.0, 1.5

The skewness of the total cost sample distributions should of course be kept in mind when interpreting these results.

Several uniformities are evident in Table 4. First, total realized costs are independent of the state-coefficient mean $E\omega$ when $q=0$; and secondly, total realized costs are independent of q when $E\omega=0$. Recalling Eq. (5) for the optimal control law, single-period costs take the form

$$W(\omega_n, \theta_n^{\text{opt}}(x_n), x_n) = (\omega_n - E\omega)^2 x_n^2$$

when $q=0$, where $\omega_n - E\omega \sim N(0, \text{SD}(\omega))$. Similarly, for all values of q one obtains

$$W(\omega_n, \theta_n^{\text{opt}}(x_n), x_n) = \omega_n^2 x_n^2$$

when $E\omega=0$, where $\omega_n \sim N(0, \text{SD}(\omega))$. Thus the two uniformities mentioned above can be expected to hold, approximately, regardless of other parameter values. The exactness of the uniformities is a result of the random-number generator, which generates $N(E\omega, \text{SD}(\omega))$ deviates from $N(0, \text{SD}(\omega))$ deviates by simply adding the desired mean.

A more significant uniformity in Table 4 is the dramatic deterioration of optimal-control-law cost performance when the standard deviation $\text{SD}(\omega)$ increases from 1.0 to 1.5, regardless of other parameter values. In contrast, total realized costs increase rather gradually along rows, as q and $E\omega$ increase. In addition, when $\text{SD}(\omega)=1$, total realized costs monotonically decrease as the time horizon N increases from 10 to 100, for all values of q and $E\omega$. In contrast, when $\text{SD}(\omega)=1.5$, total realized costs increase by a factor of approximately 10^3 as the time horizon N increases from 10 to 50, and decrease by a factor of approximately $\frac{1}{28}$ as N increases from 50 to 100.

Table 5 focuses on what appear to be the two key parameters affecting cost performance, the time horizon N and standard deviation $\text{SD}(\omega)$, enlarging the range considered for each. Somewhat surprisingly, Table 5 indicates that total realized costs are approximately independent of the time horizon N when the

TABLE 5
Total Realized Costs[a]: $q=1$, $E\omega=2.0$

$\text{SD}(\omega)$ \ N	10	40	70	100
0.05	4.832	4.839	4.777	4.807
1.0	43.859	22.384	44.664	18.833
1.5	8201.664	648,379.250	14,095,970.000	418,116.688

[a]All table entries represent averages over 50 trial runs.

TABLE 6
First Zero State or Final State[a]: $q=1$, $E\omega=2.0$

N / SD(ω)	10	40	70	100
0.5	$x_9=0$	$x_6=0$	$x_{13}=0$	$x_6=0$
1.0	$x_{11}=-0.056$	$x_{23}=0$	$x_{26}=0$	$x_{23}=0$
1.5	$x_{11}=-2.697$	$x_{41}=-0.017$	$x_{71}=-5.054$	$x_{88}=0$

[a]All table entries represent averages over 50 trial runs.

state-coefficient standard deviation SD(ω) has value 0.5. For the remaining two standard-deviation values, 1.0 and 1.5, cost performance deteriorates at $N=70$, significantly so for SD(ω)$=1.5$. In particular, what appears in Table 4 to be a monotonic relationship between realized costs and the time horizon N for SD(ω)$=1.0$ now appears to be more accurately described as a cyclic relationship. Clearly, many more simulation runs are needed in order to fully understand the dependence of total realized costs on the time horizon N.

In the final table (Table 6), a second summary characteristic is displayed: the first zero state for each averaged simulation run, if such a state value exists, or the final state value otherwise. Recalling Eq. (5) for the optimal control law and the definition (3) for single-period costs, it is clear that

$$x_n=0 \Rightarrow \theta_n^{\text{opt}}(x_n)=0 \text{ and } W(\omega_n,\theta_n^{\text{opt}}(x_n),x_n)=0.$$

Thus Table 6 gives some indication of the shape of the state, control, and single-period cost trajectories.

5. CONCLUSION

A detailed parameter-sensitivity study has been carried out for a discrete-time, finite-horizon, linear-quadratic control problem with random state coefficients. Both analytical and computer simulation studies are used to examine the sensitivity of state, control, single-period cost, and cost-to-go trajectories to changes in the time horizon N, the state equation parameters $[E\omega, \text{Var}(\omega), b]$, and the cost-function parameters $[c,q]$.

One interesting characteristic revealed by analysis is the existence of simple linear relationships between the (b,c,q)-parameter sensitivities for the optimal feedback control selections and for the corresponding cost-to-go expressions. A second interesting characteristic derived analytically is the sign determinateness of parameter sensitivities solely on the basis of the signs of the parameters and current state in all but one case, the change in control intensity with respect to the gain coefficient b.

An important characteristic revealed by simulations is the extreme skewness of the sample cost distributions when the state-coefficient standard deviation

SD(ω) takes values greater than 1.0, regardless of other parameter values. Not surprisingly in view of the observations by Athans et al. [13] (see also Theorem 6, Sec. 3), there is a corresponding dramatic deterioration in average cost performance. What remains to be explained is why, for each fixed N, the deterioration in average cost performance as SD(ω) varies from 0.5 to 1.5 is significantly more noticeable for the middle-range time horizons $N=40$ and $N=70$ than for either the short time horizon $N=10$ or the long time horizon $N=100$.

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